COMBINATORIA 1

Titolo nota 06/09/2011

Principi sciocchi.

1) Principio di equivalenza

se c'è corrisp. Biunivoca tra A e B

$|A| = |B|$

2) Principio della somma

se $A \cap B = \emptyset$ $|A \cup B| = |A| + |B|$

3) Principio del prodotto

$|A \times B| = |A| \times |B|$

Esempio $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \, p_n^{\alpha_n}$

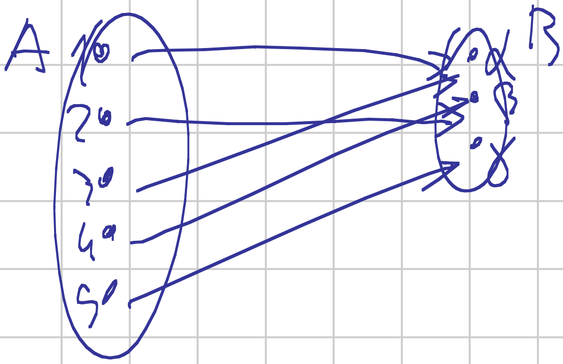
$(\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdot \ldots \cdot (\alpha_n + 1)$

Ultime cose semi-sciocche

Ci sono 3 modi (almeno)

di pensare una funzione

1) modo con frecce



2) modo con parole

Esempio: α β α β γ

rappresenta funzione dai primi

3) occupazione di B da parte di A

1, 3	2, 4	5
α	β	γ

<u>Nomenclatura</u> : $f : A \to B$

1) <u>iniettiva</u> $\forall a_1, a_2 \in A \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

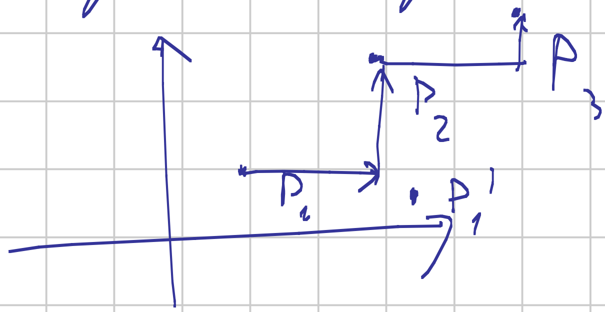
2) suriettiva $\forall b \in B \ \exists a \in A$ t.c. $f(a) = b$

3) Se A e B sono insiemi totalmente ordinati, f crescente (debolmente) (strettamente)

significa $\forall a_1, a_2 \in A \quad a_1 \le a_2 \Rightarrow$ $(a_1 < a_2)$

$\Rightarrow f(a_1) \le f(a_2)$
$(<)$

(esempio di ordine parziale



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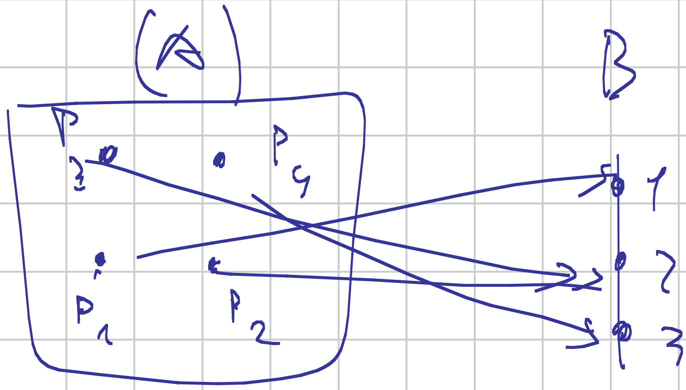
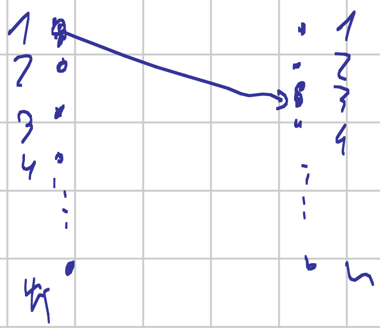


$$\boxed{P_n = n!}$$

$$\begin{cases} P_1 = 1 \\ P_n = n\,P_{n-1} \end{cases}$$



n A

m B

$$\frac{(n+m)!}{n!\cdot m!}$$

$A_1 \; A_2 \; \cdots \; A_n$

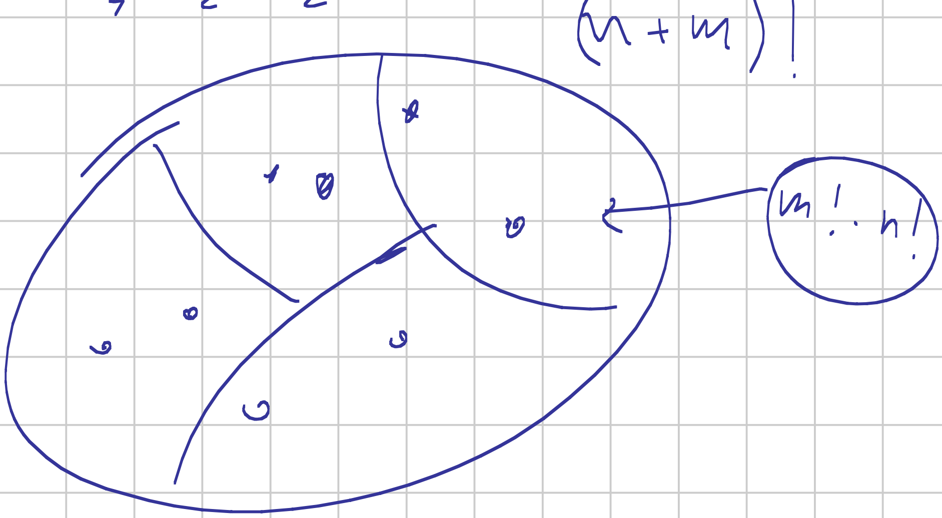
$B_1 \; B_2 \; \cdots\cdots \; B_m$ $(*)$

$(n+m)!$

Se P_1 P_2 sono parole fatte con tutti i simboli (*)

$P_1 \approx P_2 \iff$ cancellando gli indici sono uguali.

$A_1 \; B_1 \; A_3 \; B_2 \; A_2$



$$\frac{(n+m)!}{n! \cdot m!}$$

$n_1 \quad X_1$

$n_2 \quad X_2$

$n_k \quad X_k$

$$\frac{(n_1+n_2+\dots+n_k)!}{n_1! \cdot n_2! \cdots n_k!}$$

Applicazioni (permutazioni, [illegible])

Combinazioni semplici

N	oggetto 1
S	ogg 2
N	ogg 3
N	ogg 4
S	ogg 5

~~K~~ k lettere S

n − k lettere N

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

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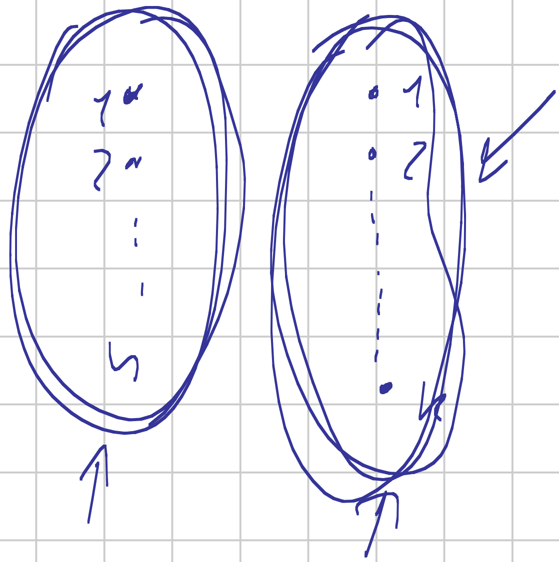
2) funzioni strett. crescenti tra I_n e I_k

con $n \leq k$

$$\binom{k}{n}$$



3) funzioni iniettive

$$\binom{k}{n} n! = \frac{k!}{\cancel{n!}\,(k-n)!} \cdot \cancel{n!}$$

4) $x_1 + x_2 + \cdots\cdots + x_k = n$

n A

k − 1 B

$$x_1 + x_2 + x_3 = 5$$

$(2, 2, 1)$, $\quad$ 5 A

$\rightarrow (0, 5, 0)$; $\quad$ 2 B

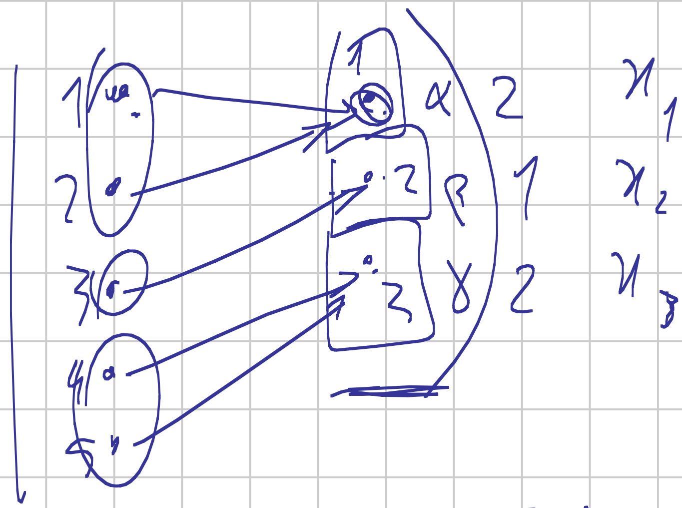
$\rightarrow (5, 0, 0)$,

AABAABA

n $\boxed{A}$, $n-1$ $\boxed{B}$

$$\frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1}$$

5) $f : I_n \rightarrow I_m$ crescenti



$$x_1 + x_2 + x_3 = 5$$

5 A

2 B

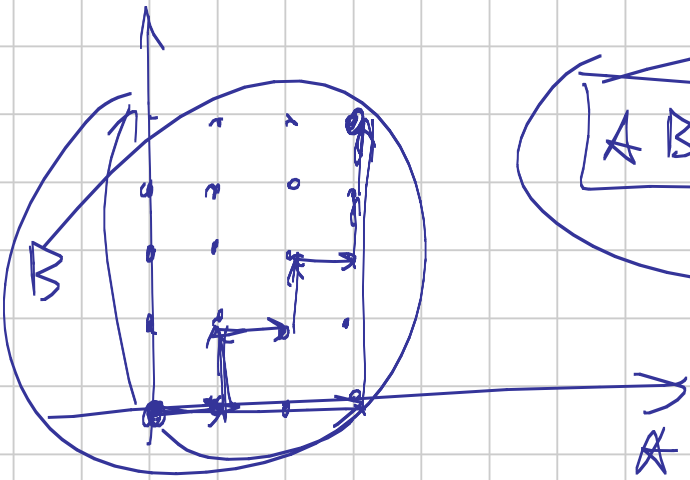
$$\frac{7!}{5!\cdot 2!} =$$

$$= \frac{7 \cdot 6}{2} = 21$$

$$f: I_n \to I_m \quad \text{crescente}$$

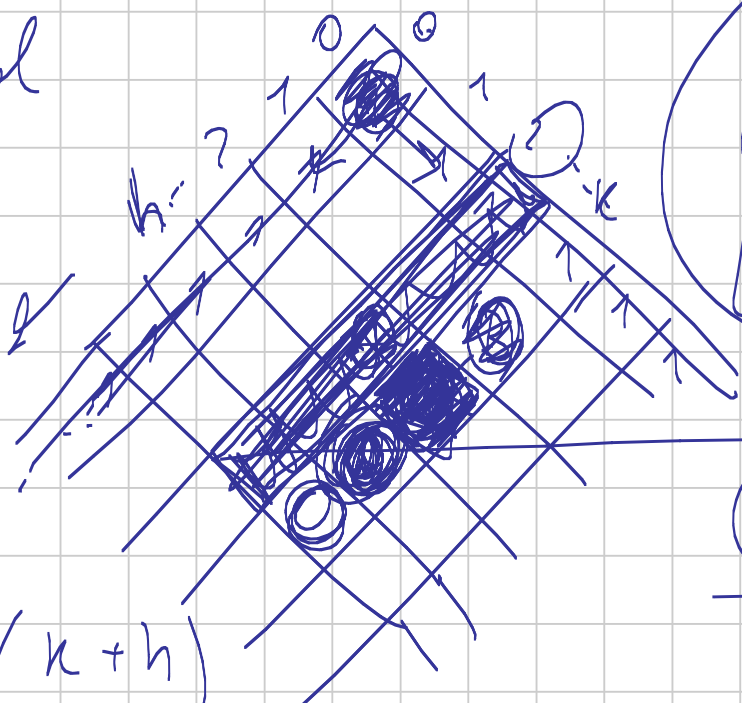
$$x_1 + x_2 + \dots + x_m = n$$

$$\binom{n+m-1}{m-1}$$



$\boxed{ABABABB}$

8) Pascal



1) sul bordo metto 1

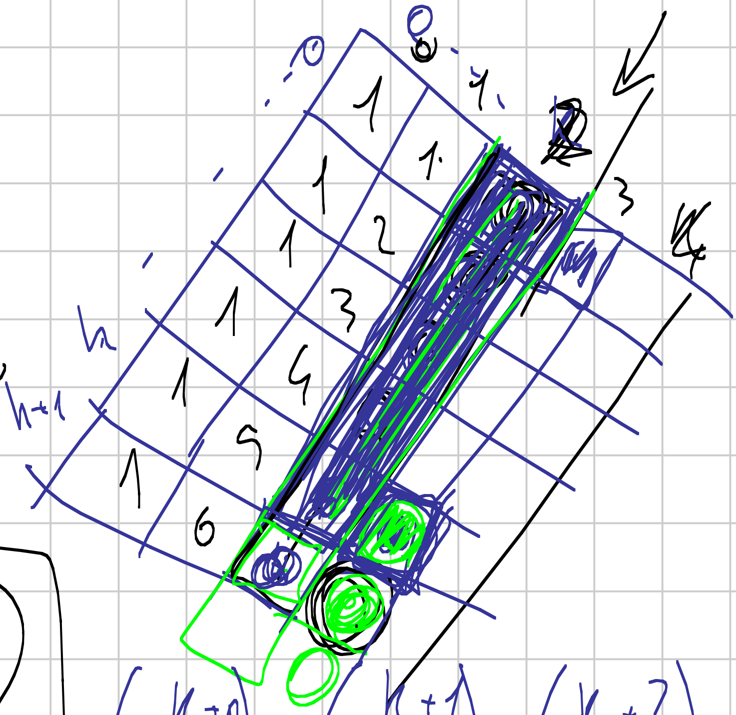
2) $C = A + B$

$$\frac{(k+h)!}{k! \cdot h!} =$$

$$= \binom{k+h}{h}$$

Oss. su Pascal

$(n-1)$



$$\boxed{\binom{n-1+3}{3}}$$

$$\binom{n}{1} \quad n$$

$$\binom{n}{2} \quad \frac{n(n+1)}{2}$$

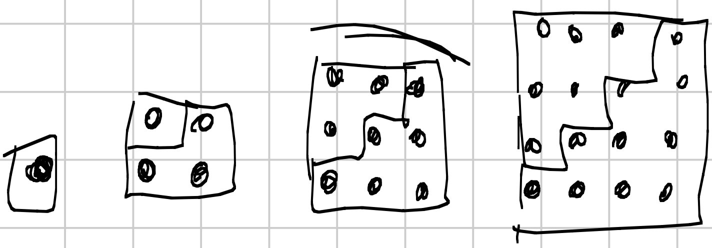
$$\binom{k+0}{k}+\binom{k+1}{k}+\binom{k+2}{k}\cdots\binom{k+n}{k}=\binom{k+n+1}{k+1}$$

Per ind. su n

1) $n=0$ ovvio

2) Passo induttivo (ovvio)

$$\sum_{k=1}^{n} k^2$$



T_1 $\quad T_2+T_1$ $\quad T_3+T_2$ $\quad T_4+T_3$

$$\sum_{k=1}^{n} T_k = T_1+T_2+\cdots+T_n = \binom{2}{2}+\binom{3}{2}+\cdots+\binom{n+1}{2}=\binom{n+2}{3}$$

$$\boxed{1^2 + 2^2 + 3^2 + \cdots + n^2}$$

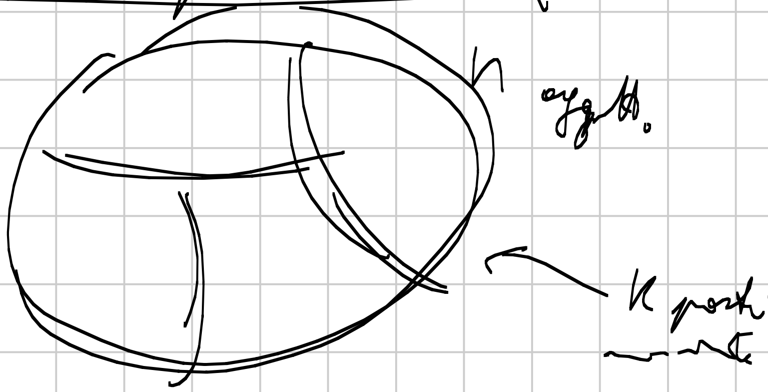
$$= (T_1 + T_2 + T_3 + \cdots + T_n) + (T_1 + T_2 + \cdots + T_{n-1}) =$$

$$= \binom{n+2}{3} + \binom{n+1}{3} = \frac{(n+2)(n+1)n}{3 \cdot 2} + \frac{(n+1)n(n-1)}{3 \cdot 2}$$

$$= \boxed{\frac{(n+1)n(2n+1)}{6}}$$

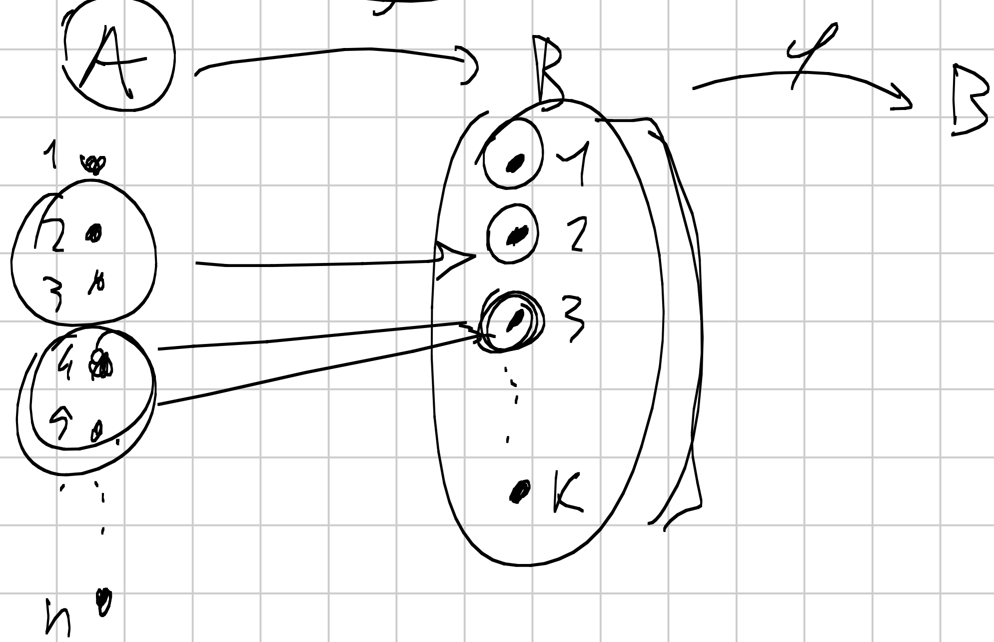
INIZIO II^a parte

9) $S_{n,k}$



$\forall b \in B$

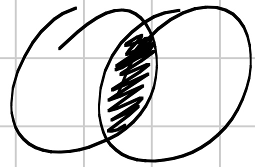
$f^{-1}(b)$



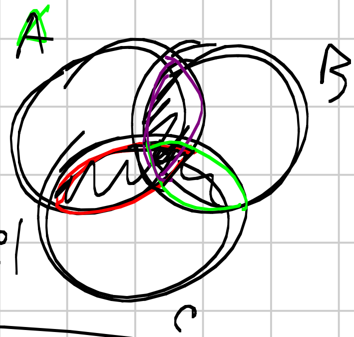
$$S_{n,k} = \frac{1}{k!} \cdot (\text{num. di funzioni suriettive da } A \to B)$$

Principio di inclusione esclusione

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



$A_1, A_2 \dots A_n$

$$\boxed{|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j\le n} |A_i \cap A_j| + \sum_{i<j<k\le n} |A_i \cap A_j \cap A_k| \pm \dots + (-1)^{k+1} \sum_{i_1<i_2<\dots<i_k\le n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|}$$

$$|(A_1 \cup A_2 \cup \dots A_n) \cup A_{n+1}| =$$

$$= \underbrace{|A_1 \cup A \cup \dots \cup A_n|} + |A_{n+1}| - |(A_1 \cup A_2 \cup \dots \cup A_n) \cap A_{n+1}|$$

$$= (*) + |A_{n+1}| - \underbrace{|(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \dots (A_n \cap A_{n+1})|}_{(\star)}$$

$$|A_i \cap A_j \cap A_{n+1}|$$

$$(\star) = \sum_{i=1}^{n} |A_i \cap A_{n+1}| - \sum_{i<j\le n} |(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})|$$

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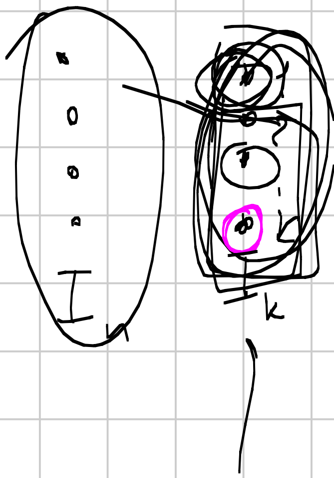
$$\sum_{i=1}^{n} |A_i| - \sum_{i<j\le n} |A_i \cap A_j| + \sum_{i<j<k\le n} |A_i \cap A_j \cap A_k|$$

$$\sum_{i=1}^{n+1} |A_i| - \sum_{i<j\le n+1} |A_i \cap A_j| + \sum_{i<j<k\le n+1} |A_i \cap A_j \cap A_k|$$

$$f : I_n \to I_k \qquad n \ge k$$

suriettive



$A_1 = \{ f \mid 1$ è scoperto $\}$

$A_i = \{ f \mid i$ è scoperto $\}$

$1 \le i \le k$

$A_1 \cup A_2 \cup \dots \cup A_k$ = insieme di tutte le f non suriettive

$$|A_1 \cup A_2 \cup \dots \cup A_k| = \underbrace{\sum_{i=1}^{k} |A_k|}_{(k-1)^n} - \sum_{i<j\le k} |A_i \cap A_j|$$

$$+\sum_{i<j<l\leq k} |A_i \cap A_j \cap A_l| - $$

$$= \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \binom{k}{4}(k-4)^n$$

$$\ldots \quad \ldots$$

$$= \cancel{\sum_{i=1}^{k} (-1)^{i+1}\binom{k}{i}(k-i)^n}$$

$$= \boxed{\sum_{i=1}^{k} (-1)^{i+1}\binom{k}{i}(k-i)^n}$$

$$\text{buone} = \text{tutte}^{\,k^n} - \text{cattive} =$$

$$= \binom{k}{0}(k-0)^n + \sum_{i=1}^{k} (-1)^{i+1}\binom{k}{i}(k-i)^n$$

$(i+2)$

$$= \sum_{i=0}^{k} (-1)^{i}\binom{k}{i}(k-i)^n$$

suriettive da I_n a I_k $n \geq k$

$$\boxed{\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n}$$

$$S_{n,k} = \frac{1}{k!} \cdot \bigcirc$$

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(10) Dimostrare che

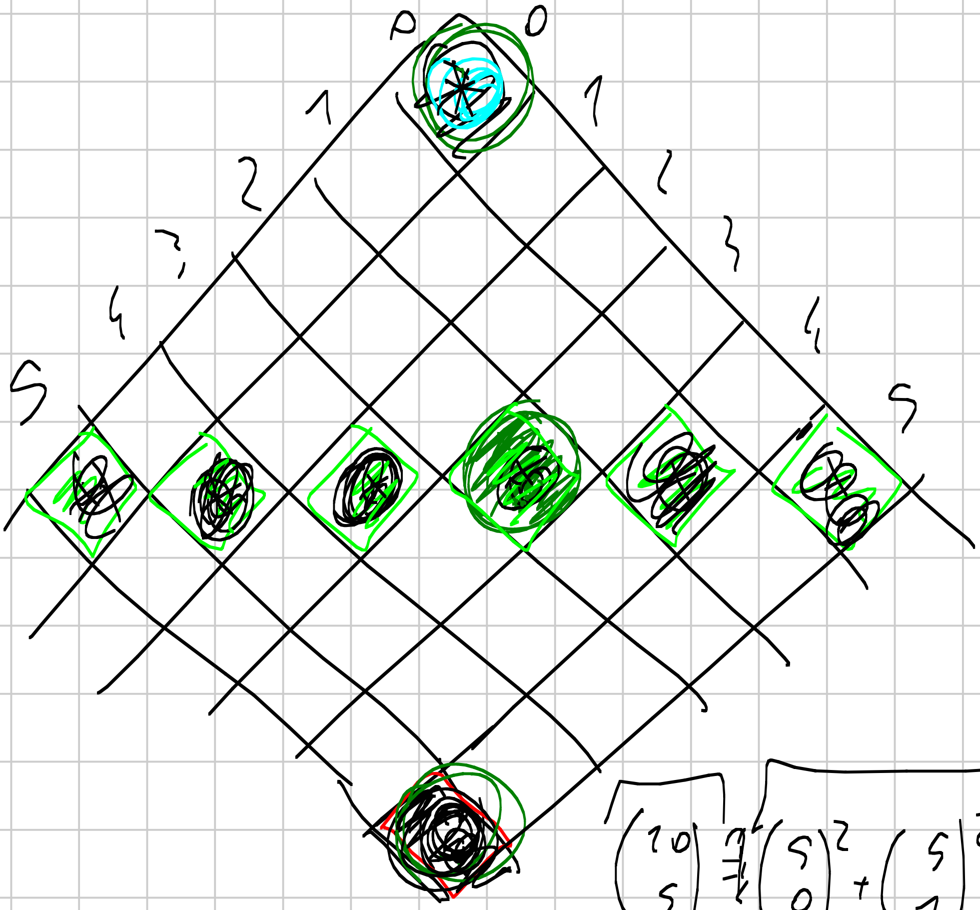
$$\boxed{\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n = n!}$$

$$\boxed{\binom{2n}{n}} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots \binom{n}{n}^2$$

$2n$

n da $2n$ $\binom{2n}{n}$ $\binom{n}{n-1}$

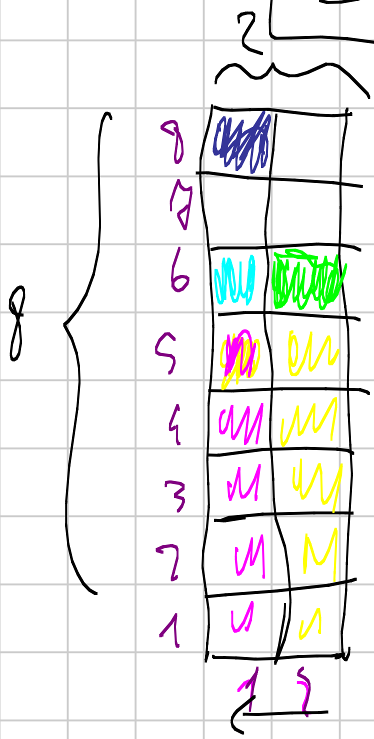


$n = 5$

$$\binom{5}{3}^2$$

$$\binom{10}{5} = \overbrace{\binom{5}{0}^2 + \binom{5}{1}^2 + \dots + \binom{5}{5}^2}$$

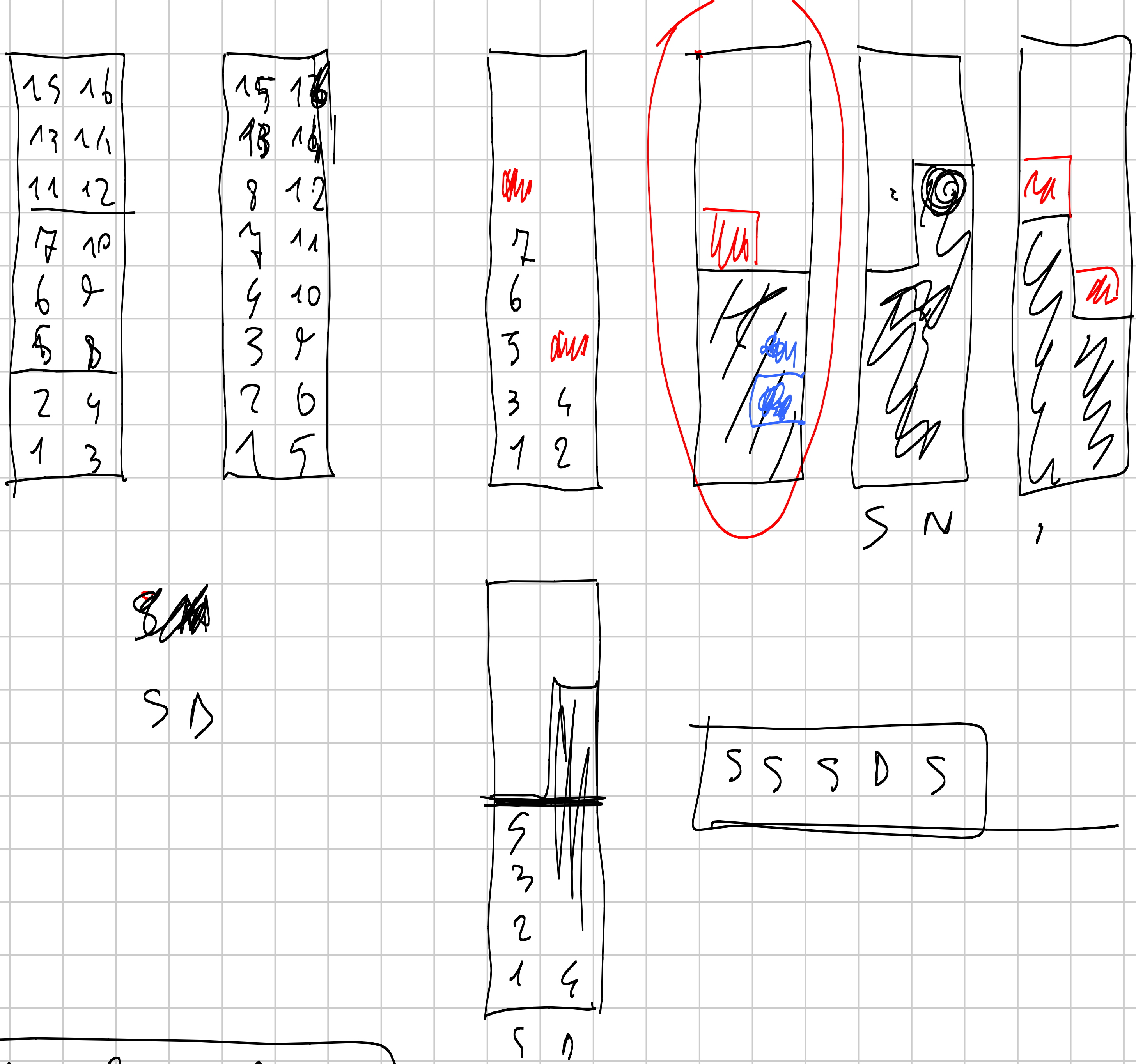
III^a Parte



$$A \xrightarrow{f} I_{16} = \underbrace{\{1, 2, \dots, 16\}}$$

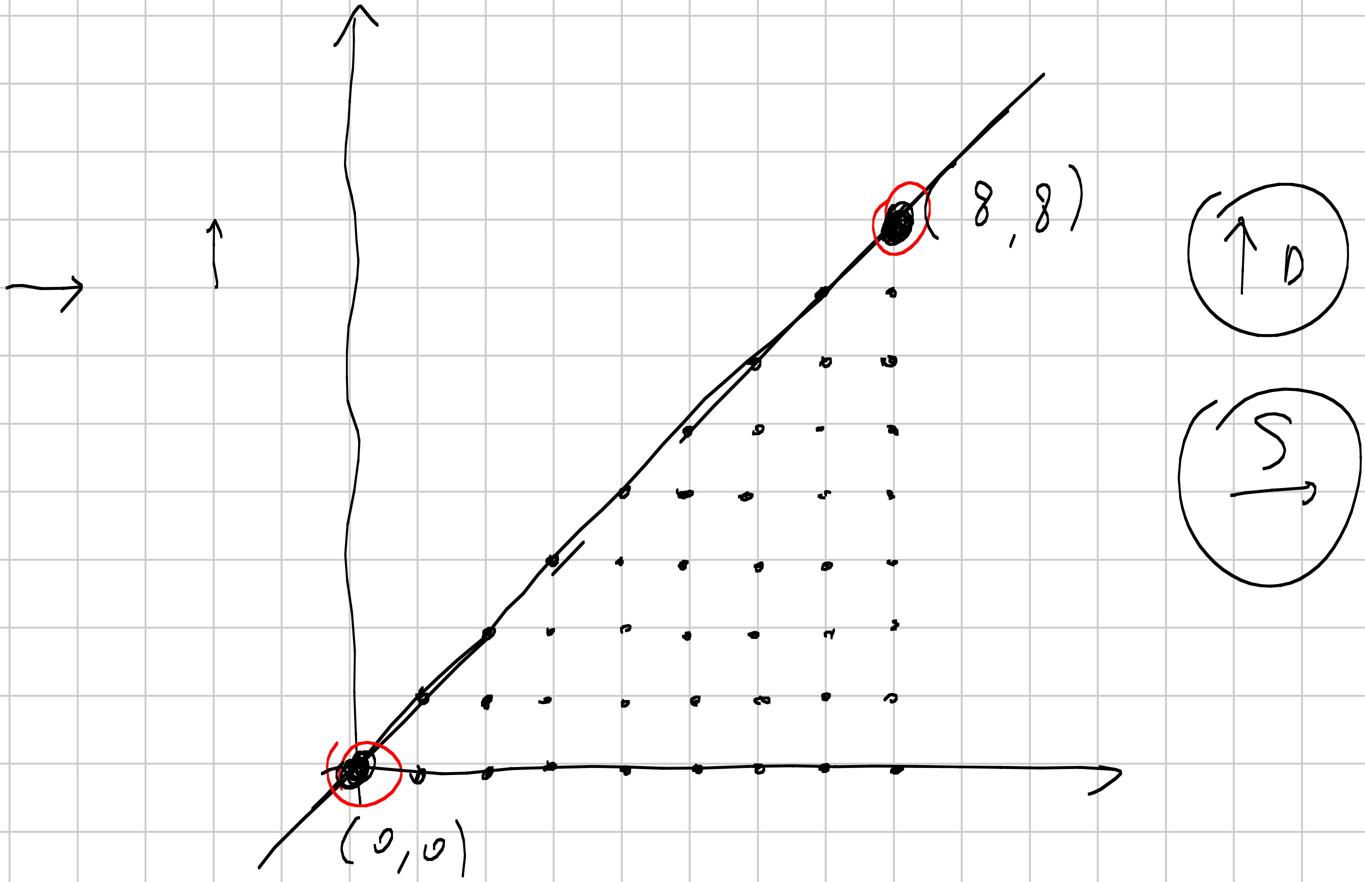
15	16
13	14
11	12
9	10
7	8
5	6
3	4
1	2

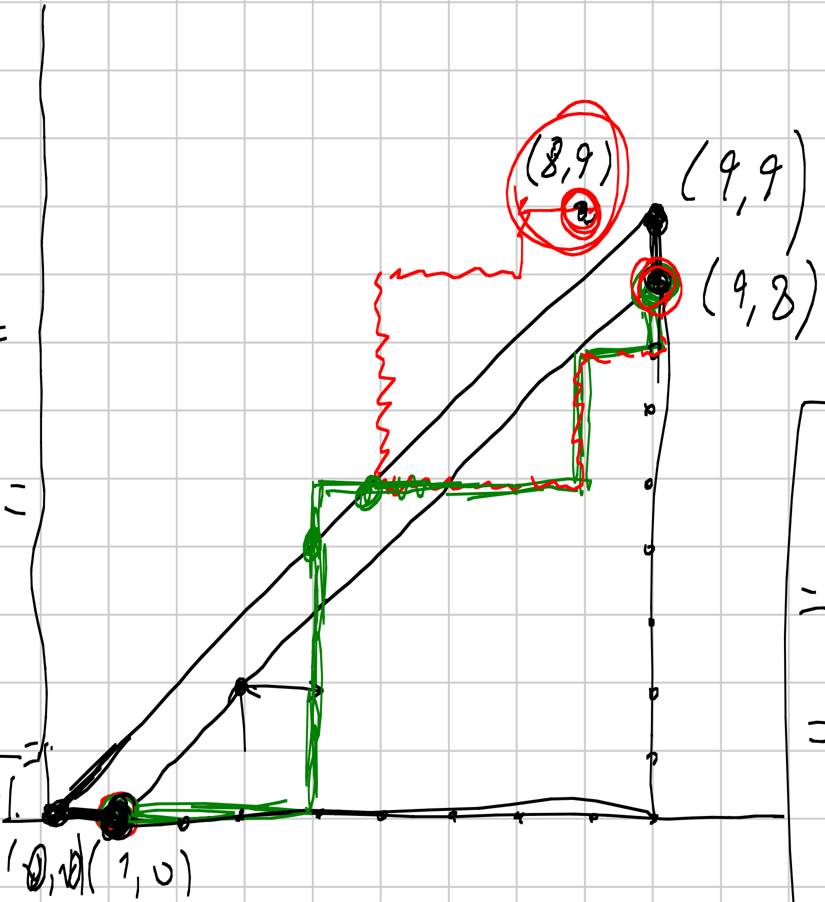
8	16
7	15
6	14
5	13
4	12
3	11
2	10
1	9



Parole fatte di

Con proprietà che
comunque prendo le
prime K lettere
le D non sono mai più
delle S.





buoni =

$= \text{tutti} - \text{cattivi} =$

$= \binom{17}{8} - 2\binom{16}{7} =$

$= \frac{17!}{8!\cdot 9!} - 2\,\frac{16!}{7!\cdot 9!} =$

$= \frac{16!}{8!\cdot 9!}\left(17 - 2\cdot 8\right) = \frac{16!}{8!\cdot 8!}\cdot\frac{1}{9} =$

$= \boxed{\frac{1}{9}\binom{16}{8}}$

cattivi =

$= \text{Catt}_1 + \text{Catt}_2$

$= \binom{16}{7} + \binom{16}{7}$

$2\cdot\binom{16}{7}$

$$\frac{1}{n}\binom{2n-2}{n-1} \qquad \frac{1}{n+1}\binom{2n}{n}$$

IMO 4 (come questo)

$$\boxed{S_n = (2n-1)\, S_{n-1}}$$

$$\boxed{S_1 = 1}$$

$$\boxed{S_n = (2n-1)(2n-3) \cdots 3 \cdot 1}$$