Senior 2011 A2m - Jack

Titolo nota 08/09/2011

$p(x) = a_0 x^n + \dots + a_n \qquad a_i \in A \qquad A = \underbrace{\mathbb{Z}[x]}$

z è radice di $p(x) \longrightarrow (x-z) \mid p(x)$ (Ruffini)

z radice di molt. k se $(x-z)^k \mid p(x)$

$p(z)=0, \; p'(z)=0 \; \dots, \; p^{(k-1)}(z) = 0$

$$\frac{d}{dx} x^m = m x^{m-1}$$

$\forall p(x) \in \mathbb{C}[x] \quad \exists z \in \mathbb{C} : \; p(z) = 0 \quad$ | Theo fond Alg

Lemma 1. $\forall p(x) \in \mathbb{R}[x], \; \partial p \equiv 1 \, (2) \quad \exists z \in \mathbb{R} : \; p(z) = 0$ (per continuità)

Lemma 2. $p(x) \in \mathbb{R}[x] \quad p(z) = 0$, allora $p(\bar{z}) = 0$

$z = a + ib, \quad \bar{z} = a - ib$

$p(z) = 0 \longrightarrow \overline{p}(z) = 0 = p(\bar{z})$

Lemma 3. (Viète) $p(x) \in \mathbb{C}[x] \qquad [x^{\partial p}]\, p(x) = 1$ monico

∂p radici $z_1, \dots, z_{\partial p}$

$n = \partial p$

$$[x^{n-j}]\, p(x) = (-1)^j \sum_{1 \le i_1 < \dots < i_j \le n} \prod_{h=1}^{j} z_{i_h} \qquad p(x) = \prod_{i=1}^{n} (x - z_i)$$

$f(x_1, \dots, x_n)$ è f. sym. di n var sse

$\forall \sigma \in S_n$ si ha $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$

F. sym elementari di n variabili

$e_1(x_1, \dots, x_n) = x_1 + \dots + x_n$

$e_2(x_1, \dots, x_n) = \sum_{i<j} x_i x_j$

Somme di potenze

$p_1(x_1, \dots, x_n) = x_1 + \dots + x_n$

$p_k(x_1, \dots, x_n) = \sum x_i^k$

Theo Sia $\{e_i\}_{i=0}^{n-1}$ che $\{p_i\}_{i=0}^{n-1}$ sono una base per l'anello delle f. sym in n variabili

Dim Ind nel grado.

Formule di Newton-Girard

$$\boxed{k \cdot e_k = \sum_{j=1}^{k} (-1)^{j-1} e_{k-j} p_j}$$

Dim.
Ind nel grado
+ trucco analitico

$\frac{d}{dx} x^m \to m x^{m-1}$ $\qquad \delta : p(x) \to p(x) - p(x+1)$

forward diff. op.

$$\begin{cases} p \in \mathbb{Z}[x] \to \delta p \in \mathbb{Z}[x] \\ \delta(pq) + p \cdot (\delta q) + q \cdot (\delta p) + (\delta p)(\delta q) = 0 \\ \partial(\delta p) = \partial p - 1 \\ [x^{\partial p - 1}](\delta p) = -\partial p \, [x^{\partial p}] p \\ \delta^{\partial p} p = (-1)^{\partial p} \cdot (\partial p)! \cdot [x^{\partial p}] p \end{cases}$$

5	10	21	32	37
5	11	11	5	
6	0	−6		
−6	−6			

metodo delle diff. finite

5 11 11 5 — polin. di grado 2

6 0 −6 — polinomio di grado 1

−6 −6 — costante

Disuguaglianze $x^2 \geq 0$.

$f : \mathbb{R}^n \to \mathbb{R}$ è detta convessa se

$\forall \lambda_1, \dots, \lambda_n \in [0,1]$, $\sum \lambda_i = 1$ si ha $f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{j=1}^{n} \lambda_j f(x_j)$

Disug. di Jensen

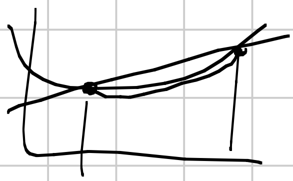
f si dice midpoint-convex se $\forall (c,d) \in [a,b]^2$ si ha

$$f\left(\frac{c+d}{2}\right) \leq \frac{f(c)+f(d)}{2}$$

Lemma midpoint-convex + continua $\Rightarrow$ convessa.

$f \in C^1([a,b])$, $f'(x)$ è una f. deb cresc $\Rightarrow$ f è convessa

$f \in C^2([a,b])$, $f''(x) \geq 0$ $\Rightarrow$ f è convessa



al di sotto delle secanti

al di sopra delle tangenti

Disug di Karamata (Hardy-Littlewood)

$(a_1, \dots, a_k) \gg (b_1, \dots, b_k)$ di numeri reali ≥ 0 debolmente decrescenti

$$\begin{cases} a_1 \geq b_1 \\ a_1 + a_2 \geq b_1 + b_2 \\ \dots \\ a_1 + \dots + a_{n-1} \geq b_1 + \dots + b_{n-1} \\ a_1 + \dots + a_n = b_1 + \dots + b_n \end{cases}$$

$\forall f$ convessa vale $\quad f \in C^1$

$$\boxed{\sum_{i=1}^{k} f(a_i) \geq \sum_{i=1}^{k} f(b_i)}$$

$$\delta_f(a,b) = \frac{f(b)-f(a)}{b-a} \qquad \delta_f(a,a) = f'(a) \qquad c_i = \delta_f(a_i, b_i)$$

$$A_i = \sum_{j=1}^{i} a_j \quad B_i = \sum_{j=1}^{i} b_j$$

$$\sum_{i=1}^{k} \left(f(a_i) - f(b_i)\right) = \sum_{i=1}^{k} c_i (a_i - b_i)$$

$$= \sum_{i=1}^{k} c_i (A_i - A_{i-1} - B_i + B_{i-1})$$

$$= \sum_{i=1}^{k-1} \underbrace{(c_i - c_{i+1})}_{\geq 0} \underbrace{(A_i - B_i)}_{\geq 0}$$

$$c_i = \delta_f(a_i, b_i) \geq \delta_f(b_i, a_{i+1}) \geq \delta_f(b_{i+1}, a_{i+1}) = c_{i+1}$$

Disug di Cauchy-Schwarz

$$\left(\sum_{j=1}^{k} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{k} a_j^2\right)\left(\sum_{j=1}^{k} b_j^2\right)$$

$$p(x) = \sum_{j=1}^{k} (a_j x + b_j)^2 \geq 0 \quad \text{allora} \quad \Delta p \leq 0.$$

$v, w \in \mathbb{R}^k$

$$\|v - w\|^2 \geq 0$$

$$\|v\|^2 + \|w\|^2 - 2\langle v, w\rangle \geq 0$$

$$\langle v, w\rangle \leq \frac{1}{2}\left(\|v\|^2 + \|w\|^2\right)$$

Trick: amplificazione o interpolazione

$$\forall \lambda \in \mathbb{R}_0^+ \quad \langle v, w\rangle \leq \frac{1}{2}\left(\frac{1}{\lambda^2}\|v\|^2 + \lambda^2\|w\|^2\right)$$

$$\lambda^2 = \|v\| / \|w\|$$

$$\langle v, w\rangle \leq \|v\| \cdot \|w\| \qquad \text{C.S.}$$

Disug. di riarrangiamento

$(\underline{a_1}, \dots, a_k)$ $(b_1, \dots, \underline{b_k})$ seq. deb. cresc. di num. reali ≥ 0

allora $\forall \sigma \in S_k$ vale
$$\underline{\sum_{j=1}^{k} a_j b_j} \geq \sum_{j=1}^{k} a_j b_{\sigma(j)} \geq \underline{\sum_{j=1}^{k} a_j b_{k+1-j}}$$

struttura di S_k + induzione

$$\sigma_1 = (n_1\ n_2)\,\sigma_2 \quad \underline{\mathrm{II}}$$

Disug di Chebyshev

$(a_1, \dots, a_k)$ $(b_1, \dots, b_k)$ non decrescenti di num. reali ≥ 0

Allora
$$k \cdot \sum_{j=1}^{k} a_k b_k \geq \left(\sum_{j=1}^{k} a_j\right)\left(\sum_{j=1}^{k} b_j\right)$$

$\sigma \in S_k$ delle forme $\left.\begin{array}{l} \sigma = (1\ 2\ \dots\ k) \\ \sigma^2 \\ \vdots \\ \sigma^k = Id \end{array}\right\}$ sommare k disug di riarrangiamento.

$$T[a_1, \dots, a_k] = \sum_{\sigma \in S_k} x_{\sigma(1)}^{a_1} \cdot x_{\sigma(2)}^{a_2} \cdot \dots \cdot x_{\sigma(k)}^{a_k} = \sum_{sym} \prod_{j=1}^{k} x_j^{a_j}$$

Schur

$\forall a, b \in \mathbb{R}^+$ $\quad T[a+2b, 0, 0] + T[a, b, b] \geq$

$$2 \cdot T[a+b, b, 0]$$

Muirhead / Bunching / Riarrang. gen.

$(a_1, \dots, a_k)$ $(b_1, \dots, b_k)$ sono seq deb decrescenti per cui

$a \gg b$

Allora $\quad T[a_1, \dots, a_k] \geq T[b_1, \dots, b_k]$

$$\sum_{sym} b^2 c^2 \geq \sum_{sym} abc^2$$

$$(2, 2, 0) \gg (2, 1, 1)$$

$(a_1, \dots a_j, a_{j+1}, \dots, a_k)$ vs $(a_1, \dots, a_j - \rho, a_{j+1}, \dots, a_s + \rho, a_{s+1}, \dots, a_k)$

$\gg$

$x_1, \dots, x_k \geq 0$ allora $\quad \frac{1}{k} \sum_{j=1}^{k} x_j \geq \left(\prod_{j=1}^{k} x_j \right)^{1/k}$

AM - GM

Hint: $\left.\begin{array}{l} \log x \text{ è concavo per } x \geq 0 \\ \frac{d}{dx} \lg x = \frac{1}{x} \quad \frac{d^2}{dx^2} \lg x = -\frac{1}{x^2} \leq 0 \end{array}\right\}$ applico Jensen e ottengo:

$$\log\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right) \geq \frac{1}{k}\sum_{j=1}^{k} \log(x_j)$$

$$\frac{1}{k}\sum_{j=1}^{k} x_j \geq e^{\frac{1}{k}\sum_{j=1}^{k}\log(x_j)} = \left(\prod_{j=1}^{k} x_j\right)^{1/k}$$

Se $m_1 > m_2$

$$\left(\frac{1}{k}\sum_{j=1}^{k} x_j^{m_1}\right)^{1/m_1} \geq \left(\frac{1}{k}\sum_{j=1}^{k} x_j^{m_2}\right)^{1/m_2}$$

$$y_j = x_j^{1/m_2}$$

$$\forall t > 1 \quad \left(\frac{1}{k}\sum_{j=1}^{k} y_j^t\right)^{1/t} \geq \frac{1}{k}\sum_{j=1}^{k} y_j$$

per omogeneità, non è restrittivo supporre

$$\sum_{j=1}^{k} x_j = k$$

$$z_i = y_i - 1$$

$$\sum_{j=1}^{k} (1+z_j)^t \geq k$$

$\vee$/ Bernoulli

$$\sum_{j=1}^{k} (1 + t z_j)$$

$$\lim_{t \to 0} \left(\sum_{j=1}^{k} a_j^t\right)^{1/t} = \left(\prod_{j=1}^{k} a_j\right)^{1/k}.$$

$(x_1, \dots, x_n)$ n-uple di num. reali ≥ 0

$$d_k = \binom{n}{k}^{-1} [t^{n-k}] \prod_{j=1}^{n} (t + x_j)$$

$$d_2 = \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{3}$$

Valgono (1) $d_{k-1}\, d_{k+1} \leq d_k^2$ Disug. Newton-McLaurin

2) $d_k^{1/k} \geq d_{k+1}^{1/(k+1)}$

$$p(x,y) = \prod_{j=1}^{n} (x + y \cdot x_j)$$

i valori di $\frac{x}{y}$ per cui p si annulla sono tutti reali positivi

$$\Delta\left(\frac{\partial^{n-2}}{(\partial x)^{n-2}} \tilde{p}(x)\right) \geq 0.$$

$d_0 = 1$ $(d_0 d_2)(d_1 d_3)^2 \cdot \ldots \cdot (d_{k-1} d_{k+1})^k$

$$\leq d_1^2 \cdot d_2^4 \cdot \ldots \cdot d_k^{2k}$$

$$d_{k+1}^k \leq d_k^{k+1} \Rightarrow 2).$$

Disug di Young $p > 1$ $\frac{1}{p} + \frac{1}{q} = 1$

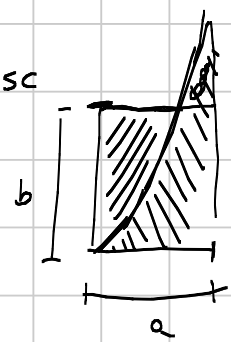
q è detto esponente coniugato di p

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

$f(x) = x^{p-1}$ $g(x) = x^{\frac{1}{p-1}} = x^{q-1}$

f, g sono funz cresc

$$\int_0^{|a|} x^{p-1}\, dx + \int_0^{|b|} x^{q-1}\, dx \geq |ab|$$



Hölder $p > 1$ $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{j=1}^{k} |x_j y_j| \leq \left(\sum_{j=1}^{k} |x_j|^p\right)^{1/p} \cdot \left(\sum_{j=1}^{k} |x_j|^q\right)^{1/q}$$

$$\|x\|_p = \left(\sum_{j=1}^{k} |x_j|^p \right)^{1/p}$$

$$\frac{\sum_k |x_k||y_k|}{\|x\|_p \, \|y\|_q} \leqslant \frac{1}{p} \underbrace{\sum_k \frac{|x_k|^p}{\|x\|_p^p}}_{=1} + \frac{1}{q} \underbrace{\sum_k \frac{|y_k|^q}{\|y\|_q^q}}_{=1} = \frac{1}{p} + \frac{1}{q} = 1$$

Minkowski

$$\left(\sum_{j=1}^{k} |x_j + y_j|^p \right)^{1/p} \leqslant \left(\sum_{j=1}^{k} |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^{k} |y_j|^p \right)^{1/p}$$

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\|x+y\|_p^p \leq \sum_{j=1}^{k} |x_j| \cdot |x_j + y_j|^{p-1} + \sum_{j=1}^{k} |y_j| \cdot |x_j + y_j|^{p-1}$$

$$\overset{\text{Hölder}}{\leqslant} \left(\|x\|_p + \|y\|_p \right) \left(\|x+y\|_p^{p-1} \right)$$

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1) $(a, b, c) \geq 0 \implies \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geqslant \frac{3}{2}$ (Nesbitt)

2) $\prod_{j=1}^{n} \left(|x_j| + |y_j| \right)^{1/n} \geqslant \prod_{j=1}^{n} |x_j|^{1/n} + \prod_{j=1}^{n} |y_j|^{1/n}$ (Mahler)

3) $(a, b, c) \geq 0 \quad a+b+c=3$

Allora $\sum_{cyc} \frac{1}{a^2} \geqslant \sum_{cyc} a^2$

4) $x_0 > 0$
$x_{k+1} = x_k^2 + x_k$ $\quad \forall n \in \mathbb{N} \quad \underline{\sum_{j=1}^{n} \frac{1}{x_j + 1}} \leq \frac{1}{x_1}.$]

1) $A = b+c$ $\quad a = \frac{1}{2}(B+C-A)$

$B = a+c$

$C = a+b$

$$\sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{1}{2}\,\frac{B+C-A}{A}$$

$x \geq 0$

$x + \frac{1}{x} \geq 2$

$$= -\frac{3}{2} + \underbrace{\sum_{cyc} \frac{B+C}{2A}}_{\geq 3} \longleftarrow \underline{\frac{A}{B} + \frac{B}{A} \geq 2}$$

$$\geq \frac{3}{2}.$$

2) AM-GM

$$\left.\begin{aligned} \prod_{k=1}^{n}\left(\frac{|x_k|}{|x_k|+|y_k|}\right)^{1/n} &\leq \frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{|x_k|+|y_k|} \\ \prod_{k=1}^{n}\left(\frac{|y_k|}{|x_k|+|y_k|}\right)^{1/n} &\leq \frac{1}{n}\sum_{k=1}^{n}\frac{|y_k|}{|x_k|+|y_k|}\end{aligned}\right\} \frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|+|y_k|}{|x_k|+|y_k|} = 1$$

4) $x_0 > 0 \quad x_k > 0 \quad \forall k$

$$x_{k+1} = x_k^2 + x_k \qquad \frac{1}{x_{k+1}} = \frac{1}{x_k^2+x_k} = \overbrace{\frac{1}{x_k(x_k+1)} = \frac{1}{x_k} - \frac{1}{x_k+1}}^{\text{Dec. in fratti semplici}}$$

Dec. di Hurwitz

$$\frac{1}{x_k+1} = \frac{1}{x_k} - \frac{1}{x_{k+1}}$$

$$\sum_{k=1}^{n}\frac{1}{x_k+1} = \sum_{k=1}^{n}\frac{1}{x_k} - \sum_{k=1}^{n}\frac{1}{x_{k+1}}$$

$x_{n+1} \geq 0$

$$\sum_{k=1}^{n}\frac{1}{x_k} - \sum_{k=2}^{n+1}\frac{1}{x_k} = \frac{1}{x_1} - \frac{1}{x_{n+1}} \leq \frac{1}{x_1}.$$

3) $\underline{a+b+c=3 \quad a,b,c \geq 0}$

$$\sum_{cyc}\frac{1}{a^2} \geq \sum_{cyc} a^2$$

$$f(x) = \frac{1}{x^2} - x^2$$

$$0 = f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a)+f(b)+f(c)}{3}$$

$f''(x) = \frac{6}{x^4} - 2$ $\quad f$ è convessa in $(0, 3^{1/4})$

$$\sum_{cyc}\left(\frac{1}{a^2} - a^2\right) = \sum_{cyc} \frac{(1-a)(1+a)(1+a^2)}{a^2} \geq 0$$

$$\sum_{cyc} \frac{(1+a)(1+a^2)}{a^2} \geq \sum_{cyc} \frac{(1+a)(1+a^2)}{a}$$

$$\sum_{cyc} (1+a)(1+a^2)b^2c^2 \geq \sum_{cyc} (1+a)(1+a^2)ab^2c^2$$

$$(a+b+c)\sum_{cyc} \underbrace{(1+a)(1+a^2)}b^2c^2 \geq 3\sum_{cyc} \underbrace{(1+a)(1+a^2)}ab^2c^2$$

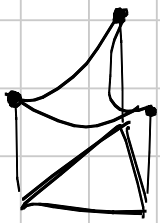
$$(-)\ b^3c^2 + b^2c^3 \qquad\qquad (-)\ 2\,ab^2c^2$$

$$(3\,2\,0) \gg (2\,2\,1)$$

per bunching, fine.

Remark 1. Studiare i casi in cui vale $=$ per tutte le disug. esposte finora $\}$

Remark 2. f convessa su un chiuso assume massimo al bordo



Remark 3. Se i punti critici di una disug sono molteplici e giacciono nella parte interna del dominio di definizione, le tecniche qui mostrate, da sole, sono insufficienti.