COMBINATORIA 2 BASIC

Titolo nota 09/09/2011

$$\{1,2,\ldots\ldots n\} \xrightarrow{\sigma} \{1,2,\ldots\ldots n\}$$

Biiettiva

$$\sigma: \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 6 & 3 & 5 & 7 & 4 & 1 & 9 & 8 \end{array}$$

Tutte le permutazioni su n elementi

$n!$

$$\sigma: \boxed{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{array}} \qquad \tau: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{array}$$

$$\sigma \circ \tau: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{array} \qquad \tau \circ \sigma: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{array}$$

$$(1\ 5\ 3)$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{array}$$

$$\sigma = (\overbrace{a_1\ a_2\ \ldots\ a_k}^{\in \{1 \ldots\ldots n\}})$$

TUTTI GLI ALTRI FISSI

$\sigma(a_1) = a_2$

$\sigma(a_2) = a_3$

$\ldots\ \sigma(a_{k-1}) = a_k$

$\sigma(a_k) = a_1$

$$(1\ 2\ 3) \qquad (4\ 5)$$

$$\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{array}$$

$$a_1 \xrightarrow{} a_2 \xrightarrow{\sigma} a_3 \xrightarrow{} \dots\dots$$

$a_k = a_h \quad h < k \quad \exists$ cassetti

$\exists$ k più piccolo con questa proprietà
minimo

$$a_k = a_1 \qquad a_k = a_h \qquad a_k = \sigma(a_{k-1})$$
$$h = 1 \qquad k > h > 1 \qquad a_h = \sigma(a_{h-1})$$

PER ASSURDO $\quad a_{k-1} = a_{h-1}$

$$a_1 \to a_2 \to \dots\dots a_{k-1}$$

$$\sigma = (a_1\ a_2\ \dots\ a_{k-1})\,\sigma'$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 6 & 3 & 5 & 7 & 4 & 1 & 9 & 8 \end{pmatrix}$$

$$(1\ 2\ 6\ 4\ 5\ 7)(8\ 9)$$

segno

$$\operatorname{sgn}(\sigma) = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j} \in \{\pm 1\}$$

$$\sigma : \{1 \dots n\} \to \{1 \dots n\}$$

$$\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau)$$

$$\operatorname{sgn}(\sigma \circ \tau) = \prod_{1 \le i < j \le n} \frac{\sigma \circ \tau(i) - \sigma \circ \tau(j)}{i - j} =$$

$$= \prod_{1 \le i < j \le n} \underbrace{\frac{\sigma \circ \tau(i) - \sigma \circ \tau(j)}{\tau(i) - \tau(j)}}_{\operatorname{sgn} \sigma} \underbrace{\frac{\tau(i) - \tau(j)}{i - j}}_{\operatorname{sgn} \tau}$$

$\tau(i) = k \qquad k > \ell$

$\tau(j) = \ell$

$$k < \ell \qquad \frac{\sigma(k) - \sigma(\ell)}{k - \ell} = \frac{\sigma(\ell) - \sigma(k)}{\ell - k}$$

Il segno di σ conta la parità del numero di coppie (i, j) con $1 \le i < j \le n$ tali che $i < j$ ma $\sigma(i) > \sigma(j)$

$$(a_1 \ a_2 \ \dots \ a_k) = (a_1 \ a_2)(a_2 \ a_3) \qquad (a_{k-1} \ a_k)$$

$$\begin{matrix} a_1 & a_2 & \dots & a_{k-1} & a_k \\ a_2 & a_3 & \dots & a_k & a_1 \end{matrix}$$

$a_{k-2} \quad a_{k-1} \quad a_k$

$a_k \quad a_{k-1}$

$a_{k-1} \quad a_{k-2}$

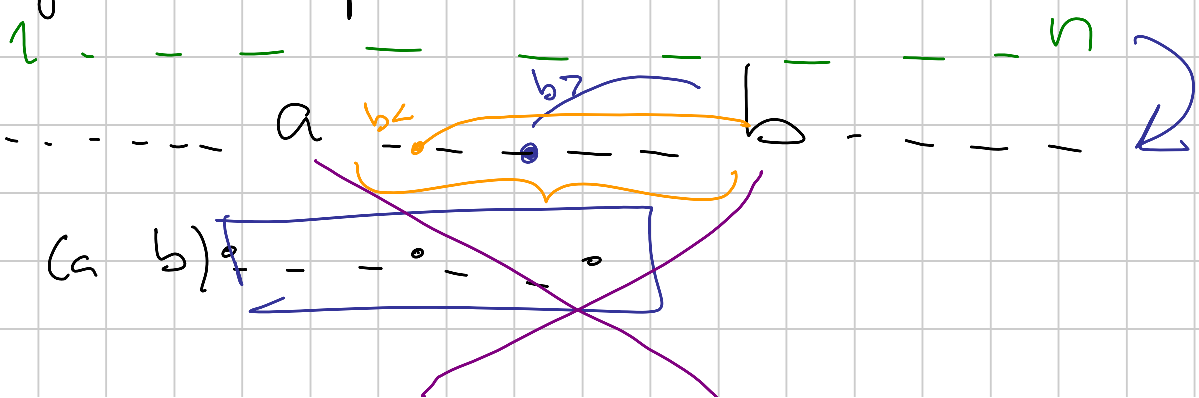
$a_{k-2} \quad a_{k-3}$

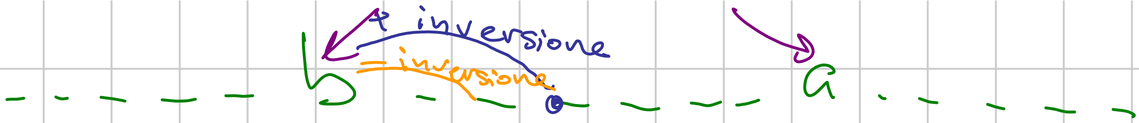
a_1

pari: Sgn $+1$: prodotto di pari trasposizioni

dispari: Sgn -1 : prodotto di dispari trasposizioni

Segno parità del numero di inversioni





parità del numero di inversioni = parità del numero di trasposizioni

$$(a_1\ a_2\ \dots\ a_k) = (a_1\ a_2)(a_2\ a_3) \quad (a_{k-1}\ a_k)$$

Un ciclo di lunghezza pari è una permutazione dispari

____________________ dispari è

________________ pari

?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

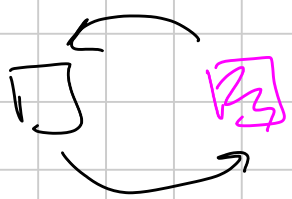
1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

NO

MOSSA = TRASPOSIZIONE

VOGLIAMO OTTENERE UNA PERMUTAZIONE DISPARI

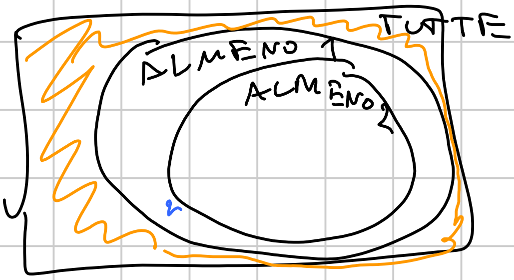
NUMERO DISPARI DI MOSSE



NO

PERMUTAZIONI SENZA PUNTI FISSI

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \binom{n}{1}(n-1)!$$

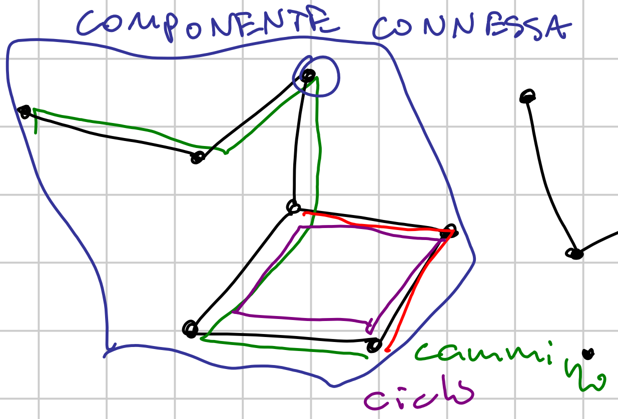


$$n! - \frac{n}{1}(n-1)! + \frac{n(n-1)}{2!}(n-2)! - \frac{n(n-1)(n-2)}{3!}(n-3)! + \dots + (-1)^n \frac{n!}{n!} 0!$$

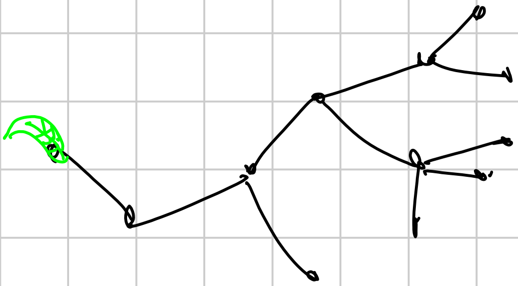
$$\boxed{n! \underbrace{\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}\right)}_{\sim \frac{1}{e}}}$$

L'intero più vicino a $\frac{n!}{e}$

$e = 2,71828 \dots$



vertici
archi



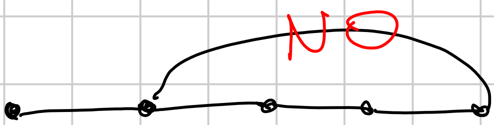
ALBERO

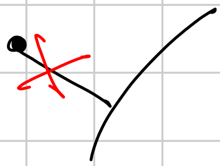
togliendo un arco: non più connesso

aggiungendo un arco: ottengo un ciclo

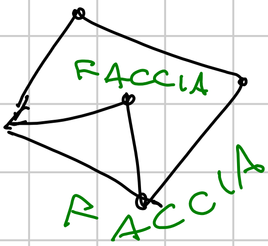
$V = e + 1$

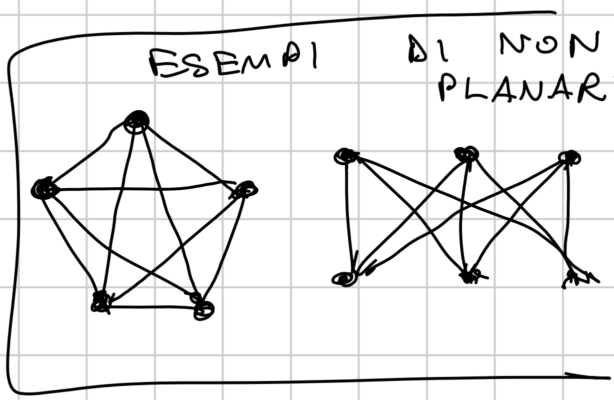
„edges" ~ archi





$1 = 0 + 1$

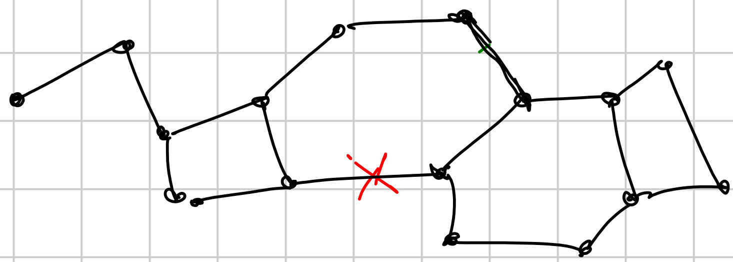




GRAFI PLANARI CONNESSI (NON VUOTI)

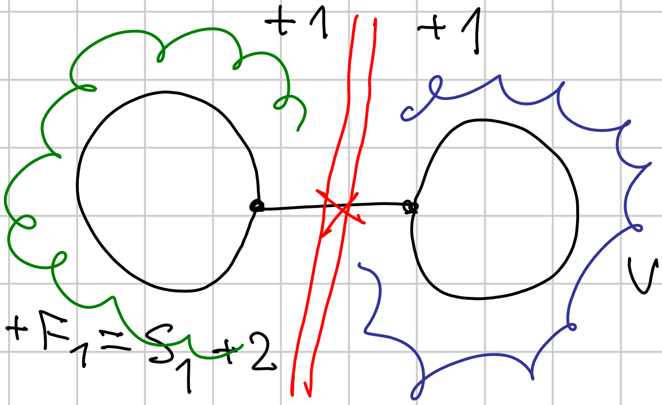
$$V+F=S+2$$

$\bullet \quad 1+1=0+2$



$$V+F=S+2$$

$+1 \qquad +1$



$V_1+F_1=S_1+2$

$V_2+F_2=S_2+2$

$$(V_1+V_2)+F=(S_1+S_2+1)+2$$

SPERIAMO

$$(V_1+V_2)+(F_1+F_2-1)=(S_1+S_2+1)+2+1$$

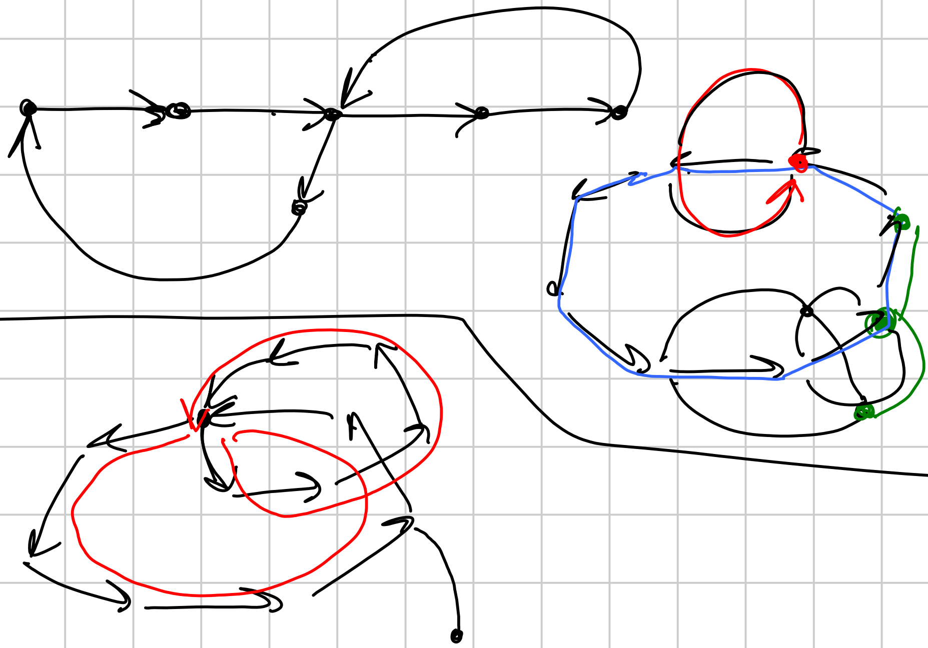
IP IND $\begin{cases} V_1+F_1=S_1+2 \\ V_2+F_2=S_2+2 \end{cases}$

$$(V_1+V_2)+(F_1+F_2)=(S_1+S_2)+4$$

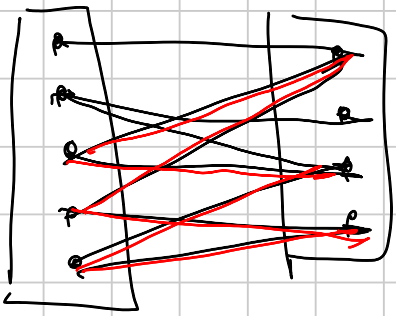
CICLO EULERIANO ESISTE SSE OGNI VERTICE HA ORDINE PARI

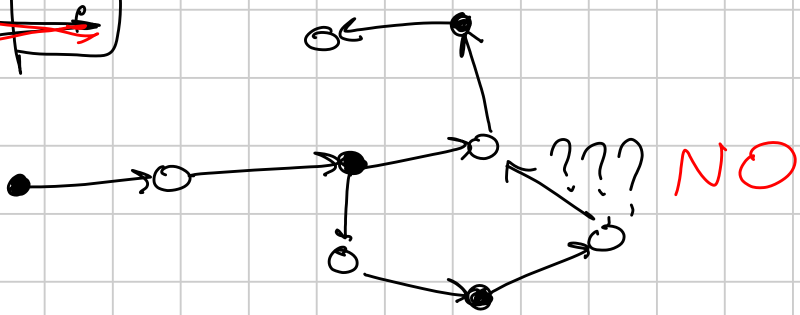
CICLO DI EULERO ⟹ CARATTERIZZAZIONE

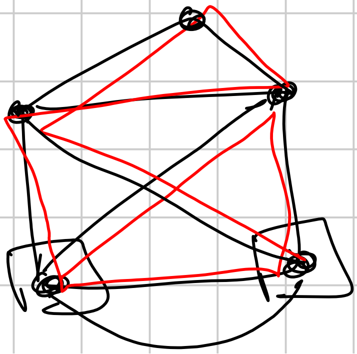
CICLO: ESCE ED ENTRA IN OGNI VERTICE



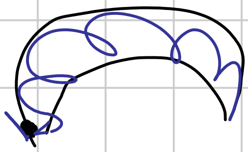
IL CICLO PIÙ GRANDE COMPRENDE TUTTI GLI ARCHI



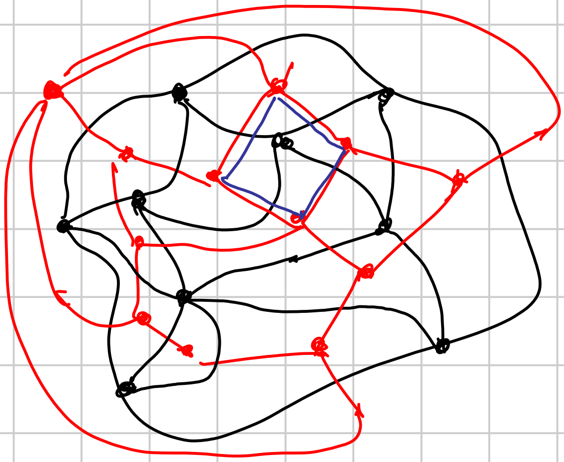




$$\sum_{\text{vertici}} \underbrace{O(v)}_{\text{NUMERO DI ARCHI DAL VERTICE}} = 2e$$

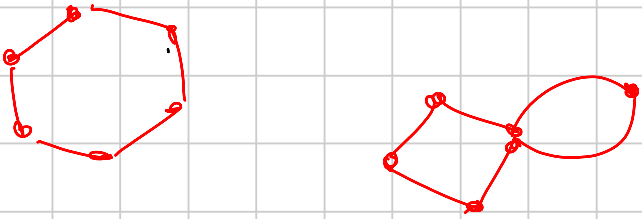


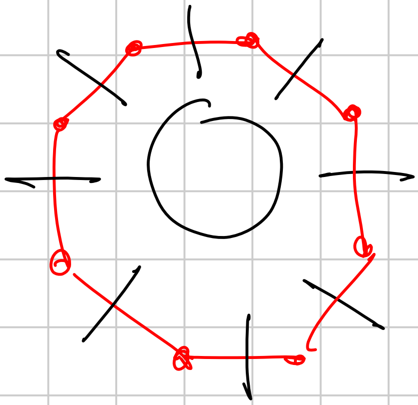
- CARTINA CON 2 COLORI
- FERROVIA LUNGO I CONFINI



GRAFO DEI CONFINI = CICLO DI EULERO

GRAFO DEGLI STATI BIPARTITO



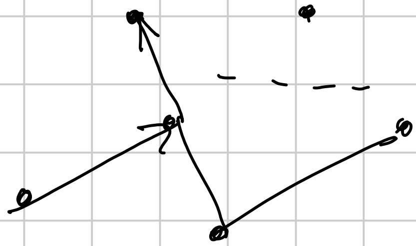


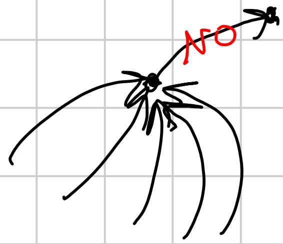
$$\sum_{\substack{\text{vertici grafo confini}\\ \text{DENTRO IL CICLO}}} o(v) = \text{PARI} =$$

= 2 archi interni + 1 archi corrispondenti agli archi del ciclo

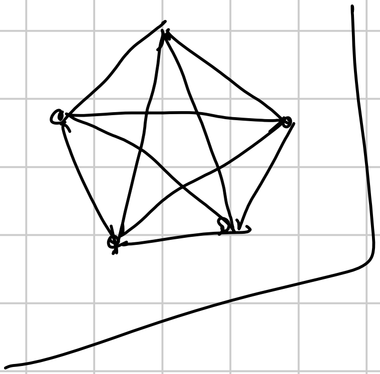
SONO PARI

GRAFI ORIENTATI





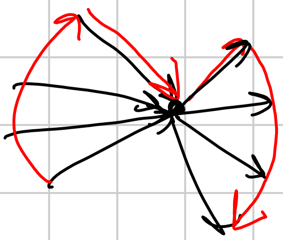
IP: PER OGNI COPPIA, (a,b) POSSIAMO ANDARE da a a b o VICEVERSA



GRAFO COMPLETO

OGNI ARCO È ORIENTATO

OGNI VERTICE UNA E UNA SOLA VOLTA



INVARIANTI

$a_1 \quad a_2 \quad a_3 \quad \dots \quad a_{n-1} \quad a_n \in \mathbb{R}$

$a_1 \quad \frac{a_2+a_3+a_{n-1}}{3}, \quad \frac{a_2+a_3+a_{n-1}}{3} \quad \dots \quad \frac{a_2+a_3+a_{n-1}}{3}, \quad a_n$

SOMMA È UN INVARIANTE

SOMMA DEI QUADRATI CALA

GNOMI CONFORMISTI

Gennaio, Febbraio, ... Dicembre

Azzurro Rosa

Alcune coppie: Amici

Si ridipinge da conformisti

Processo è finito

Felicità (gnomo) = amici con casa stesso colore − amici con casa dell'altro colore

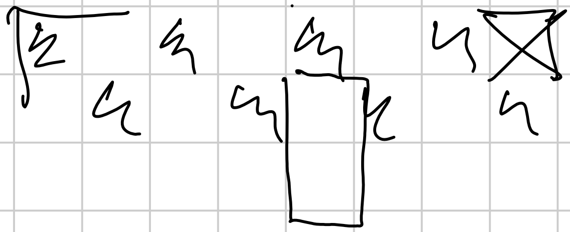
$$F = \sum_{\text{gnomi}} f(\text{gnomo}) \qquad - \to +$$

$$m > n$$

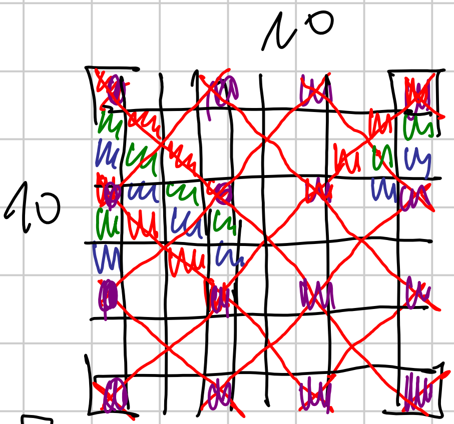
$$+2m - 2n = 2(m-n) > 0$$

$$F \leq \sum_{\text{gnomi}} N^o \text{ amici (gnomo)}$$

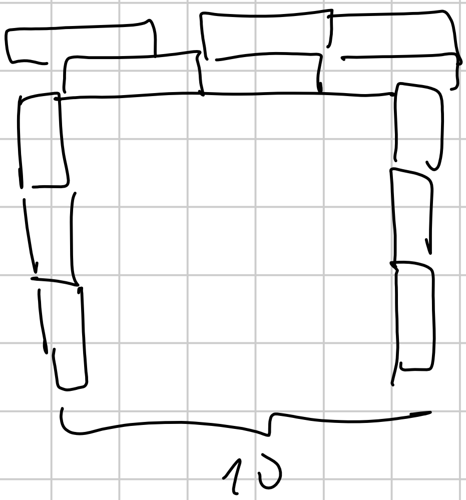
Prima o poi non si ridipinge più

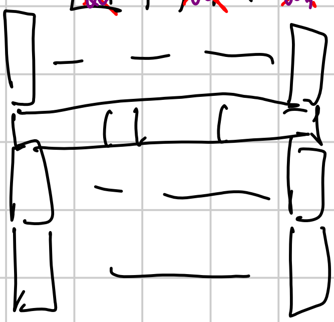


NO



34 33 33





2011×2011

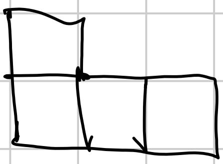
2011 NO

$4a + 9b$

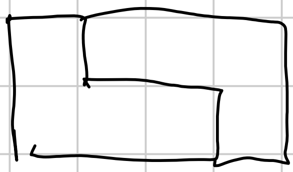
> 23

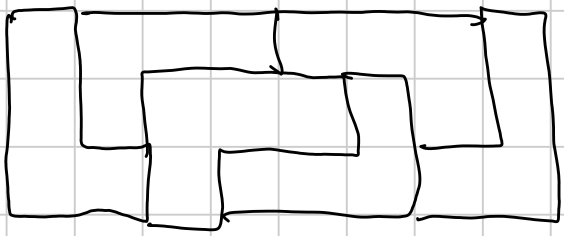
$(m, n) = 1$

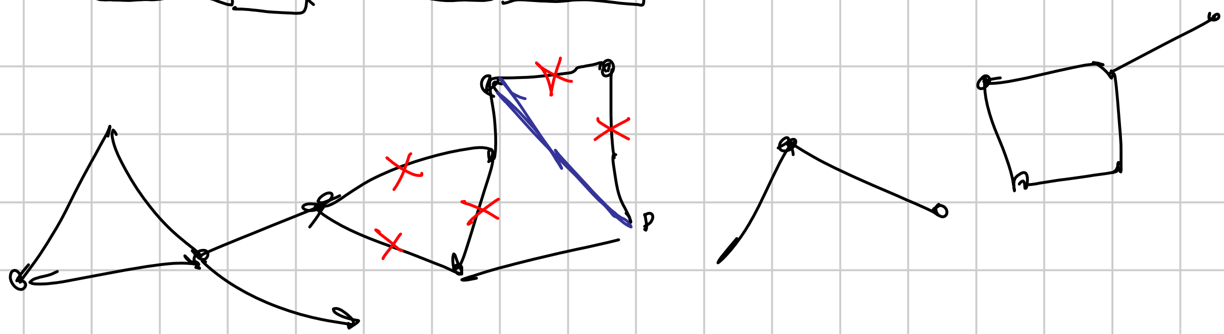
$mn - m - n$ l'ultimo



un lato pari → strisce perpendicolari a esso
stesso numero bianche e nere (pari) caselle



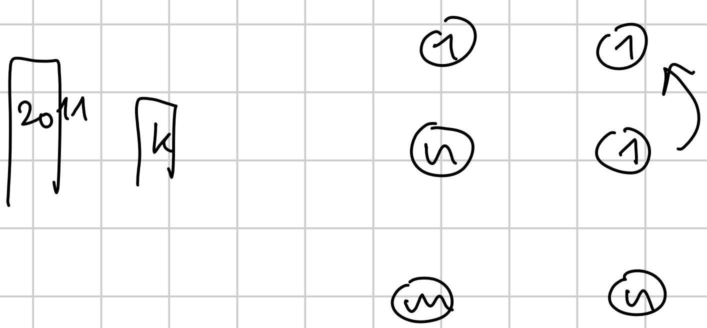




Alla fine sappiamo quanti vertici con arco uscente

Quanti archi

Sappiamo mosse fatte parità



$m \neq n$

$m > n$

↓

$n \quad n$

perdente: qualunque mossa risulta in una config vincente per l'avversario vincente: $\exists$ mossa che risulta in una config perdente per l'avversario