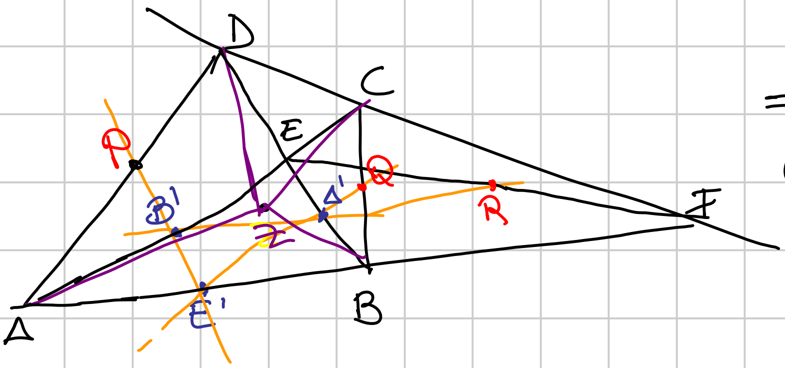
# G3 MEDIUM

-Maria-

Titolo nota 08/09/2010

- Linea di Gauss e quadrilateri
- Geo proiettiva: - birapporti
  - polarità
- Ex/Incerchi
- Simmediane ...




P, Q, R pti medi
$\Rightarrow$ P, Q, R allineati (LINEA DI GAUSS)

Dim 1: geo analitica (+ affinità) es.

Dim 2: Menelao sul triangolo $A'E'B'$.

$$\frac{E'P}{PB'} \cdot \frac{B'R}{RA'} \cdot \frac{A'Q}{QE'} = -1$$

Omotetia manda $AE'P$ in $ABD$

$$\frac{E'P}{PB'} = \frac{BD}{DE} \qquad \frac{B'R}{RA'} = \frac{AF}{FB} \qquad \frac{A'Q}{QE'} = \frac{EC}{CA}$$

$$\frac{BD}{DE} \cdot \frac{AF}{FB} \cdot \frac{EC}{CA} = -1$$

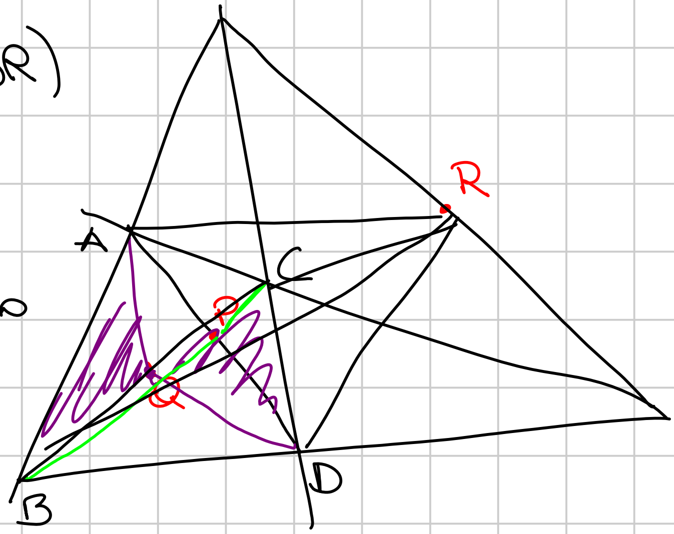
Menelao su $BEA$, retta $DFC$

Dim 3: luogo di $Z$ t.c.

$$(ABZ) + (CDZ) = (ACZ) + (BDZ)$$

$(ABR) + (CDR) = (ACR) + (BDR)$

(esercizio)



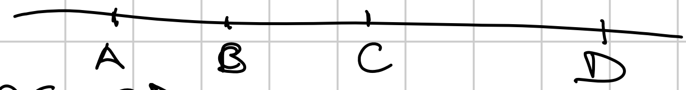

$\mathbb{R}^2 \longrightarrow \mathbb{R}$

$Z \longrightarrow (ABZ)$ è un piano

- Contiene $P, Q, R$
- Non contiene i vertici (a meno di casi da verif a mano)

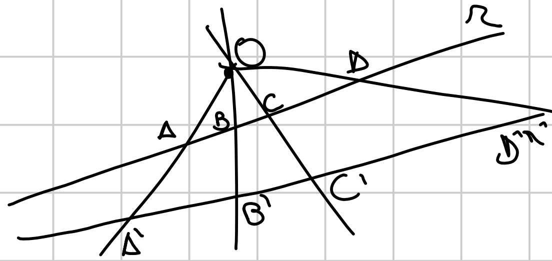
$\Rightarrow$ è una retta.

# BIRAPPORTI




$$(A,B,C,D) = \frac{AC \cdot BD}{BC \cdot AD}$$

(possono essere $+\infty$)




Prop:

$(A,B,C,D) = (A',B',C',D')$

Dim:

$$\frac{AC}{\sin A\hat{O}C} = \frac{AO}{\sin \hat{C}} \qquad \frac{BC}{\sin B\hat{O}C} = \frac{OC}{\sin \hat{B}} = \frac{BO}{\sin \hat{C}}$$

$$\frac{BD}{\sin B\hat{O}D} = \frac{DO}{\sin \hat{B}} = \frac{BO}{\sin \hat{D}} \qquad \frac{AD}{\sin A\hat{O}D} = \frac{AO}{\sin \hat{D}}$$

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{\sin A\hat{O}C \cdot \sin B\hat{O}D}{\sin B\hat{O}C \cdot \sin A\hat{O}D}$$

Posso definire il bir di 4 rette concorrenti

$(ABCD) = \lambda$ $\qquad$ $(BACD) \overset{?}{=} \dfrac{BC \cdot AD}{AC \cdot BD} = \dfrac{1}{\lambda}$

$(ABDC)$ $\nearrow$

I possibili birapporti $\left\{ \lambda, \dfrac{1}{\lambda}, 1-\lambda, 1-\dfrac{1}{\lambda}, \dfrac{1}{1-\lambda}, 1-\dfrac{1}{1-\lambda} \right\}$

(esercizio)

$$(ABCD) + (ACBD) = 1$$

$$\frac{(C-A)(D-B)}{(C-B)(D-A)} + \frac{(B-A)(D-C)}{(B-C)\ (D-A)} \overset{?}{=} 1$$

$$(C-A)(D-B) - (B-A)(D-C) \overset{?}{=} (C-B)(D-A).$$

Se $B=C$ si annulla il 1° membro

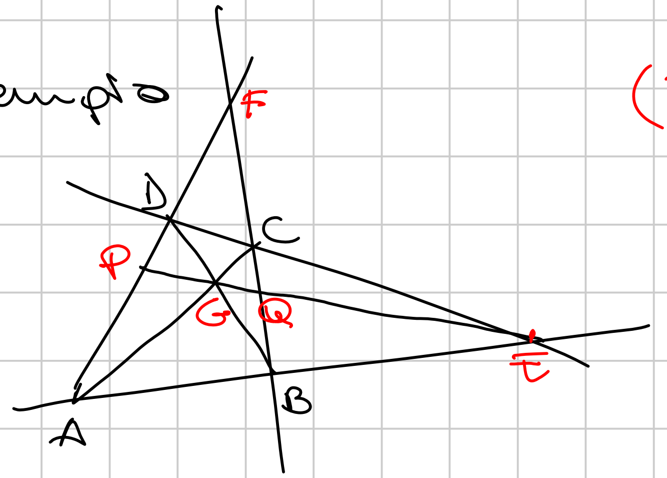
$(B-C) \mid$ 1° membro

BD

~~Def~~: $(A,B,C,D) = -1$

$\rightsquigarrow$ QUATERNA ARMONICA

Esempio



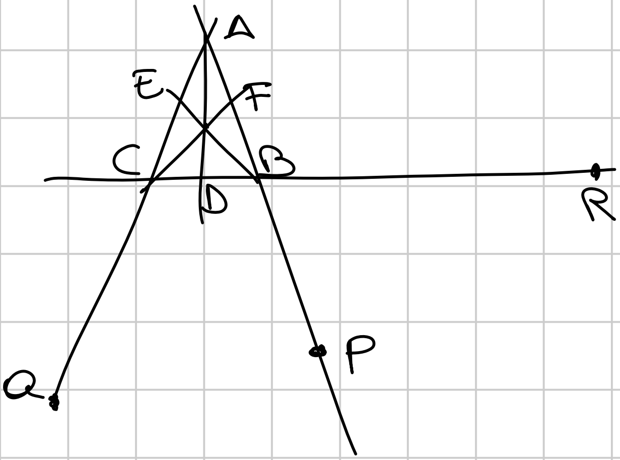

$(E,G,P,Q) = -1$

$(E,G,P,Q) = (D,A,P,F)$ centro C retta FD

$= (G,E,P,Q)$ centro B retta EG $= \dfrac{1}{(EGPQ)}$

$(E,G,P,Q) = $ 1 NO / $-1$ SÌ

Esercizio




$(ABFP) = -1$

$(ACEQ) = -1$

$(CBDR) = -1$

$\Rightarrow$ P, Q, R all.

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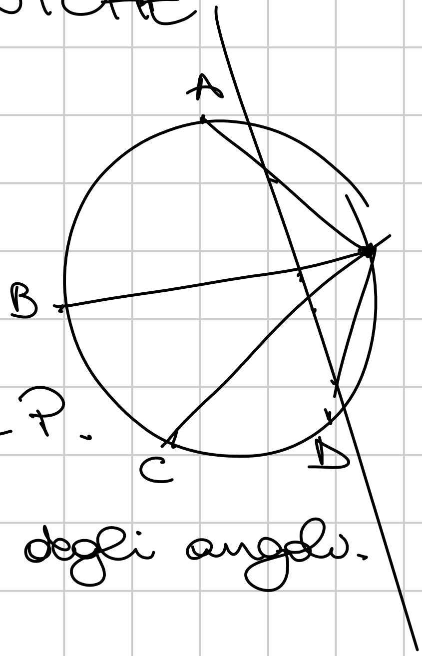
BIRAPPORTI E CONICHE

$\Gamma$ (conica) circo

$(A, B, C, D)_P =$
bir di $PA$ $PB$ $PC$ $PD$




Prop: non dipende da $P$.

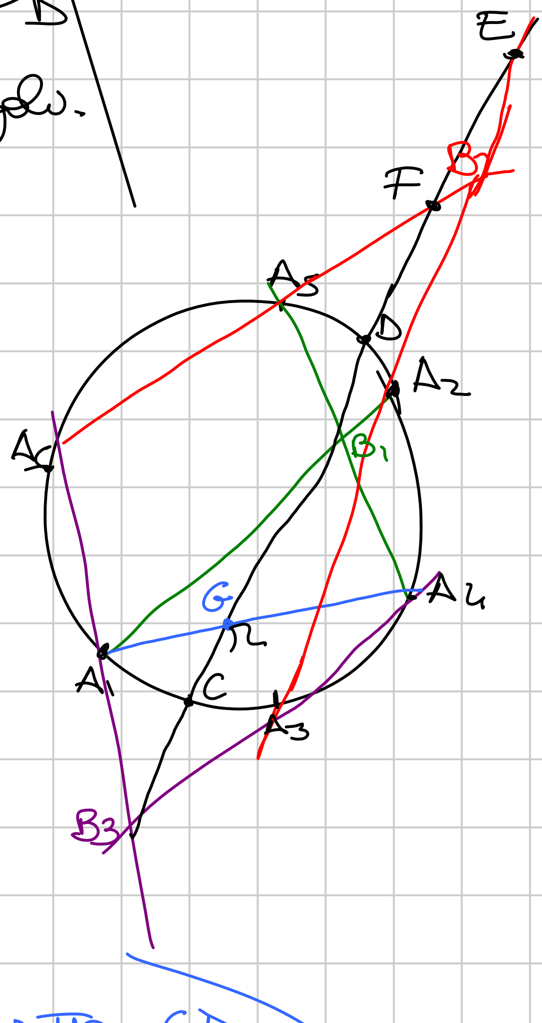
Dim:
Era scritto in funz degli angoli.

Teo di Pascal:

$A_1, \ldots, A_6$ su (conica) circo.

$A_i A_{i+1} \cap A_{i+3} A_{i+4} = B_i$
$i = 1, \ldots, 3$

$B_i$ sono allineati.




Dim:

$(C B_3 F D) \overset{\longrightarrow}{=} (C\ A_1\ A_5\ D)$ centro $A_6$, $\Gamma$

$\overset{\longrightarrow}{=} (C\ G\ B_1\ D)$ centro $A_4$ sulla retta $CD$

$(C B_3 E D) =$ $(C\ A_4\ A_2\ D)$ centro $A_3, \Gamma$ $= (C\ G\ B_1\ D)$ centro $A_1$

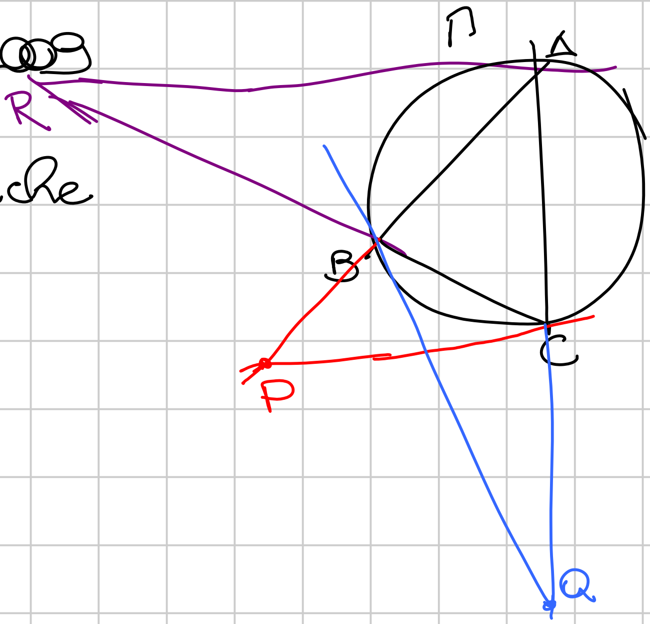
$\Rightarrow F = E$.

Cesenatico 5 2009

Esempio

$AB \cap t_C$, cicliche

sono allineati




Pascal su
AABBCC

Esempio 2:

A', B', C' pti medi degli archi

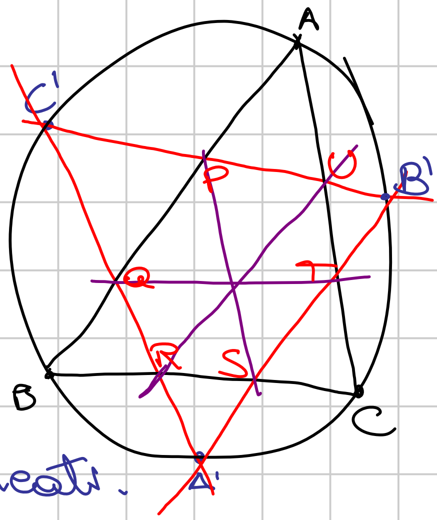
Tesi: PS QT UR concorrono.

NELL'INCENTRO.

Pascal

A A'C' B BC

$AA' \cap BB' = I$, R, U sono allineati.




# POLARITÀ

Γ (conica) circo

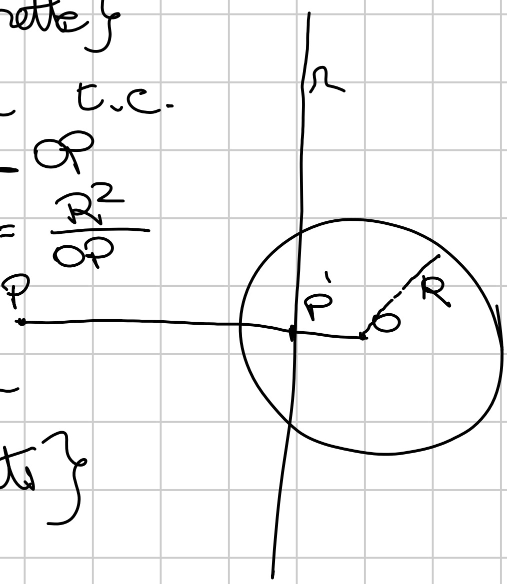
MAPPA DI DUALITÀ

{punti del piano} ⟶ {rette}

P ⟶ r t.c.

$r \perp OP$

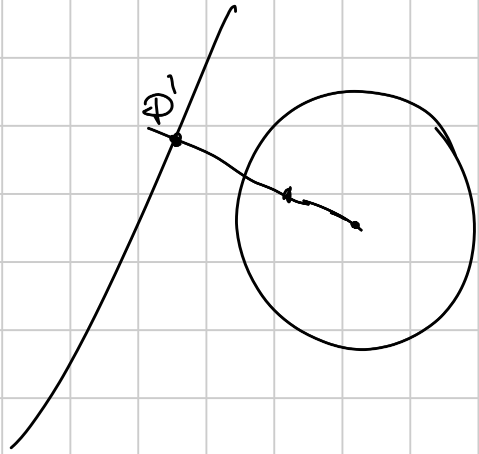
$OP' = \frac{R^2}{OP}$




Questa mappa è invertibile.

{rette} ⟶ {punti}

r ⟶ P

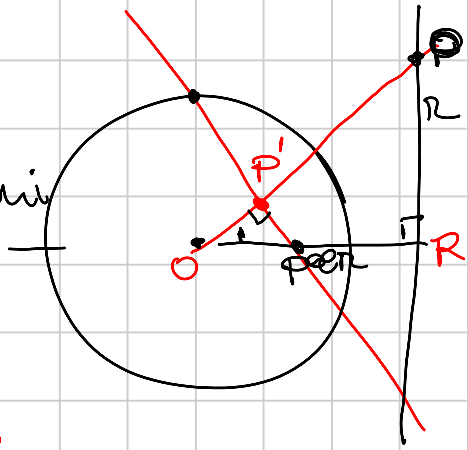



Proprietà

① $P \in \mathcal{C} \iff P \in pol_{\mathcal{C}} P$

② (Mantiene) Rovescia le inclusioni

$P \in r \iff pol P \ni pol\, r$




$OP' \cdot OP = R^2$

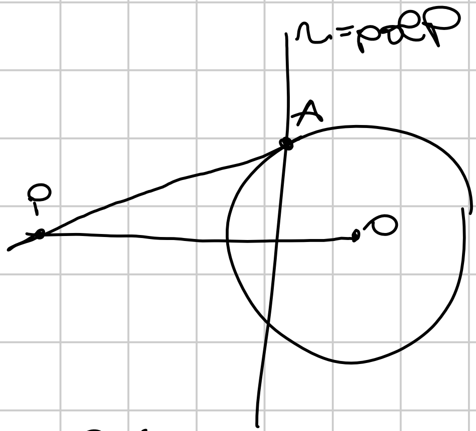
È suff vedere $O\, polR\, P' \sim ORP$

$\overline{O\, polR}\; \overline{OR} = R^2$

③ $pol_{\mathcal{C}}(P) \cap \mathcal{C} = \{A, B\}$

PA, PB sono tangenti

$A \in pol\, r$ $\quad polA \ni P$



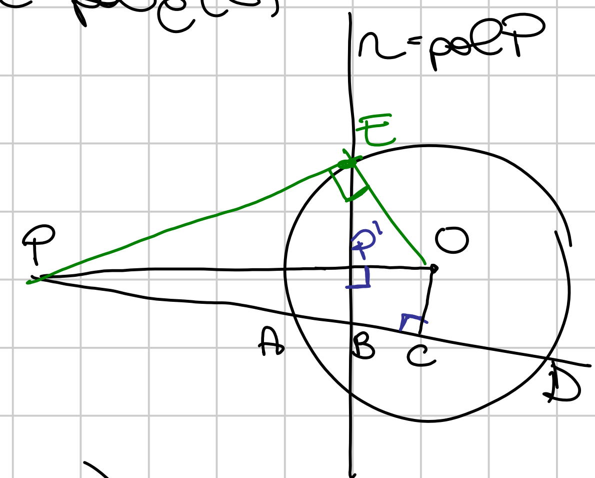

④ $pol_{\mathcal{C}}(P) \cap pol_{\mathcal{C}}(Q) = pol_{\mathcal{C}}(PQ)$

⑤ retta per $pol_{\mathcal{C}}(r)$ e $pol_{\mathcal{C}}(s)$
$= pol\,(r \cap s)$

Esercizio

① C = pto medio AD

$PA \cdot PD = PB \cdot PC$



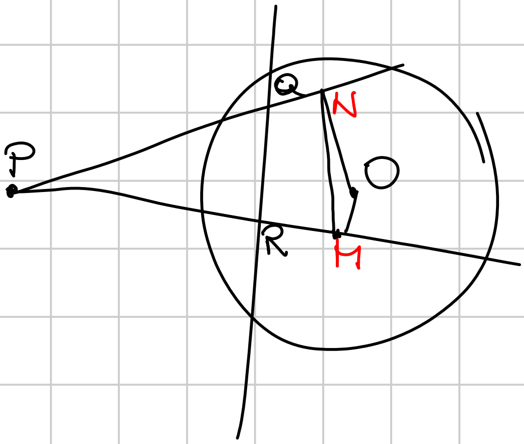

② $(AD\, PB) = -1$ (x casa)

$PA \cdot PD = \mathrm{pot}_\Gamma(P)$

$PB \cdot PC = PP' \cdot PO \quad = OP^2 - R^2$

Euclide su OPE $\quad PE^2 = PP' \cdot PO$

RMNQ è ciclico

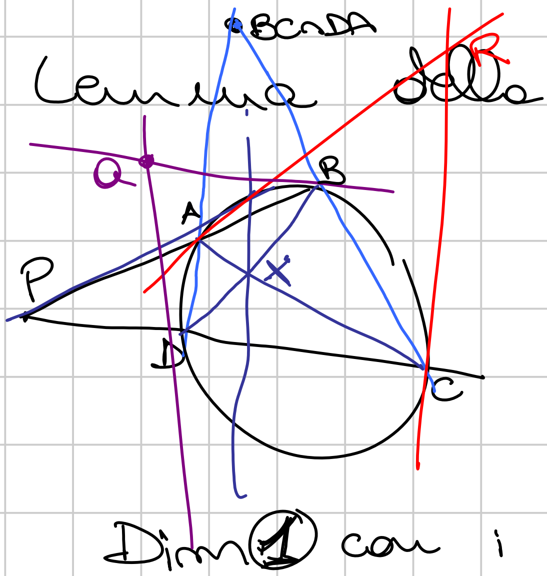



$PQ \cdot PN = \mathrm{Pow}_\Gamma P$

$= PR \cdot PM$.

Lemma della polare!

$X \in \mathrm{pol}\, P$




Dim ① con i Birapporti sulle rette PB e PC.

Dim ② Pascal su AABCCD

$AA \cap CC = R \qquad AB \cap CD = P \qquad BC \cap DA$

Pascal su BBADDC

$BB \cap DD = Q \qquad P \qquad BC \cap AD$

$\Longrightarrow$ $P, \underset{\text{pol}\, BD}{Q}, \underset{\text{pol}(AC)}{R}$, $BC \cap AD$ sono all.

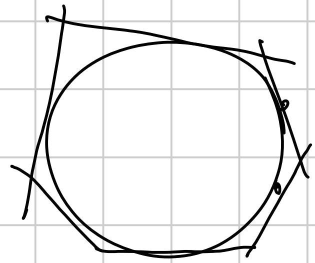
$P \in$ retta per pol BD e pol AC

$\Longleftrightarrow AC \cap BD \in \mathrm{pol}\, P$

Teo di Br.

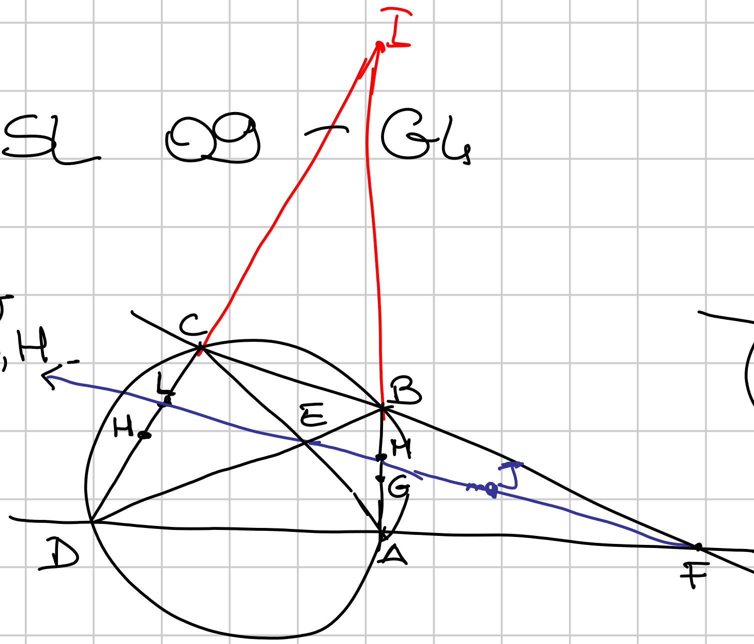
$C_1, \dots, C_6$ esagono circoscr. a una conica

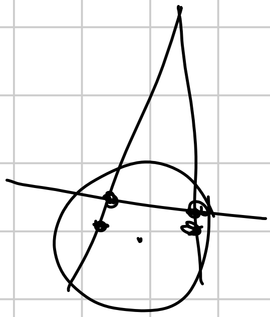
$\Rightarrow C_iC_{i+3}$ concorrono.



Esempio IMOSL 09 - G4

Tesi: EF è tang. al cerchio per E, G, H.



Sol 1 simm risp alla bis + omotetia.

Sol 2: J pto medio di EF

$\Rightarrow$ H, G, J all. (retta Gauss).

Tesi: $JE^2 = JG \cdot JH$.

$(F, E, M, L) = -1$

$$FM \cdot EL = (-\underbrace{EM}_{=ME}) \cdot FL \qquad EJ = JF.$$

$$(EJ + JM)(JL - EJ) = (EJ - JM)(EJ + JL)$$

$$\cancel{EJ \cdot JL} - EJ^2 + JM \cdot JL - \cancel{JM \cdot EJ} =$$

$$= EJ^2 - JM \cdot JL - \cancel{JM \cdot EJ} + \cancel{EJ \cdot JL}$$

$$EJ^2 = JM \cdot JL.$$

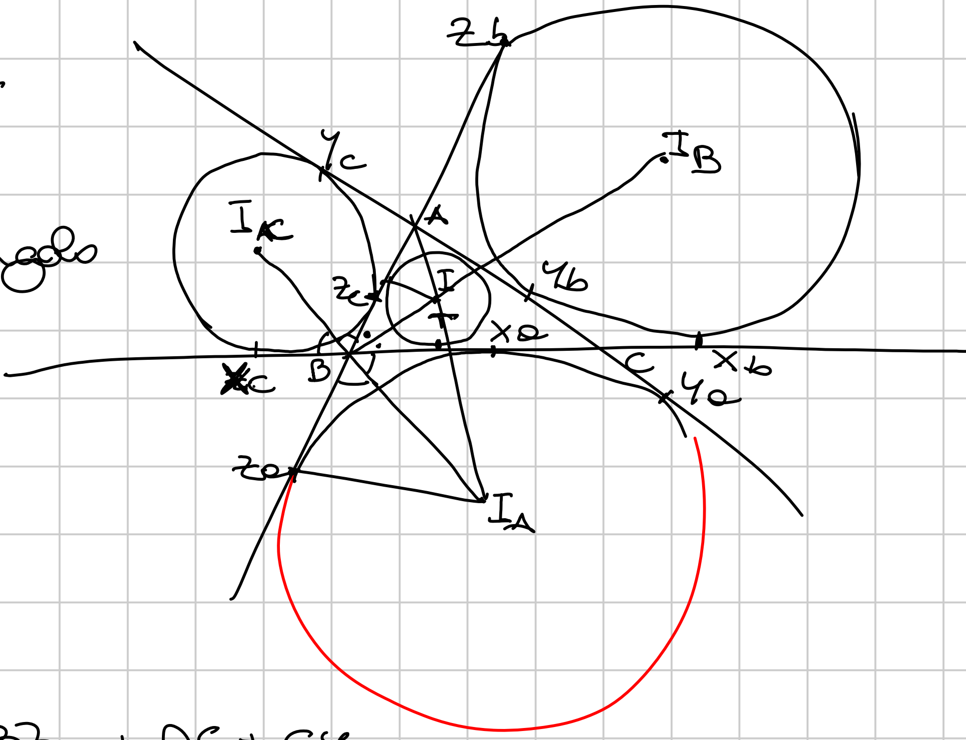
Resta da dim: $JG \cdot JH = JM \cdot JL$

$\Longleftarrow$ HLGM ciclico

$\Longleftarrow$ $IL \cdot IH = IM \cdot IG$

ok per esercizio di prima.

# IN / EX-CERCHI.




(1) $I_A, B, I_C$ all.

$\Rightarrow$ ABC è il triangolo ortico di $I_A I_B I_C$

$BI_B \perp I_A I_C$

(2) $AZ_a = \dfrac{a+b+c}{2}$

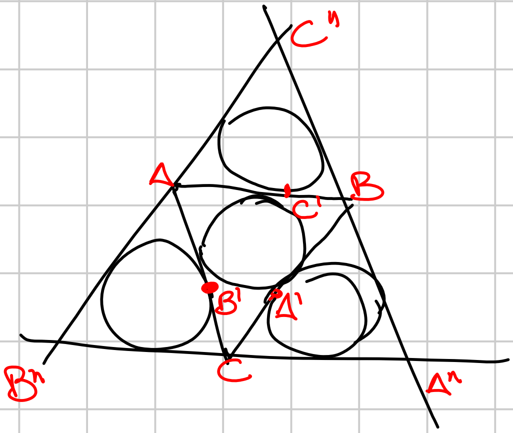
$AZ_a = AY_a$

$AZ_a + AY_a = AB + BZ_a + AC + CY_a$
$= c + BX_a + b + CX_a = a+b+c$

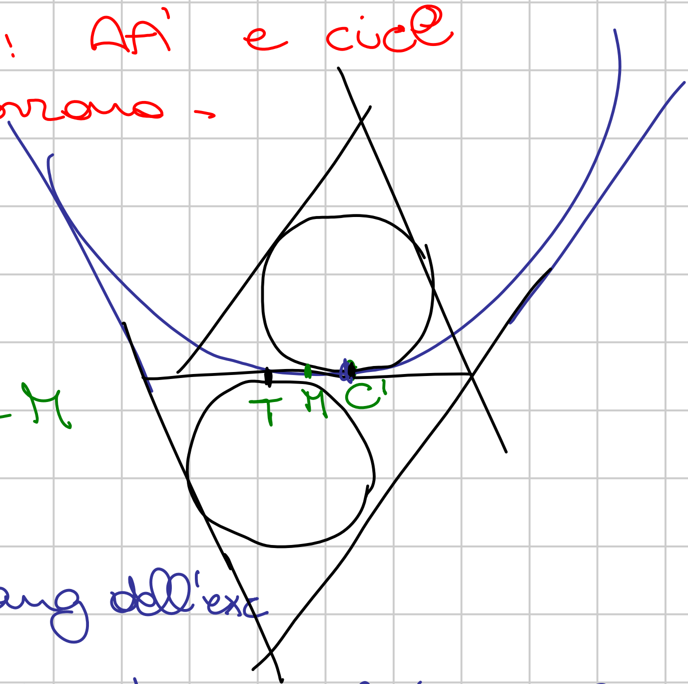
(3) $BX_a = CT$

$\overset{\shortparallel}{BZ_a} - c = \dfrac{a+b-c}{2}$

(4) $AX_a, BY_b, CZ_c$ concorrono.
(per Ceva + (3)) PUNTO DI NAGEL.




Fatto: AA' e cicl concorrono.



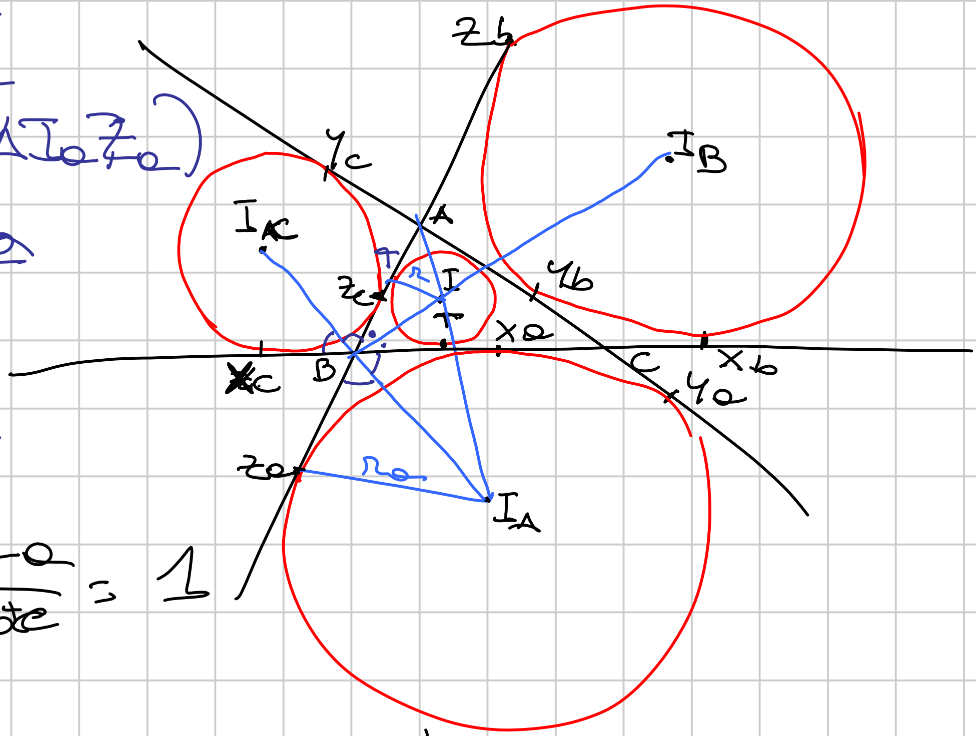

simm rispetto a M

TM=TC' per

$\Rightarrow$ C' è il pto di tang dell'exc

$\Rightarrow$ concorrono nel pto di Nagel del triangolo mediale.

(5) $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$




$\frac{r}{r_a}=\frac{AT}{AZ_a}$

(sim $\Delta IT$, $\Delta I_aZ_a$)

$=\frac{2\cdot\frac{b+c-a}{2}}{a+b+c}$

$=\frac{b+c-a}{a+b+c}$

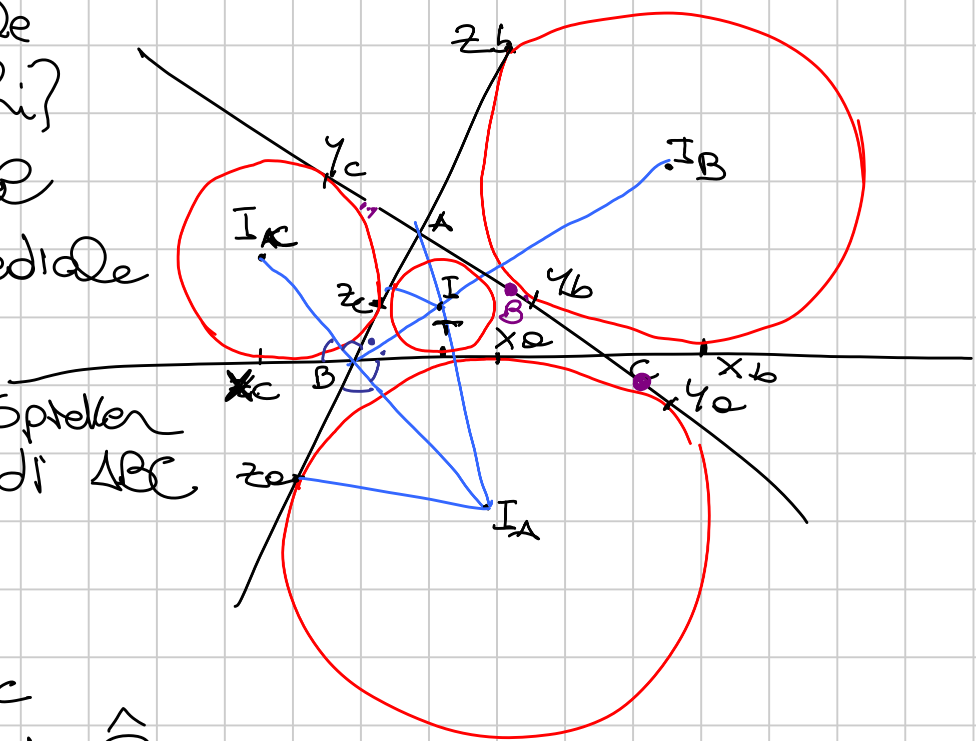
$\sum_{cyc}\frac{r}{r_a}=\sum_{cyc}\frac{b+c-a}{a+b+c}=1$

$r_a=I_aZ_a=\frac{a+b+c}{2}\,tg\frac{\alpha}{2}$

(6) centro radicale dei 3 excerchi?

= incentro del triangolo mediale di ABC

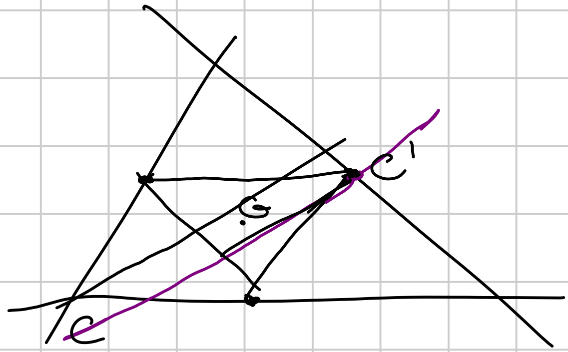
= S = p.to di Spieker di ABC




Asse rad $I_AI_C$

// Bisettr di $\hat{B}$

Basta dim che
l'asse rad di $I_B I_C$
passa per B'

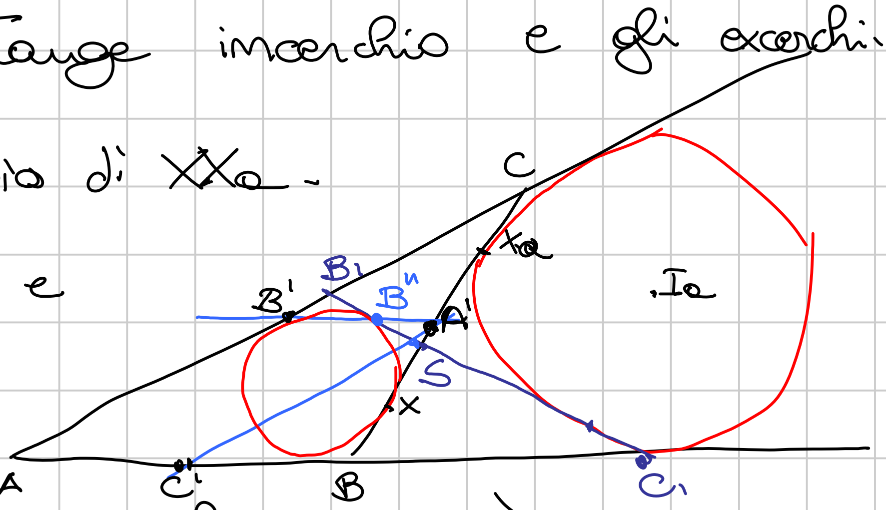



Potenze!
$B'Y_c^2 = B'Y_a^2$

# Teo di Feuerbach

Circonf di F. Tange incerchio e gli excerchi.

Oss A' è pto medio di $XX_a$.




Inversione di centro A' e raggio $A'X = AX_a$

incerchio $\longrightarrow$ sè    excerchio $\longrightarrow$ sè

$T_1 \longrightarrow$ sè    Feuerbach $\xrightarrow{?}$ $B_1C_1$

Definiamo $B_1C_1$ la retta tang a in e ex cerc.

$B_1C_1$ è simm rispetto alla bis di $\hat{A}$ di BC.

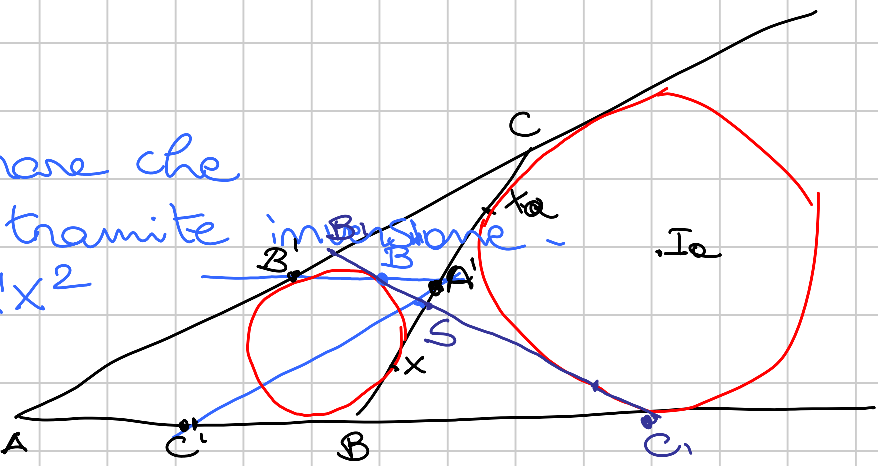
S = ? Piede della bisettrice

~~A posteriori~~ S è l'inverso del piede dell'altezza da A

$B'' = A'B' \cap B_1C_1$

Dobbiamo mostrare che B' va in B'' tramite inversione.

$A'B' \cdot A'B'' = A'X^2$



$A'X = CX - CA' = \frac{a+b-c}{2} - \frac{a}{2} = \frac{b-c}{2}$

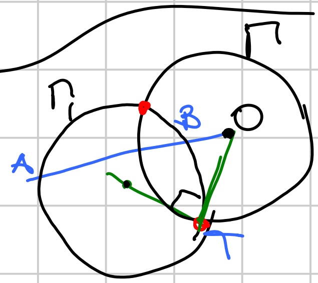
$A'B' = \frac{c}{2}$

Resta da dim

$A'B'' \overset{?}{=} \frac{(b-c)^2}{2c}$

A e B si scambiano?

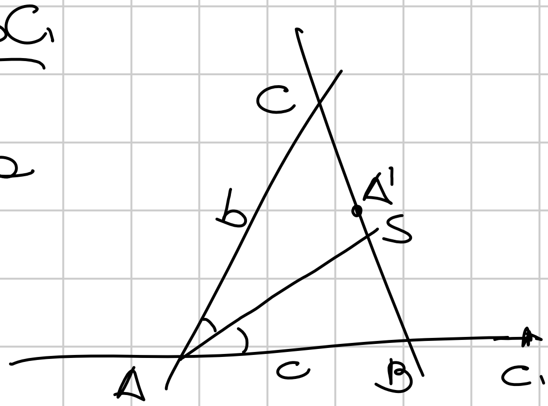
$AO \cdot BO = r_\Gamma^2$
$\overset{\shortparallel}{OT^2}$




Similitudine $A'B''S$ e $BC_1S$

$\frac{A'B''}{A'S} = \frac{BC_1}{BS} \Rightarrow A'B'' = \frac{A'S \cdot BC_1}{BS}$

$BS = \frac{c}{b+c} \cdot a \qquad CS = \frac{b}{b+c} \cdot a$




$\begin{cases} \frac{BS}{CS} = \frac{c}{b} \\ BS + CS = a \end{cases}$

$A'S = CS - CA' = \frac{ab}{b+c} - \frac{a}{2}$

$BC_1 = b - c$

$$\frac{\frac{\cancel{2}ab - \cancel{ab} - ac}{2\cancel{(b+c)}} \cdot (b-c)}{\frac{ca}{\cancel{b+c}}} = \frac{\cancel{a}(b-c)^2}{2\cancel{a}c}$$

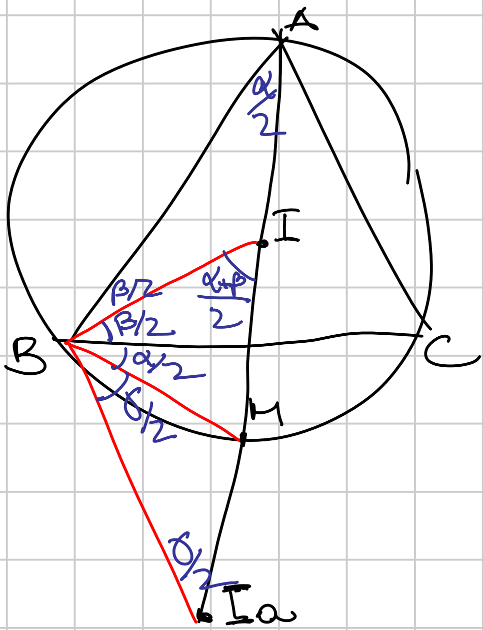
Lemma

$BICI_a$ è ciclico con centro M.

$I_a\hat{B}I = 90°$

BIM è isoscele

$IM \underset{ok}{=} MB = MI_a$

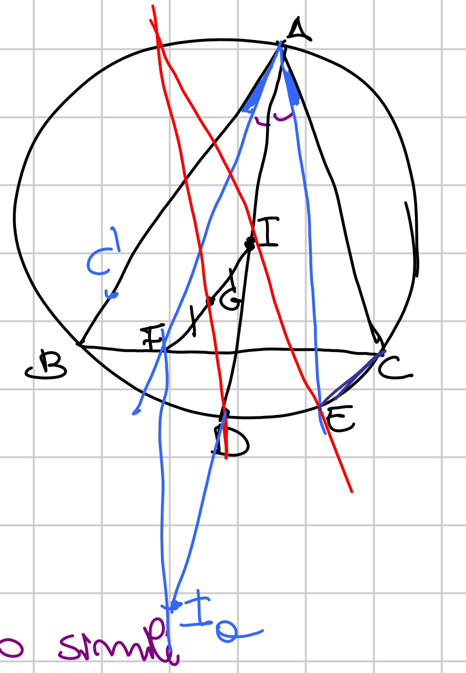


# IMO 2010 - 2.

Tesi: $DG \cap IE \in$ circonf circoscr.

$\Leftrightarrow$ D pto A E è ciclico

$\Leftrightarrow$ $G\hat{D}I \overset{?}{=} I\hat{E}A$



$FI_0 \parallel GD$

Tesi: $F\hat{I_0}I \overset{?}{=} I\hat{E}A$

$\Leftrightarrow$ $AFI_0$ e $AIE$ sono simili

$$\frac{AF}{AI_0} \overset{?}{=} \frac{AI}{AE}$$

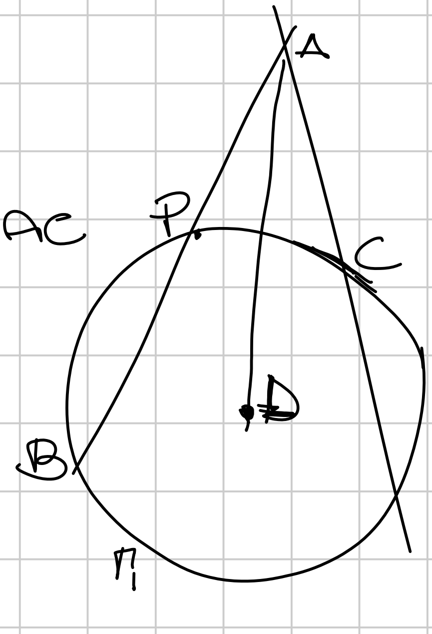
ABF e AEC sono simili $\Rightarrow$ $\frac{AB}{AF} = \frac{AE}{AC}$

$AE \cdot AF = AB \cdot AC$

Tesi: $AI \cdot AI_0 = AB \cdot AC$.

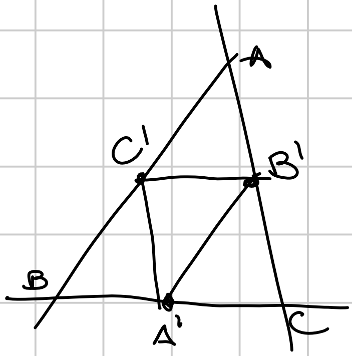
$AP = AC$

$pot_{\Gamma} A = AP \cdot AB = AB \cdot AC$



Teorema:

$$\overset{1/3}{I \text{---}} \overset{1/6}{G \text{---}} \overset{1/2}{S \text{-----}} N$$




I = incentro di $\Delta ABC$

S = incentro di $A'B'C'$

N = punto di Nagel di $\Delta ABC$

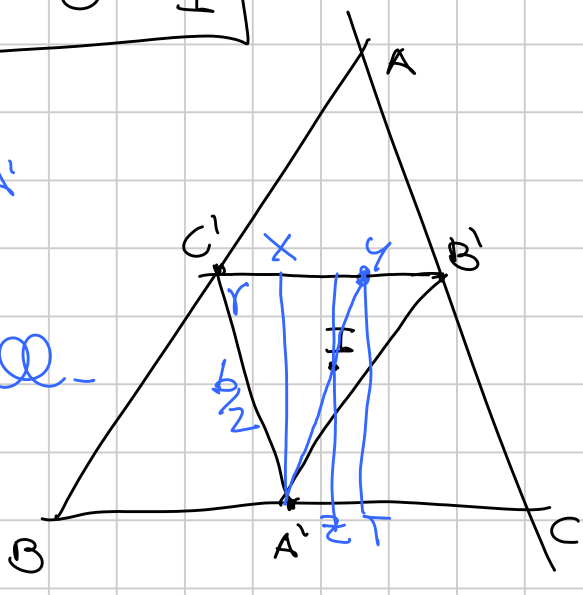
G = baric di $\Delta ABC$

$\Rightarrow$ sono all.

Idea: omotetia centro G rapporto $-\frac{1}{2}$

G fisso $I \to S$

$\boxed{\text{Pto di Nagel di } A'B'C' \text{ è } I}$




Y pto di tang dell'exc $\Gamma_{A'}$

$C'Y = \frac{a+c-b}{4}$

Voglio dim $A', I, Y$ sono all.

$\Leftrightarrow$ Talete su $A'IZ$ e $A'YT$

ovvero $\frac{IZ}{A'Z} = \frac{YT}{A'T}$

$IZ = r$

$A'Z = BZ - BA' = \frac{a+c-b}{2} - \frac{a}{2} = \frac{c-b}{2}$

$YT = \frac{1}{2} h_a = \frac{1}{2} b \sin\gamma$

$A'T = XY = CY - CX = \frac{a+c-b}{4} - \frac{b}{2}\cos\gamma$

$$\frac{\cancel{2}r}{c-b} = \frac{\cancel{\frac{1}{2}}\, b \sin\gamma \; : \cancel{\tfrac{1}{2}}}{a+c-b-2b\cos\gamma}$$

$$r \cdot \frac{a+b+c}{2} = \text{area} = \frac{1}{2} ab\sin\gamma$$

$$\Rightarrow r = \frac{ab\sin\gamma}{a+b+c}$$

$$\frac{a}{(c-b)(a+b+c)} \overset{?}{=} \frac{1}{a+c-b-2b\cos\gamma}$$

$$a^2 + \cancel{ac} - \cancel{ab} - 2ab\cos\gamma \overset{?}{=} \cancel{ac} - \cancel{ab} + c^2 - b^2$$

OK