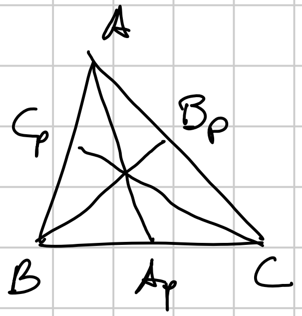
COORDINATE BARICENTRICHE – 2

Titolo nota 07/09/2011

$S_\theta = 2[ABC]\cot\theta$ $\qquad S_{\theta\varphi} = S_\theta \cdot S_\varphi$

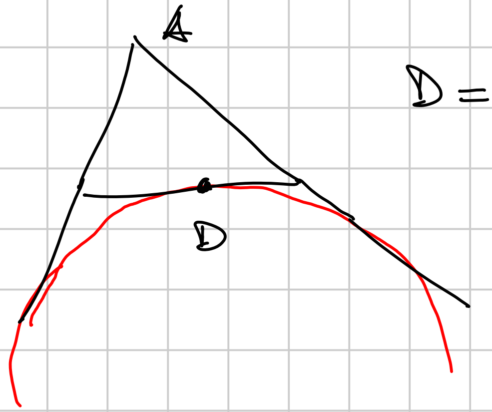
G, O, H, I, N

<u>Oss</u>: $P = (u:v:w) \iff$



$A_P = (0 : v : w)$
$B_P = (u : 0 : w)$
$C_P = (u : v : 0)$

<u>Es</u>: Punto di NAGEL (forse) $\qquad s = \dfrac{a+b+c}{2}$



$D = (0 : s-b : s-c)$

$N_a = (s-a : s-b : s-c)$

$G_e = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right)$

- $P = (u : v : w)$ $\qquad P' = \left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right)$ coniug. isogonale

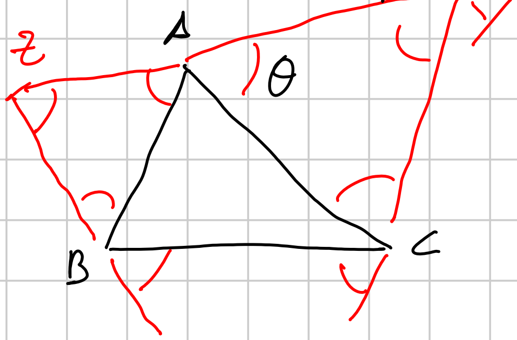
$P^* = \left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right)$ coniug. isotomico

<u>Oss</u> (+): $X = (* : q : r) \quad Y = (p : * : r) \quad Z = (p : q : *)$

AX, BY, CZ concorrono in $(p : q : r) = P$

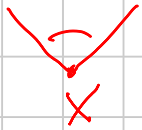
$\overset{\Delta}{ABC}$, $\overset{\Delta}{XYZ}$ sono prospettici in P

<u>Es</u>:



$\theta = \frac{\pi}{3} \qquad \cot\theta = \frac{1}{\sqrt{3}}$

$X = \left(-a^2 : \frac{S}{\sqrt{3}} + S_C : \frac{S}{\sqrt{3}} + S_B\right)$



$$Y = \left(S_C + \frac{S}{\sqrt3} : -b^2 : S_A + \frac{S}{\sqrt3} \right)$$

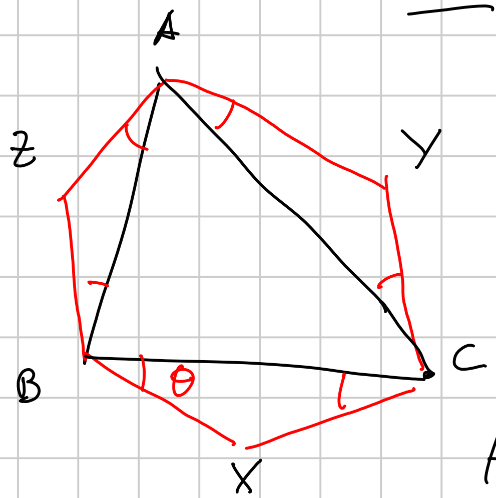
$$Z = \left(S_B + \frac{S}{\sqrt3} : S_A + \frac{S}{\sqrt3} : -c^2 \right)$$

$$X = \left(\frac{-a^2}{\left(S_C + \frac{S}{\sqrt3}\right)\left(S_B + \frac{S}{\sqrt3}\right)} : \frac{1}{S_B + \frac{S}{\sqrt3}} : \frac{1}{S_C + \frac{S}{\sqrt3}} \right)$$ idem per gli altri due

$$AX \cap BY \cap CZ = \left(\frac{1}{S_A + \frac{S}{\sqrt3}} : \frac{1}{S_B + \frac{S}{\sqrt3}} : \frac{1}{S_C + \frac{S}{\sqrt3}} \right)$$

$$= \left(\frac{1}{\sqrt3 S_A + S} : \text{—} \right)$$

— ∘ —



$$X = \left(-a^2 : S_C + S_\theta : S_B + S_\theta \right)$$

$$X = \left(* : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta} \right)$$

$$\Downarrow$$

AX, BY, CZ concorrono in

punto di KIEPERT di angolo θ (di XYZ) $\rightarrow$

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta} \right)$$

— ∘ —

$$P_t = (u+t : v+t : w+t) \qquad t \in \mathbb{R}$$

$$x - y + \frac{v-u}{v-w}(y-z) = 0$$

$$P'_t = \left(\frac{a^2}{u+t} : \frac{b^2}{v+t} : \frac{c^2}{w+t} \right) \quad \Big\downarrow$$

$$\frac{a^2}{x} - \frac{b^2}{y} + \frac{v-u}{v-w}\left(\frac{b^2}{y} - \frac{c^2}{z} \right) = 0$$

$$\begin{cases} a^2yz - b^2xz + \dfrac{v-u}{v-w}\left(b^2xz - c^2xy\right) = 0 \\[2ex] x+y+z=0 \end{cases}$$

$$\begin{cases} x-y+\dfrac{v-u}{v-w}(y-z)=0 \\[2ex] a^2yz+b^2xz+c^2xy=0 \end{cases}$$

$$a^2(v+t)(w+t)+b^2(u+t)(w+t)+c^2(u+t)(v+t)=$$

$$= t^2(a^2+b^2+c^2)+t\left(u(b^2+c^2)+v(a^2+c^2)+w(a^2+b^2)\right)+$$

$$+\, a^2vw+b^2uw+c^2uv$$

$$\Big[u^2b^4+u^2c^2+2u^2b^2c^2$$
$$+v^2a^4+v^2c^4+2v^2a^2c^2$$
$$+w^2a^4+w^2b^4+2w^2a^2b^2+$$
$$2uva^2b^2+2uvc^2b^2$$
$$+2uvc^2a^2+2uvc^4+\Big]-4\big(a^4vw+a^2b^2uw+a^2c^2uv+a^2b^2vw+b^4uw+b^2c^2uv$$
$$+c^2a^2vw+c^2b^2uw+c^4uv\big)$$
$$+2uwa^2b^2+2uwb^4+$$
$$+2uwc^2a^2+2uwc^2b^2+$$
$$+2vwa^4+2vwa^2b^2$$
$$+2vwc^2a^2+2vwc^2b^2$$

<u>Es:</u> Retta di Eulero

$$(1:1:1) \qquad (S_{BC}:S_{CA}:S_{AB})$$

$$0=\det\begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{vmatrix} 1 & 1 & 1 \\ S_{BC} & S_{CA} & S_{AB} \end{vmatrix} \end{array} = x(S_{AB}-S_{CA})-y(S_{AB}-S_{BC})+z(S_{CA}-S_{BC})=$$

$$=x(S_{AB}-S_{CA})+y(S_{BC}-S_{AB})+z(S_{CA}-S_{BC})=$$

$$=\sum_{cyc} S_A(S_B-S_C)\,x$$

<u>Retta IO</u>: $(a:b:c) \qquad (a^2S_A:b^2S_B:c^2S_C)$

$$0=\sum_{cyc}(b^2S_Bc-c^2S_Cb)\,x=\sum_{cyc} bc(bS_B-cS_C)\,x=0$$

$$bS_B - cS_C = b\,ac\,\cancel{\sin\beta}\cdot\frac{\cos\beta}{\cancel{\sin\beta}} - c\,\frac{a^2+b^2-c^2}{2} =$$

$$\cancel{ac}\,\frac{a^2+c^2-b^2}{2\cancel{ac}}$$

$$= \frac{ba^2+bc^2-b^3-ca^2-cb^2+c^3}{2} = (b-c)\,\frac{a^2-bc-c^2-bc-b^2}{2} =$$

$$= (b-c)\,\frac{a^2-(b+c)^2}{2} = -2(b-c)\,s\,(s-a)$$

$$\sum_{cyc} bc\,(b-c)\,(s-a)\,x = 0 \qquad \sum_{cyc} \frac{(b-c)(s-a)}{a}\,x = 0$$

$$F_{\pm} = \left(\frac{1}{\sqrt{3}S_A \pm S} : \frac{1}{\sqrt{3}S_B \pm S} : \frac{1}{\sqrt{3}S_C \pm S}\right)$$

$$\begin{cases} \sum_{cyc} (S_B - S_C)(3S_{AA} - S^2)\,x = 0 \\ \sum_{cyc} S_A(S_B - S_C)\,x = 0 \end{cases}$$

$$\begin{cases} p_1x + q_1y + r_1z = 0 \\ p_2x + q_2y + r_2z = 0 \end{cases} \qquad (q_1r_2 - q_2r_1 : r_1p_2 - p_1r_2 : p_1q_2 - q_1p_2)$$

$$S_B(S_C-S_A)(S_A-S_B)(3S_{CC}-S^2) - S_C(S_A-S_B)(S_C-S_A)(3S_{BB}-S^2) =$$

$$= (S_C-S_A)(S_A-S_B)\left[S_B(3S_{CC}-S^2) - S_C(3S_{BB}-S^2)\right] =$$

$$= (S_C-S_A)(S_A-S_B)\left[3S_{BCC} - S_BS^2 - 3S_{BBC} + S_CS^2\right] =$$

$$= (S_C-S_A)(S_A-S_B)\left[3S_{BC}(S_C-S_B) + S^2(S_C-S_B)\right] =$$

$$= (\quad)(\quad)(\,)\,\left[3S_{BC} + S^2\right]$$

$$\left(3S_{BC} + S^2 : \underline{\qquad\quad}\right)$$

$$H = \left(\frac{S_{BC}}{S^2} : \frac{S_{CA}}{S^2} : \frac{S_{AB}}{S^2} \right) \qquad S_{BC} + S_{CA} + S_{AB} = S^2$$

$$G = \left(\frac{1}{3} : \frac{1}{3} : \frac{1}{3} \right) \qquad \frac{S_{BC}}{S^2} + \frac{1}{3} = \frac{3S_{BC} + S^2}{3S^2}$$

$$\frac{1}{\sqrt{3}S_A \pm S} \qquad S^2 \pm \sqrt{3}S(\underbrace{S_B + S_C}_{=a^2}) + 3S_{BC}$$

$$6S^2 \pm \sqrt{3}S(a^2+b^2+c^2)$$

$$\frac{S^2 + \sqrt{3}Sa^2 + 3S_{BC}}{6S^2 + \sqrt{3}S(a^2+b^2+c^2)} + \frac{S^2 - \sqrt{3}Sa^2 + 3S_{BC}}{6S^2 - \sqrt{3}S(a^2+b^2+c^2)} =$$

$$= 6S^4 + \cancel{6\sqrt{3}S^3a^2} + 18S^2S_{BC} - \cancel{\sqrt{3}S^3(a^2+b^2+c^2)} - 3S^2a^2(a^2+b^2+c^2) -$$

$$- \cancel{3\sqrt{3}SS_{BC}(a^2+b^2+c^2)} + 6S^4 - \cancel{6\sqrt{3}S^3a^2} + 18S^2S_{BC} + \cancel{\sqrt{3}S^3(a^2+b^2+c^2)}$$

$$- 3S^2a^2(a^2+b^2+c^2) + \cancel{3\sqrt{3}SS_{BC}(a^2+b^2+c^2)} =$$

$$= 12S^4 + 36S^2S_{BC} - 6S^2a^2(a^2+b^2+c^2) =$$

$$= 2S^2 + 6S_{BC} - a^2(a^2+b^2+c^2) \simeq (b^2-c^2)^2$$

$$\left((b^2-c^2)^2 : (a^2-c^2)^2 : (a^2-b^2)^2 \right)$$

<u>Es</u>: $K(\pm\theta)$ la retta tra loro è $\sum_{cyc} (S_B - S_C)(S_{AA} - S^2\cot^2\theta)x = 0$

Al variare di θ queste rette passano per un punto fisso che è il punto di Lemoine.

—— o ——

$x+y+z=0 \qquad \mathbb{P}^2(\mathbb{R}) = \{ [x:y:z] \mid x,y,z \in \mathbb{R}$ non tutti nulli$\}$

$v, w \in \mathbb{R}^3 \setminus \{0\} \quad v \sim w$ sse $\exists \lambda \in \mathbb{R}$ t.c. $\lambda v = w$

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^3 \setminus \{0\} / \sim$$

retta $= \{ p(x,y,z) = 0 \}$

r p omogeneo di 1° grado.

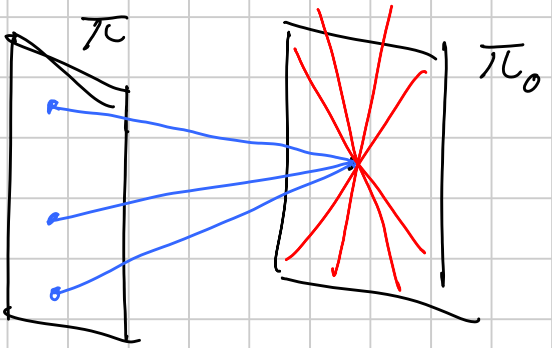
$$\mathbb{P}^2(\mathbb{R}) \setminus \text{retta} \simeq \mathbb{R}^2$$

$$(\mathbb{R}^3 \setminus \{0\}) \setminus \{ p(x,y,z) = 0 \}$$

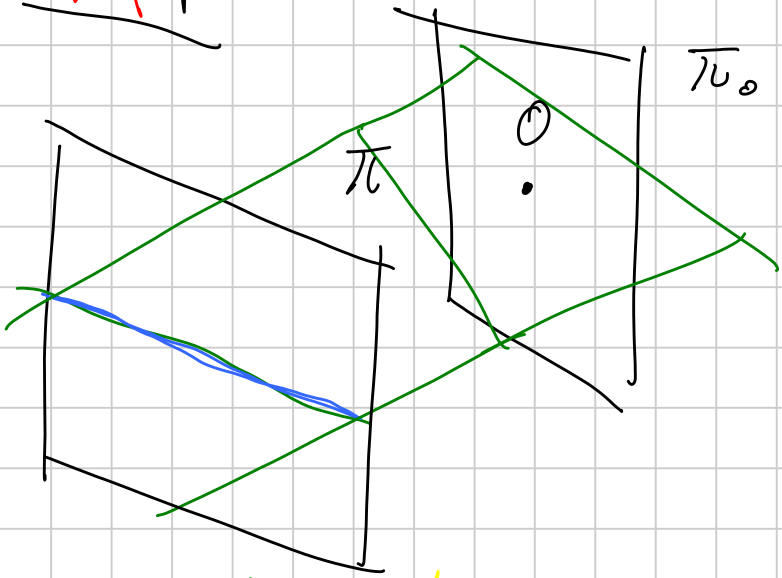
$$\pi_0 = \{ \alpha x + \beta y + \gamma z = 0 \}$$

$$\pi = \{ \alpha x + \beta y + \gamma z = 1 \}$$

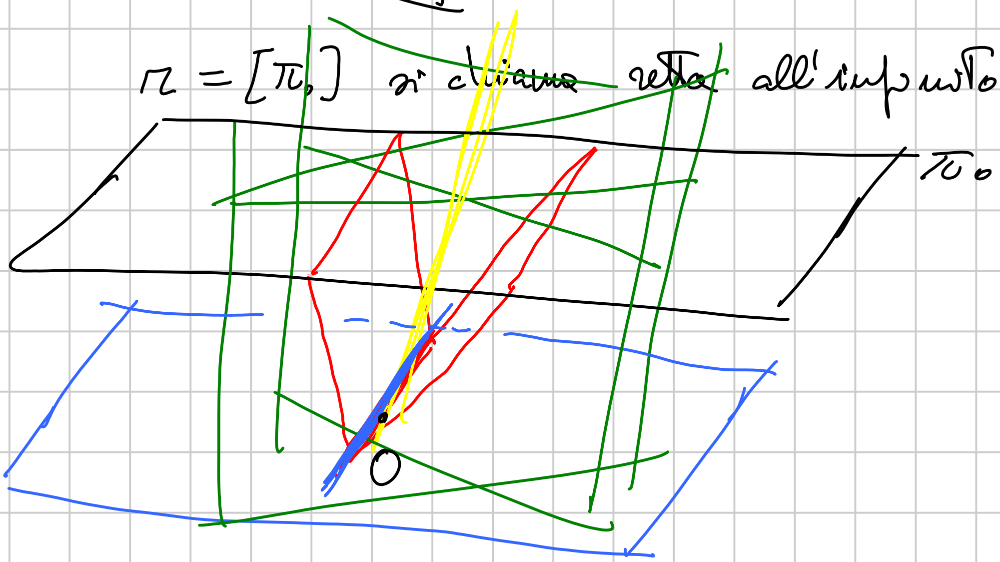
$\pi \subseteq \mathbb{R}^3 \setminus \pi_0$ $\quad \forall [x:y:z] \in \mathbb{P}^2(\mathbb{R})$ $\exists$ al più un punto $P \in \pi$ t.c. $[P] = [x:y:z]$



$\Rightarrow \forall [x:y:z] \in \mathbb{RP}^2 \setminus r$ $\exists!$ $P \in \pi$ t.c. $[P] = [x:y:z]$

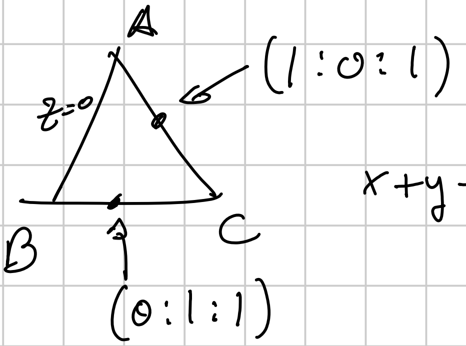


$r = [\pi_0]$ si chiama retta all'infinito di π.



in baricentriche

$z=0$



$x+y-z=0$

$$\begin{cases} z=0 \\ x+y-z=0 \end{cases} \qquad \begin{cases} z=0 \\ x+y=0 \end{cases} \qquad (1:-1:0)$$

$$\begin{cases} px+qy+rz=0 \\ x+y+z=0 \end{cases} \qquad (q-r : r-p : p-q) \in \mathcal{L}^\infty$$

$[u:v:w]$

$$\det\begin{vmatrix} x & y & z \\ u & v & w \\ q-r & r-p & p-q \end{vmatrix} =$$

$$= x\big(v(p-q)-w(r-p)\big) + y\big(w(q-r)-u(p-q)\big) + z\big(u(r-p)-v(q-r)\big) = 0.$$

Oss: $A_H = \left(0 : \frac{1}{S_B} : \frac{1}{S_C}\right) = (0 : S_C : S_B)$

$A = (1:0:0) = (S_B+S_C : 0 : 0) = (a^2 : 0 : 0)$

$A_H - A = (-a^2 : S_C : S_B)$

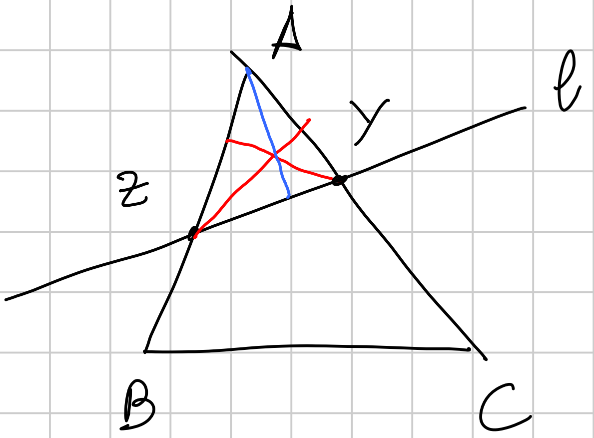
$$\begin{vmatrix} x & y & z \\ -a^2 & S_C & S_B \\ u & v & w \end{vmatrix} = 0$$

$\|$

$-(S_B v - S_C w)x + (S_B u + a^2 w)y - (S_C u + a^2 v)z = 0$

Se interseco con BC $\{x=0\}$ $\quad A_{(P)} = (0 : S_C u + a^2 v : S_B u + a^2 w)$

$$\Rightarrow \text{Tri pedale di } P = \begin{pmatrix} 0 & S_C u + a^2 v & S_B u + a^2 w \\ S_C v + b^2 w & 0 & S_A v + b^2 w \\ S_B w + c^2 w & S_C w + c^2 v & 0 \end{pmatrix} = A$$



Fatto: se ℓ ha pt. all'∞ $(f:g:h)$ allora le perp hanno pt. all'∞
$(f':g':h') = (S_B g - S_C h : S_C h - S_A f : S_A f - S_B g)$

$$(f:g:h) \perp (f':g':h')$$
$$\Updownarrow$$
$$S_A f f' + S_B g g' + S_C h h' = 0$$

Es: Il Tri circumceviano è simile al Tri pedale

$P = (u:v:w)$ $\quad AP = \{ vz - wy = 0 \}$

$$\begin{cases} vz - wy = 0 \\ a^2 yz + b^2 xz + c^2 xy = 0 \end{cases}$$

Risolviamo il problema isogonalmente coniugato

$$\begin{cases} vc^2 y - wb^2 z = 0 \\ x + y + z = 0 \end{cases}$$

$$x + z\left(1 + \frac{wb^2}{vc^2}\right) = 0 \qquad y = \frac{wb^2}{vc^2} z$$

$$x + z\left(\frac{vc^2 + wb^2}{vc^2}\right) = 0 \qquad x = -(vc^2 + wb^2) \quad y = wb^2 \quad z = vc^2$$

$$\left(-(vc^2 + wb^2) : wb^2 : vc^2\right)$$

$$\left(\frac{-a^2 vw}{c^2 v + b^2 w} : v : w\right)$$ A-vertice del Tri circumceviano

---

Tornando al Tri pedale $\quad \det A = (u+v+w)(S^2)(a^2 vw + b^2 wu \ldots)$

se $P \in$ circoscritta $\Rightarrow P = \left(\frac{a^2}{f} : \frac{b^2}{g} : \frac{c^2}{h}\right)$ $\quad (f:g:h) \in \mathcal{L}^\infty$

se $(f':g':h') \perp (f:g:h) \Rightarrow x\frac{f}{f'} + y\frac{g}{g'} + z\frac{h}{h'} = 0$ è la retta di Simson

$(f':g':h') =$ retta di Simson di $P \cap \mathcal{L}^\infty$

$$f' = S_B g - S_C h$$

$$\frac{a^2}{f'} = a^2 (S_C h - S_A f)(S_A f - S_B g) =$$

$$= a^2 (S_{CA} hf - S_{CB} hg - S_A^2 f^2 + S_{AB} fg)$$

$$(a^2 gh : b^2 hf : c^2 fg)$$

$$\mathcal{C} = \{ a^2 yz + b^2 xz + c^2 xy - (x+y+z)\underbrace{(px+qy+rz)}_{\text{asse radicale tra cfr e cfr.circosc.}} = 0 \}$$

$$\begin{cases} \mathcal{C} \\ x = 0 \end{cases} \Rightarrow a^2 yz - (y+z)(qy+rz) = 0$$

$$(0 : s-c : s-b)$$

$$qy^2 + zy(q+r-a^2) + rz^2 = k((s-c)z - (s-b)y)^2$$

$k = 1$

$q = (s-b)^2 \qquad p = (s-a)^2$

$r = (s-c)^2$

inc. $a^2 yz + b^2 xz + c^2 xy - (x+y+z)((s-a)^2 x + (s-b)^2 y + (s-c)^2 z) = 0$

A-ex $\qquad s^2 \qquad (s-b)^2 \qquad (s-c)^2$

cfr. di F: $a^2 yz + b^2 xz + c^2 xy - \frac{1}{2}(x+y+z)(S_A x + S_B y + S_C z) = 0$

$$\sum_{cyc} \left((s-a)^2 - \frac{1}{2} S_A \right) x = 0 \qquad \left(\frac{b+c-a}{2}\right)^2 - \frac{1}{2}\left(\frac{b^2+c^2-a^2}{2}\right) =$$

$$= \frac{1}{4}\left[b^2 + c^2 + a^2 + 2bc - 2ac - 2ab - b^2 - c^2 + a^2 \right] =$$

$$= \frac{1}{2}\left(a^2 - a(b+c) + bc\right) = \frac{1}{2}(a-b)(a-c)$$

$\sum' (a-b)(a-c)\,x = 0$ $\qquad \sum' \frac{x}{b-c} = 0$ Tg a cf. inscr. e Feuer.

Tg. Tre cf. A-exs. e F. $\frac{x}{b-c} + \frac{y}{a+c} - \frac{z}{a+b} = 0$

— o —

retta Eulero $\longrightarrow$ Jerabek (iperbole)

OK $\longrightarrow$ iperbole di Kiepert $\sum (b^2-c^2)\,yz = 0$

$\{\alpha yz + \beta xz + \gamma xy = 0\}$ passano per i vertici

Coniche inscritte: 1. devono passare o per 3 punti allineati [illegible]

o nei piedi di 3 ceviane concorrenti

$P = (p:q:r)$

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} - \frac{2yz}{qr} - \frac{2zx}{rp} - \frac{2xy}{pq} = 0$$