G2 - METODI ALGEBRICI

Titolo nota 07/09/2011

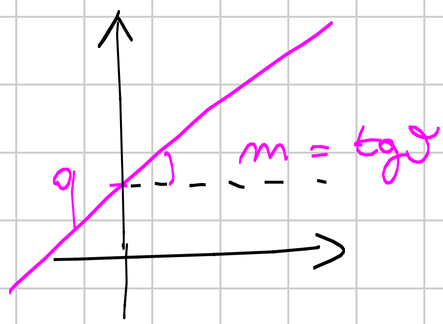
GEOMETRIA
- metodi SINTETICI
- metodi ALGEBRICI
  - COORDINATE CARTESIANE
  - VETTORI
  - NUMERI COMPLESSI

COORDINATE

- RETTA

$$y = mx + q$$

coeff angolare



mancano le parallele all'asse y!

$$ax + by + c = 0$$

- rette // hanno = coeff angolare
- rette ⊥ hanno coeff ang. m_1, m_2 |

$$m_1 m_2 = -1$$

- CIRCONFERENZA

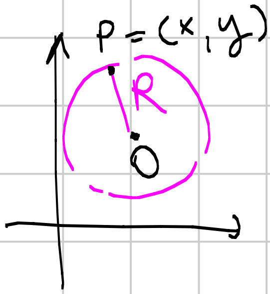
$x^2 + 3\cancel{y^2} + x + 5 = 0$ NO

$2x^2 + 2y^2 = 8$ OK

$$x^2 + y^2 + ax + by + c = 0$$

ACHTUNG! non è sempre una circ. (reale)

e.g. $x^2 + y^2 + 1 = 0$



O: (x_0, y_0)

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

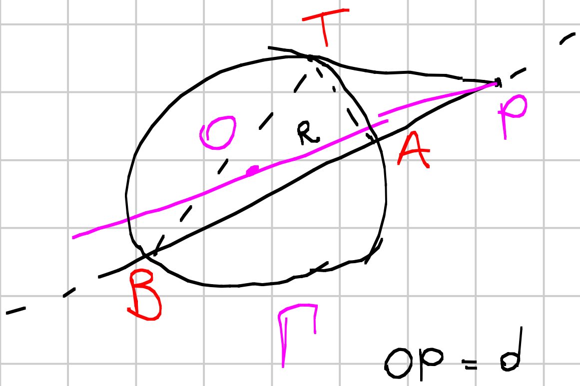
$$\left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4} + \left(y + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0$$

CONDIZIONE: $\frac{a^2}{4} + \frac{b^2}{4} \geq c$

CENTRO: $\left(-\frac{a}{2}, -\frac{b}{2}\right)$

RAGGIO: $R^2 = \frac{a^2}{4} + \frac{b^2}{4} - c$

- POTENZA di pto rispetto a circ.



$$\text{pow}_\Gamma(P) = PA \cdot PB$$

dimostra $PT^2 = PA \cdot PB$

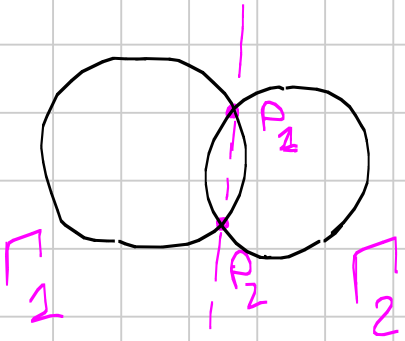
$OP = d$

$$\text{pow}_\Gamma(P) = (d - R)(d + R) = d^2 - R^2$$

Γ $(x - x_0)^2 + (y - y_0)^2 = R^2$ $\quad P$ (a, b)

$$\text{pow}_\Gamma(P) = (a - x_0)^2 + (b - y_0)^2 - R^2$$

- ASSE RADICALE



luogo dei punti P tali che

$$pow_{\Gamma_1}(P) = pow_{\Gamma_2}(P)$$

Γ_1 $x^2+y^2+a_1x+b_1y+c_1=0$

Γ_2 $x^2+y^2+a_2x+b_2y+c_2=0$

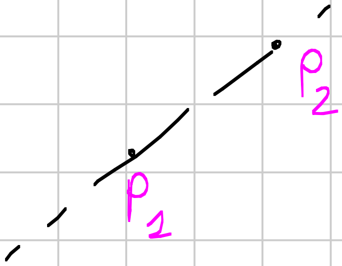
$(x,y) \mid$

$$\cancel{x^2}+\cancel{y^2}+a_1x+b_1y+c_1 = \cancel{x^2}+\cancel{y^2}+a_2x+b_2y+c_2$$

$$(a_1-a_2)x+(b_1-b_2)y+c_1-c_2=0$$

è l'equazione di una retta!

- LA RETTA per 2 PUNTI



$P_1 : (a_1, b_1)$

$P_2 : (a_2, b_2)$

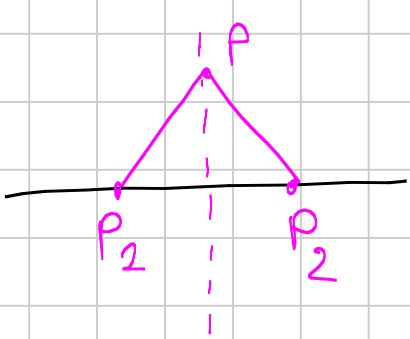
$\alpha x + \beta y + \gamma = 0$

$$\frac{x-a_1}{a_2-a_1} = \frac{y-b_1}{b_2-b_1}$$

← primo grado (in x, y), si annulla in P_1, P_2

→ è la retta per P_1, P_2

- L'ASSE di un SEGMENTO

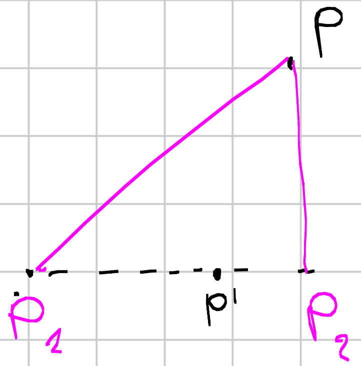


$P_1\ (a_1, b_1)$ $\quad$ $P_2\ (a_2, b_2)$

$P\ (x,y) \mid$

$$(x-a_1)^2+(y-b_1)^2 = (x-a_2)^2+(y-b_2)^2$$

- CIRCONFERENZA di APOLLONIO

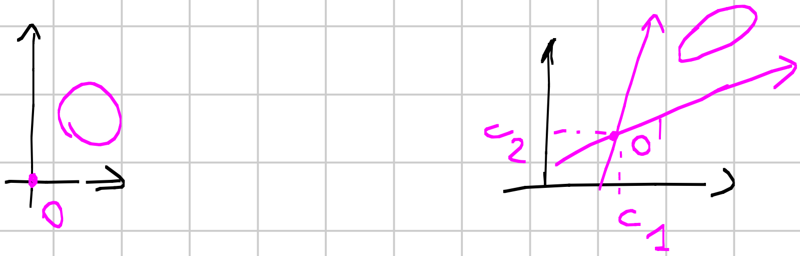


$$PP_1 = \lambda PP_2$$
$$\lambda > 0$$

(esattamente come prima: pol. di 2° grado in x e y, coeff x^2 = coeff y^2, non ci sono termini 'misti' (in xy))

PROBLEMI "INVARIANTI PER AFFINITÀ"

$$(x, y) \longmapsto (a_1x + b_1y + c_1,\ a_2x + b_2y + c_2)$$



AFFINITÀ

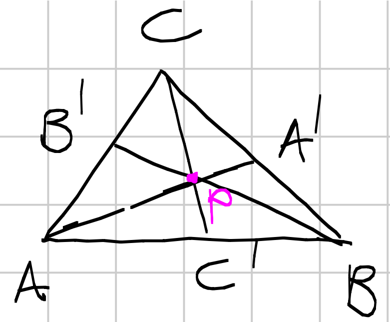
mandano RETTE in RETTE
CIRCONFERENZE in ?
ELLISSI

conservano parallelismo
rapporti fra aree
rapporti di segmenti su stessa retta

NON conservano angoli
lunghezze

3 punti non allineati $\xrightarrow{\exists \text{ affinità}}$ 3 punti non allineati

CESE 6 . 2004



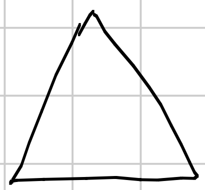
$$x = \frac{AP}{PA'}$$

$$y = \frac{BP}{PB'}$$

$$z = \frac{CP}{CP'}$$

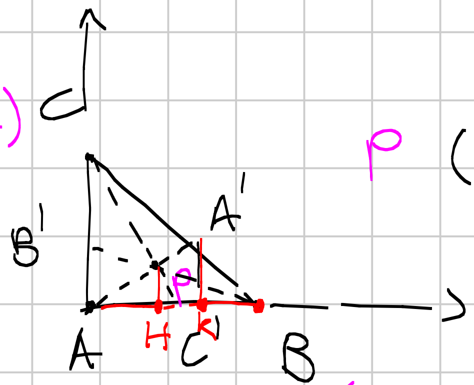
$$\boxed{\begin{aligned} xyz &= \\ &= x+y+z+2 \end{aligned}}$$

è invariante per affinità!



equilatero (mah...)

(0,1)



(0,0) (1,0)

P (a,b)

AB $y = 0$

AC $x = 0$

BC $y = -x + 1$

AA' $\frac{x}{a} = \frac{y}{b}$

BB' $\frac{x-1}{a-1} = \frac{y}{b}$

CC' $\frac{x}{a} = \frac{y-1}{b-1}$

$x = \frac{a}{b} y = -\frac{a}{b}x + \frac{a}{b}$

$x = \frac{a}{a+b}$ $\quad y = \frac{b}{a+b}$

$\left(\frac{a}{a+b}, \frac{b}{a+b}\right)$ A'

$\left(0, \frac{b}{1-a}\right)$ B' $\quad y = \frac{b}{1-a} \quad x = 0$

$\left(\frac{a}{1-b}, 0\right)$ C'

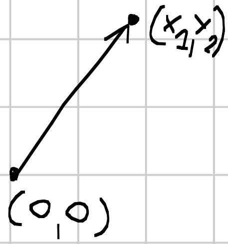
$$\frac{AP}{PA'} = \frac{\cancel{a}}{\cancel{a}\left(-1 + \frac{1}{a+b}\right)} = \frac{a+b}{1-a-b}$$

$$\frac{BP}{PB'} = \frac{1-a}{a}$$

$$\frac{CP}{CC'} = \frac{1-b}{b}$$

VETTORI

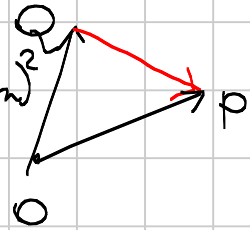
$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ ← coordinate



norma $\|(x_1, x_2, \dots, x_n)\|^2 =$
$= x_1^2 + x_2^2 + \dots + x_n^2$

$$\|\vec{P} - \vec{Q}\|^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2$$

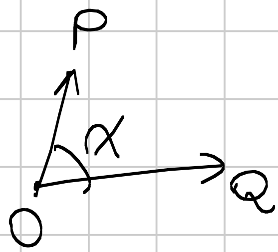
$\vec{P} = (x_1, \dots, x_n)$, $\vec{Q} = (y_1, \dots, y_n)$



prodotto scalare

$$\vec{P} \cdot \vec{Q} = x_1y_1 + x_2y_2 + \dots + x_ny_n$$

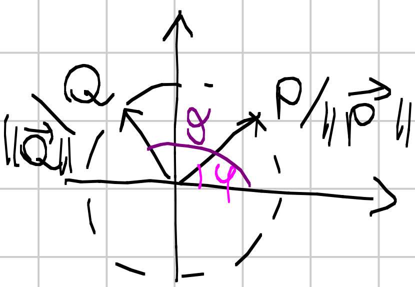
$$\vec{P} \cdot \vec{Q} = \boxed{\|\vec{P}\| \, \|\vec{Q}\|} \cos\alpha$$



dimostrazione nel piano

$$\vec{P} \cdot \lambda\vec{Q} = \lambda \, \vec{P} \cdot \vec{Q}$$

$$\vec{P} \cdot \vec{Q} = \boxed{\|\vec{P}\|\|\vec{Q}\|} \underbrace{\left(\frac{\vec{P}}{\|\vec{P}\|} \cdot \frac{\vec{Q}}{\|\vec{Q}\|}\right)}_{?}$$



$\vec{P}/\|\vec{P}\| = (\cos\varphi, \operatorname{sen}\varphi)$

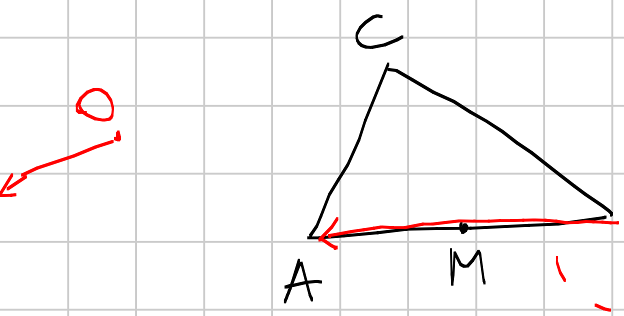
$\vec{Q}/\|\vec{Q}\| = (\cos\vartheta, \operatorname{sen}\vartheta)$

$? = \cos\varphi\cos\vartheta + \operatorname{sen}\vartheta \operatorname{sen}\varphi =$
$= \cos(\vartheta - \varphi) = \cos\alpha$

$$\|\vec{P}\|^2 = \vec{P}\cdot\vec{P}$$

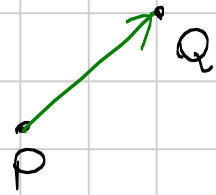
$$\vec{P}\cdot\vec{Q} = 0$$
$$\vec{P}\perp\vec{Q}$$

$$\|\vec{P}+\vec{Q}\|^2 = \|\vec{P}\|^2 + \|\vec{Q}\|^2 + 2(\vec{P}\cdot\vec{Q})$$

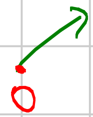


$$\vec{A},\ \vec{B},\ \vec{C}$$

$$\vec{A}-\vec{B}$$



$$\vec{P} + \vec{PQ} = \vec{Q}$$



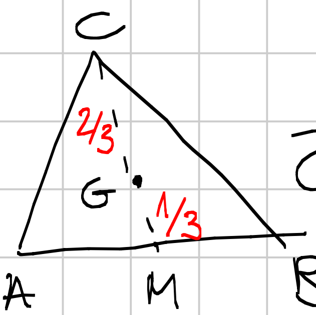
punto medio di AB $\quad \dfrac{\vec{A}+\vec{B}}{2}$

retta AB $\quad \vec{A} + \lambda(\vec{B}-\vec{A}) \quad \lambda\in\mathbb{R}$

segmento AB $\quad \lambda\in[0,1]$

punto medio $\quad \lambda = \dfrac{AM}{AB} = \dfrac{1}{2}$

BARICENTRO



$$\vec{G}$$

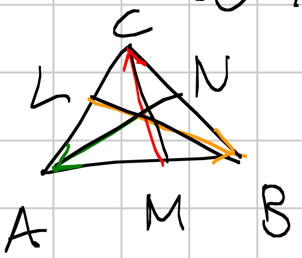
$$\vec{C} + \lambda(\vec{M}-\vec{C}) = \quad \text{retta CM (partendo da C)}$$
$$= \vec{C} + \lambda\left(\frac{\vec{A}+\vec{B}}{2} - \vec{C}\right)$$

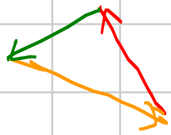
$$\lambda = \frac{CG}{CM} = \frac{2}{3}$$

$$\vec{G} = \vec{C} + \frac{1}{3}(\vec{A}+\vec{B}) - \frac{2}{3}\vec{C} = \boxed{\frac{1}{3}(\vec{A}+\vec{B}+\vec{C})}$$

ESERCIZIO

le mediane "formano un triangolo"





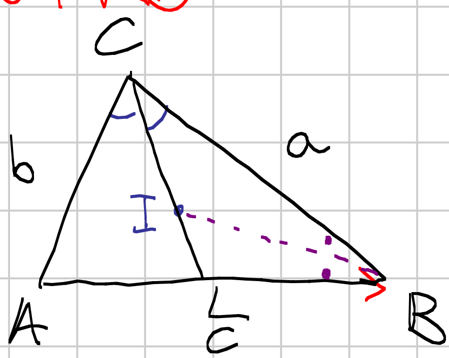
$$(\vec C - \vec M) + (\vec A - \vec N) + (\vec B - \vec L) = 0$$

TESI

$$\vec A + \vec B + \vec C \underbrace{- \frac{1}{2}\vec A - \frac{1}{2}\vec A}_{-\vec A} - \dots = 0$$

$$\vec C - \cancel{\frac{\vec A}{2}} - \frac{\vec B}{2} + \cancel{\vec A} - \frac{\vec B}{2} - \frac{\vec C}{2} + \vec B - \frac{\vec C}{2} - \cancel{\frac{\vec A}{2}}$$

INCENTRO



$b : a+b = AL : BL + AL$

$b : a = AL : BL$

$$\vec L = \vec A + \lambda(\vec B - \vec A)$$

$$\lambda = \frac{AL}{AB} = \frac{b}{a+b}$$

O

$$\vec L = \frac{b}{a+b}\vec B + \frac{a}{a+b}\vec A$$

$$\frac{a}{a+b} = 1 - \lambda, \quad \lambda = \frac{b}{a+b}$$

retta CL

$$\vec C + \mu(\vec L - \vec C)$$

$$\vec I = \vec C + \mu(\vec L - \vec C) \text{ con } \mu = \frac{CI}{CL}$$

Teo della bisettrice su BCL mi dà

$a : BL = CI : IL$

$a : BL + a = CI : IL + CI$

$$\mu = \frac{a}{BL + a} = \frac{\cancel{a}}{\cancel{a}\left(\frac{c}{a+b} + 1\right)} = \frac{a+b}{a+b+c}$$

$$\vec{LB} = \vec{B} - \vec{L} = -\vec{A}\frac{a}{a+b} + \vec{B}\frac{a}{a+b} =$$

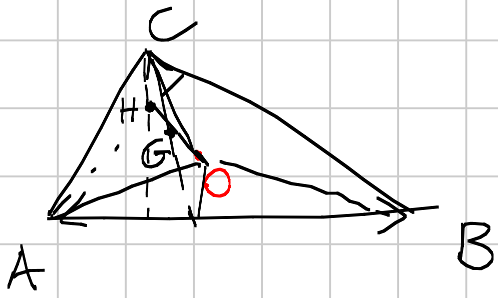
$$= \frac{a}{a+b}(\vec{B} - \vec{A})$$

$$\|\vec{LB}\| = \frac{a}{a+b}c$$

$$\vec{I} = \vec{C} + \frac{a+b}{a+b+c}\left(\frac{a}{a+b}\vec{A} + \frac{b}{a+b}\vec{B} - \vec{C}\right)$$

$$= \underbrace{\frac{1}{a+b+c}\left(a\vec{A} + b\vec{B} + c\vec{C}\right)}$$

CIRCOCENTRO



$$\|\vec{A}\| = \|\vec{B}\| = \|\vec{C}\| = R$$

SE origine la metto nel circocentro

ORTOCENTRO

$\vec{O}$ è l'origine $\rightarrow$

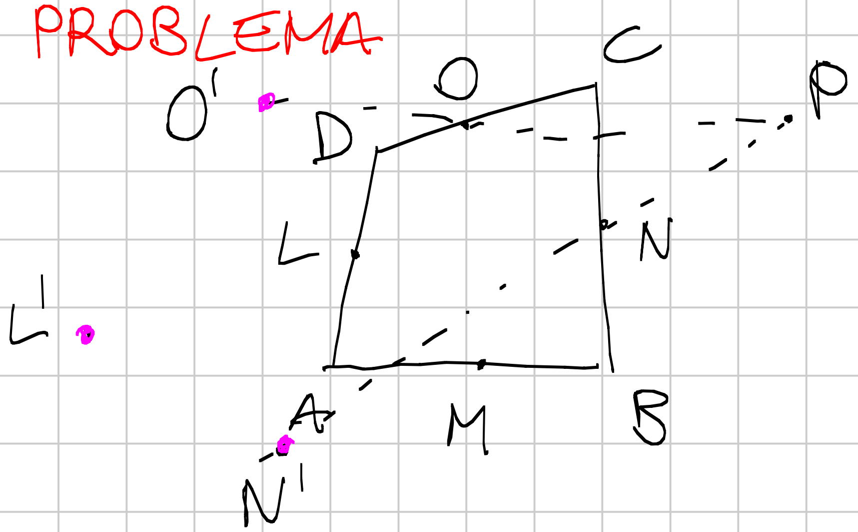
retta OH $\quad \lambda\left(\frac{\vec{A} + \vec{B} + \vec{C}}{3}\right)$

punto H è su OH con $\lambda = \frac{OH}{OG} = 3$

$\vec{H} = \vec{A} + \vec{B} + \vec{C}$

SE $\vec{O}$ è l'origine

PROBLEMA



tesi

$N'M'L'O'$

è un parallelogrammo!

$N \quad \dfrac{\vec{B}+\vec{C}}{2} \qquad N' \quad \vec{B}+\vec{C}$

se origine in P

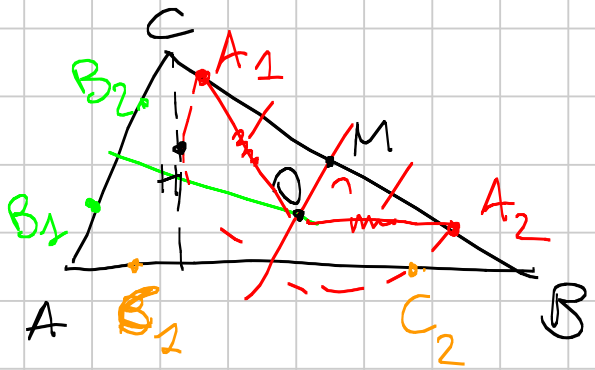
M'

$M' \quad \vec{A}+\vec{B} \qquad L' \quad \vec{A}+\vec{D}$

$O' \quad \vec{D}+\vec{C}$

$$\boxed{\begin{aligned}\overrightarrow{N'O'} &= \vec{O'} - \vec{N'} = \vec{D} - \vec{B}\\ \overrightarrow{M'L'} &= \vec{L'} - \vec{M'} = \vec{D} - \vec{B}\end{aligned}}$$

IMO 2008. 1



$A_1A_2C_2C_1B_1B_2$ è ciclico

origine in O!

$$\|\vec{A}_1\|^2 = \|\vec{B}_2\|^2$$

$$\overset{\shortparallel}{OM^2 + MA_1^2}$$

$$\left\| \frac{\vec{B}+\vec{C}}{2} \right\|^2 + \left\| \vec{A}+\vec{B}+\vec{C} - \frac{\vec{B}+\vec{C}}{2} \right\|^2$$

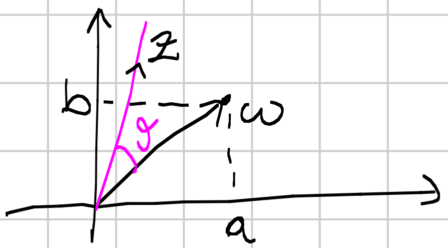
$$\left\| \frac{\vec{B}+\vec{C}}{2} \right\|^2 + \left\| \vec{A} + \frac{\vec{B}+\vec{C}}{2} \right\|^2$$

$$\|B_1\|^2 = \left\| \frac{\vec{A}+\vec{C}}{2} \right\|^2 + \left\| \vec{B} + \frac{\vec{A}+\vec{C}}{2} \right\|^2$$

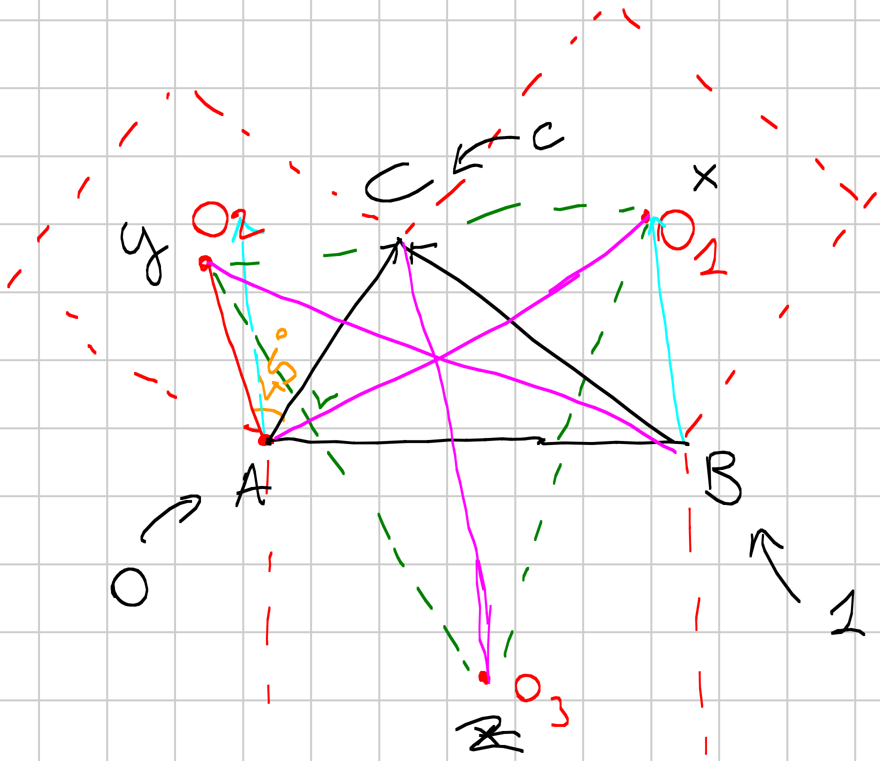
$$\|\vec{A}\|^2 = \|\vec{B}\|^2 = \|\vec{C}\|^2$$

$\square$

NUMERI COMPLESSI



$$w = a + ib$$
$$z = w \cdot e^{i\vartheta}$$



AO_1, BO_2, CO_3 concorrono.

$AO_1 = O_2O_3$

$$y = c\, e^{i\frac{\pi}{4}} \frac{\sqrt{2}}{2} = c\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) =$$
$$= \frac{1}{2}c(1+i)$$

$$z = 1 e^{-i\frac{\pi}{4}} \frac{\sqrt{2}}{2} = \frac{1}{2}(1-i)$$

$$x = (c-1)e^{-i\frac{\pi}{4}}\frac{\sqrt{2}}{2} + 1 =$$
$$= \frac{1}{2}(c-1)(1-i) + 1 =$$
$$= \underbrace{\frac{1}{2}(c-1+2)}_{\frac{1}{2}(c+1)} + i\left(\frac{1-c}{2}\right)$$

$$y - z = \frac{1}{2}c(i+1) - \frac{1}{2}(1-i) = \frac{c-1}{2} + i\left(\frac{c+1}{2}\right)$$

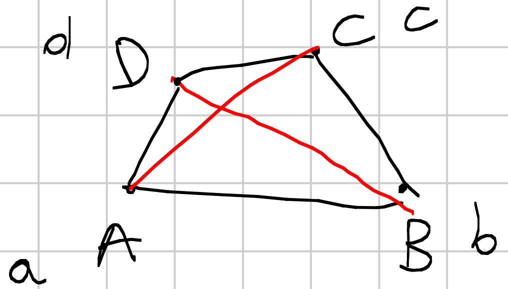
ruoto x di $90°$

ottengo

$$ix = -\frac{(1-c)}{2} + i\,\frac{1+c}{2} =$$

$$= \frac{c-1}{2} + i\left(\frac{c+1}{2}\right) = y - z$$

DISUGUAGLIANZA di TOLOMEO



$AB \cdot DC + BC \cdot AD \geq AC \cdot BD$

$=$ vale SSE ABCD è ciclico

$$|b-a||d-c| + |c-b||d-a| =$$

$$= |(b-a)(d-c)| + |(c-b)(d-a)| \geq$$

disug. triangolare

$$|(b-a)(d-c) + (c-b)(d-a)| =$$

$$= |\cancel{bd} - ad + \cancel{ac} - bc + cd - \cancel{bd} - \cancel{ca} + ba|$$

$$= |c(d-b) - a(d-b)| = |(c-a)(d-b)|$$

Caso di $=$

$$\frac{(b-a)(d-c)}{(c-b)(d-a)} \in \mathbb{R}$$