G1 medium, Senior 2011

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Titolo nota

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$$\cos(a+b) = \cos a \cdot \cos b - \sin a \sin b$$
$$\cos(2a) = 2\cos^2 a - 1$$
$$\cos(3a) = \cos(a+2a) = \cos a \cos 2a - \sin a \sin 2a$$
$$= \cos a\,(2\cos^2 a - 1) - \sin a\,(2\sin a \cos a)$$
$$= 2\cos^3 a - \cos a - 2\sin^2 a \cos a \qquad (\text{Pit } \sin^2 a = 1-\cos^2 a)$$
$$= 2\cos^3 a - \cos a - 2(1-\cos^2 a)\cos a$$
$$= 4\cos^3 a - 3\cos a.$$

Claim: $\forall n \in \mathbb{N}$ $\cos(nx)$ è un poly di grado n in $\cos x$.

Def. $T_n(x) = \cos(n \arccos x)$

$$T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x) \qquad \text{(rel. ricorr.)}$$

$x = \cos y$

$$\cos((n+2)y) = 2\cos y \cos((n+1)y) - \cos(ny)$$

$$\cos((n+2)y) + \cos(ny) = 2\cos y \cos((n+1)y)$$

Briggs $\quad \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_2(x) = 2x^2 - 1 \end{cases}$$

Poly di Chebyshev del 1° tipo.

Claim. $T_n(x) = \dfrac{(x-\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^n}{2} = \tilde{T}_n(x)$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right)$$

$\tilde{T}_0(x) = 1 \quad \tilde{T}_1(x) = x$

Per induzione su n $\quad \tilde{T}_{n+2}(x) = 2x\,\tilde{T}_{n+1}(x) - \tilde{T}_n(x).$

$$(x-\sqrt{x^2-1})^n = \sum_{j=0}^{n} \binom{n}{j} x^j (-1)^{n-j} (x^2-1)^{\frac{n-j}{2}}$$

$$= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \underline{\underline{(-1)^j}} (x^2-1)^{j/2} \qquad \|$$

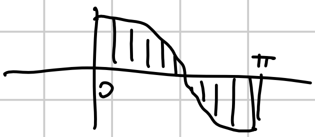
$$(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n = 2\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} x^{n-2j} (x^2-1)^j \qquad |$$

Quali sono le radici del polinomio $T_n(x)$?

Per quali valori di x si ha

$$\cos(n \arccos x) = 0 \quad ?$$

$$n \arccos x \in \left\{(2k+1)\frac{\pi}{2}\right\} \qquad \arccos x \in \left\{(2k+1)\frac{\pi}{2n}\right\}$$



$$x \in \left\{\cos\left((2k+1)\frac{\pi}{2n}\right)\right\} =: E_n$$

$$|E_n| = n.$$

Tutte le radici di $T_n(x)$ sono reali.

Def. $$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$$

$$U_{n+2}(x) = 2x\, U_{n+1}(x) - U_n(x) \qquad \begin{cases} U_0(x) = 1 \\ U_1(x) = 2x \end{cases}$$

$$U_2(x) = 4x^2 - 1$$

Le radici di $U_n(x)$ sono

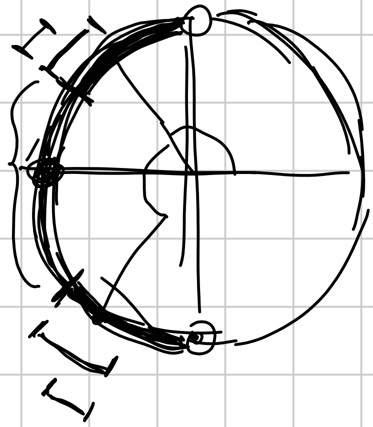
$$\cos\left(\frac{\pi k}{n+1}\right) \quad \text{per } k = 1, \dots, n$$

T_n, U_n soddisfano rel. ric.

$$\left.\begin{array}{l} \text{conosciamo le radici} \\ \text{conosciamo i coefficienti} \end{array}\right\} \xrightarrow{\text{Viète}} \underline{\sum_{k=1}^{n} \cos^3\left(\frac{\pi k}{n+1}\right)} \; \_\_$$

(1) Determinare tutti i $\theta \in [0, 2\pi)$ tali per cui

$\forall n \in \mathbb{N}$ si ha $\cos(2^n \theta) < 0$. $(*)$



$n=0$ $\cos\theta < 0$; $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ NO

$n=1$ $\cos 2\theta < 0$

$n=2$ $\cos 4\theta < 0$

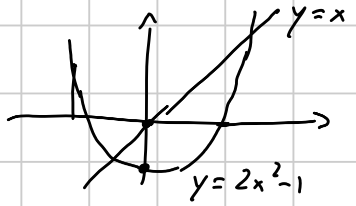
Claim: gli unici θ per cui vale $(*)$ sono $\theta = \frac{2\pi}{3}$, $\theta = \frac{4\pi}{3}$.

$$2\pi\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \dots\right) \qquad 4\pi\left(\frac{1}{2} - \frac{1}{4} + \dots\right)$$

$\theta \in (\pi/2, 3\pi/2)$ $n=0$ si verifica $\cos\theta < 0$, $\cos\theta = t$

$f : t \to 2t^2 - 1$ Quali punti di $[-1, 0)$ sono tali per

$\forall n \geq 1$ $f^{(n)}(t) < 0$



Disuguaglianze geometriche

Claim A, B, C angoli di un triangolo $A, B, C \geq 0$, $A+B+C = \pi$ allora

$$\boxed{\prod_{cyc} \sin\frac{A}{2} \leq \frac{1}{8}} . \qquad \text{Jensen (convessità)}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \qquad \log(\sin t) \text{ concava per } t \leq \frac{\pi}{2}$$

Briggs

$$\boxed{\sum_{cyc} \cos A \leq \frac{3}{2}} \xrightarrow[\text{coseno}]{\text{Carnot}} \sum_{cyc} \frac{b^2+c^2-a^2}{2bc} \leq \frac{3}{2}$$

$$2\sum_{sym} ab^2 \leq \sum_{sym}(a^3+abc)$$ Disuguaglianza di Schur.

$p(a,b,c)$

$$a(a-b)(a-c)+b(b-a)(b-c)+c(c-b)(c-a)\geq 0$$

WLOG $a \geq b \geq c$

$$\underbrace{(a-b)}_{\geq 0}\underbrace{\big(a(a-c)-b(b-c)\big)}_{\geq 0}+\underbrace{c(b-c)(a-c)}_{\geq 0} \overset{?}{\geq} 0$$

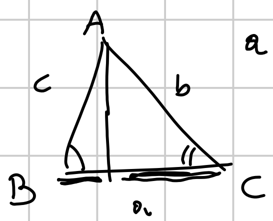
Schur - Vornicu

$a,b,c,x,y,z \in \mathbb{R}$ $\quad a \geq b \geq c$ e $x \geq y \geq z$ o $x \leq y \leq z$

$k>0$ $\quad f(z)$ funzione monotona o convessa

$$f(x)(a-b)^k(a-c)^k+f(y)(b-a)^k(b-c)^k+f(z)(c-a)^k(c-b)^k \geq 0.$$

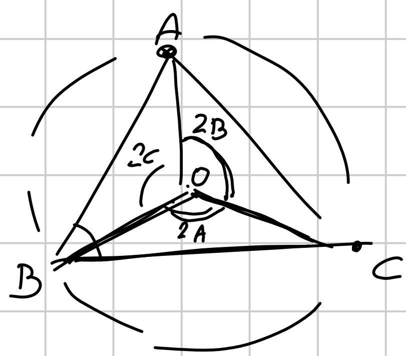
(+ potente delle disug di "bunching").



$a = c\cos B + b\cos C$

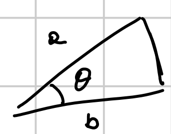
$$\sum_{cyc} a\cos A \leq \frac{a+b+c}{2} = p$$

$$R\sum_{cyc} \sin(2A) \leq \frac{\Delta}{r}$$



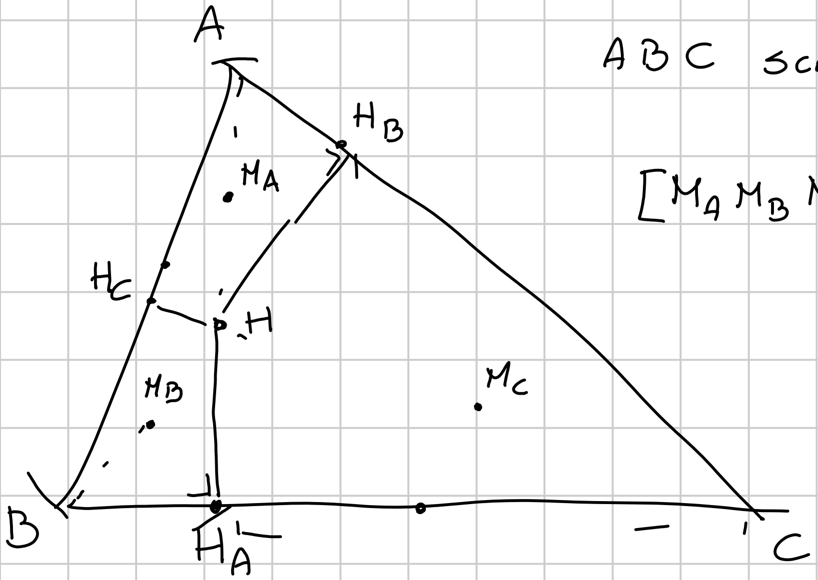
$$\sum_{cyc} R^2 \sin(2A) \leq \frac{\Delta R}{r}$$

$$2\Delta \leq \frac{\Delta R}{r}$$



$\frac{1}{2}ab\sin\theta$

(Eulero). $\quad R \geq 2r$ $\quad$ [OBC] $\frac{1}{2}R^2\sin(2A)$



ABC scaleno e acutangolo

$$[M_A M_B M_C] \geq [H_A H_B H_C].$$

$$\Updownarrow$$

$$H \not\equiv O.$$

CH_AH_B è simile al triangolo CAB $\Rightarrow$ $H_AH_B = c \cdot \cos C$

$$R(H_AH_BH_C) = \frac{1}{2} R(ABC).$$

$$\prod_{cyc} \cos A \leq \frac{1}{8}$$

$$\sum_{cyc} \sin^2 A \leq \frac{9}{4}$$

$$9R^2 \geq (a^2 + b^2 + c^2)$$

O, H, G sono allineati per il T. di Eulero

$\uparrow$ $H = A + B + C$ in un sistema centrato nel circocentro O

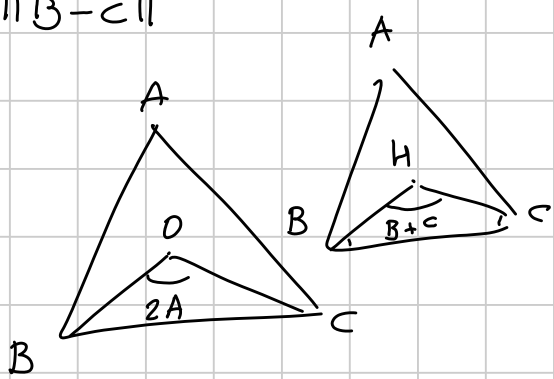
$$\|H\|^2 = \|A+B+C\|^2 = \sum_{cyc} \|A\|^2 + \sum_{cyc} 2(A \circ B) \quad \text{(teo. coseno)}$$

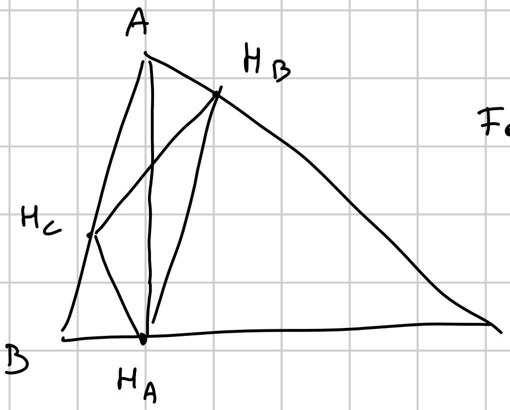
$$= 3R^2 - \sum_{cyc} \left(\|B-C\|^2 - \|B\|^2 - \|C\|^2\right)$$

$$= 9R^2 - \sum_{cyc} \|B-C\|^2$$

- $OH^2 = 9R^2 - (a^2+b^2+c^2).$

$$\begin{cases} 2A = B+C \\ \text{e cyc} \end{cases} \Rightarrow A = B = C = \frac{\pi}{3}.$$





Fatto noto: $\widehat{H_C H_A A} = \widehat{A H_A H_B}$

A, B, C sono excentri di $H_A H_B H_C$

$\begin{cases} \text{t. partenza} \\ \text{t. ortico} * \end{cases} \longrightarrow \begin{cases} \text{t. partenza} * \\ \text{t. degli excentri} \end{cases}$

$$[I_A I_B I_C] = \frac{ab \cos(C/2)}{2 \sin(A/2) \sin(B/2)} = 8R^2 \prod_{cyc} \cos\frac{A}{2}$$

$$= 2R^2 \sum_{cyc} \sin A \quad \text{(Briggs)}$$

$$= 2pR$$

$R \geq 2r$ (Eulero).

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a, b, c lati di un triangolo.

$a^2 + b^2 + c^2 = 8R^2 \longrightarrow$ ABC è rettangolo.

α, β sono angoli acuti $\in (0, \frac{\pi}{2})$

Se $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta)$

allora $\sin^2\alpha + \sin^2\beta = 1$.

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r_a il raggio della circ. ex-inscritta ad ABC relativa al vertice A

allora $4R + r = \sum_{cyc} r_a$.

A_n poligono regolare di n lati di lunghezza unitaria.

Quanto vale il prodotto delle lunghezze di tutte le diagonali di A_n?

$|OH|^2 = 9R^2 - (a^2+b^2+c^2)$ Se $a^2+b^2+c^2 = 8R^2$

$|OH|^2 = R^2$

$H \in \Gamma(ABC)$

I simmetrici di H rispetto ai lati appartengono a $\Gamma(ABC)$

$H \in$ lati   almeno 2 lati sono anche altezze

ABC è rettangolo.

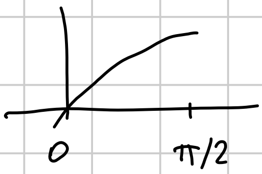
$$a^2+b^2+c^2 = 8R^2$$

$$\sum_{cyc} \sin^2 A = 2 \qquad \sum_{cyc} \cos 2A = -1 \xrightarrow{\text{Briggs}} \prod_{cyc} \cos A = 0$$

almeno 1 tra A, B e C è pari a $\pi/2$.

$$\underline{\sin^2\alpha + \sin^2\beta} = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$= \sin\alpha \sin\left(\frac{\pi}{2}-\beta\right) + \sin\beta \sin\left(\frac{\pi}{2}-\alpha\right)$$



$\alpha+\beta = \pi/2$

$$\alpha+\beta > \frac{\pi}{2} \quad \begin{cases} \alpha > \frac{\pi}{2}-\beta \\ \beta > \frac{\pi}{2}-\alpha \end{cases}$$

$$\sin\alpha > \sin\left(\frac{\pi}{2}-\beta\right)$$
$$\sin\beta > \sin\left(\frac{\pi}{2}-\alpha\right)$$

$$\alpha+\beta < \frac{\pi}{2}$$

$$r_a = p \tan\frac{A}{2} = p\,\frac{2bc\sin A}{2bc\,(1+\cos A)} = \frac{4p\Delta}{(b+c-a)(b+c+a)} = \frac{\Delta}{p-a}$$

moltiplichiamo LHS e RHS per Δ ed applichiamo Erone

Erone

$$abc + (p-a)(p-b)(p-c) = \sum_{cyc} p(p-b)(p-c) \qquad \Delta^2 = p(p-a)(p-b)(p-c)$$

$$V(a_1, \dots, a_n) = \begin{pmatrix} 1 & a_1 & \dots & a_1^{n-1} \\ \vdots & & & \\ 1 & a_n & \dots & a_n^{n-1} \end{pmatrix}$$

$$\det V(a_1,\dots,a_n) = \prod_{1\le i<j\le n} (a_j - a_i)$$

$\cdot\ \zeta = e^{2\pi i/n} = e(1/n) \qquad e(\alpha) = e^{2\pi i \alpha}$

$\zeta^k \quad k = 0, 1, \dots, n-1 \qquad \prod_{1 \le i < j \le n} (\zeta^j - \zeta^i)$

$$\det \begin{pmatrix} 1 & \zeta & \zeta^2 & \dots & \zeta^{n-1} \\ 1 & \zeta^2 & \zeta^4 & \dots & \zeta^{2n-2} \\ \vdots & & & & \\ 1 & \dots & & & \end{pmatrix}$$

$\cdot\ \zeta^n = 1$

$\det A = \det A^T$

$\det(A \cdot A^T)$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$$A \cdot A^T = \begin{pmatrix} n & & & \\ & n & & 0 \\ 0 & & n & \\ & & & \ddots\ n \end{pmatrix}$$

$\det(A \cdot A^T) = n^n$

$\det A = n^{n/2}$

$\forall z \in \mathbb{C} \setminus \{0\}$

$$\frac{\sin z}{z} = \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

$$\cos z = \prod_{m=0}^{+\infty} \left(1 - \frac{4z^2}{\pi^2 (2m+1)^2}\right)$$

Prodotti di Weierstress

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\mathbb{Z}/p\mathbb{Z}^* \overset{\mathbb{F}_p}{=}$ è un gruppo $\qquad$ i quadrati vengono detti residui quadratici

$\quad \|$

$\langle g \rangle \qquad |\mathbb{Z}/p\mathbb{Z}^*| = p-1 \qquad \left|\text{residui quadratici } \underset{\sim}{} \mathbb{F}_p\right| = \frac{p-1}{2}$

a è un residuo quadratico sse $a^{\frac{p-1}{2}} \equiv 1 \pmod p$

$$\begin{matrix} +1 \\ -1 \end{matrix} > \left(\frac{a}{p}\right) \text{ simbolo di Legendre}$$

$$G(a,p) = \sum_{x \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{x}{p}\right) e(ax/p) \qquad e(\alpha) = e^{2\pi i \alpha}$$

$$\left(\frac{0}{p}\right) = 0$$

$$\left(\frac{a}{p}\right) G(a,p) = \sum_{x \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{ax}{p}\right) e(\underline{a}x/p) \qquad \left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \checkmark$$

$$= \sum_{x \in \mathbb{Z}/p\mathbb{Z}} \left(\frac{x}{p}\right) e(x/p)$$

$$= G(1,p)$$

$$\zeta = e^{2\pi i/p}$$

$$\sum_{a=0}^{p-1} G(a,p)G(-a,p) = \sum_{\underline{a=0}}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \left(\frac{j}{p}\right)\left(\frac{k}{p}\right) \zeta^{a(k-j)}$$

$$= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \left(\frac{j}{p}\right)\left(\frac{k}{p}\right) p \; \delta(j,k)$$

$$= \sum_{n=0}^{p-1} \left(\frac{n}{p}\right)^2 p \quad = \quad p(p-1).$$

$$= (p-1)\left(\frac{-1}{p}\right) G(1,p)^2$$

$$G(a,p) = \left(\frac{a}{p}\right) G(1,p)$$

$$\begin{cases} G(1,p) = \left(\frac{-1}{p}\right) \sqrt{p} \\ G(a,p) = \left(\frac{-a}{p}\right) \sqrt{p} \end{cases}$$

$$\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{8\pi}{7} = \frac{1}{2} \sqrt{7}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$q \in \mathbb{R}$ Si dice algebrico in $\mathbb{Q}$

se $\exists\, p(x) \in \mathbb{Q}[x]$

per cui $p(q) = 0$

Se $q \in \mathbb{Q}$

$\cos(2\pi \cdot q)$

Qual è il minimo grado di un polinomio $p \in \mathbb{Q}[x]$ tale per cui $p(q) = 0$.

$$\begin{cases} p(q) = 0 \\ q(q) = 0 \end{cases} \qquad \partial p > \partial q \qquad \exists r : \ r(q) = 0, \ \partial r < \partial p$$

$\cos \frac{2\pi}{7}$ $\qquad T_7(x) = x \cdot Q(x)$

$\partial Q = 6$

$$\cos \frac{2\pi}{7} = \frac{(e^{2\pi i/7} + e^{-2\pi i/7})}{2}$$

$$\left(\frac{x^7 - 1}{x - 1}\right) = \prod_{j=1}^{6} \left(x - e^{\frac{2\pi i}{7} j}\right)$$

$\parallel$

$x^6 + x^5 + \dots + 1$

$\parallel$

$$x^3 \left(\left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)\right)$$

Claim

$x^k + \frac{1}{x^k}$ è un polinomio di grado k in $\left(x + \frac{1}{x}\right)$

$$\left(x^3 + \frac{1}{x^3}\right) - \left(x + \frac{1}{x}\right)^3$$

$$\frac{x^7 - 1}{x - 1} = x^3 \cdot \underline{Q\left(x + \frac{1}{x}\right)} \qquad \partial Q = 3$$

$A, B, C > 0 \qquad f(A,B,C) = C(A+B+C) + \left(\frac{A-B}{2}\right)^2$

$$\sqrt{f(A,B,C)} + \sqrt{f(B,C,A)} + \sqrt{f(C,A,B)} \geq 2 \max\left(\sqrt{f(A,B,C)}, \sqrt{f(B,C,A)}, \sqrt{f(C,A,B)}\right).$$

Provere che $\forall \varepsilon > 0 \quad \exists A_\varepsilon \subseteq \mathbb{N}$ per cui

$$\left| 1 - \sum_{a \in A_\varepsilon} \frac{\sin(a^2)}{a} \right| \leq \varepsilon.$$

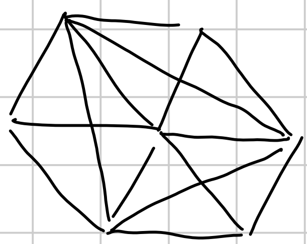
Scherzone!!!

$$\sum \frac{\sin n^2}{n}$$

$a = B + C \qquad b = A + C \qquad c = A + B$

Stewart $\quad m_a^2 = \frac{1}{2} b^2 + \frac{1}{2} c^2 - \frac{1}{4} a^2$

Le mediane di un t. formano un triangolo:



$\square$.

$\{a_n\}_{n \in \mathbb{N}} \qquad \left(\sum_{n \in \mathbb{N}} a_n \right)$ convergente se $\exists \lim_{N \to +\infty} \left(\sum_{n=1}^{N} a_n \right)$

assolutamente convergente $\exists \lim_{N \to +\infty} \left(\sum_{n=1}^{N} |a_n| \right)$

Teoremuccio. Se una serie è convergente ma non assolutamente convergente se ne può riarrangiare l'ordine di somma in modo che la nuova serie converga ad una qualunque costante.

$$\sum_{n=1}^{N} a_n b_n = A_N b_N - \sum_{n=1}^{N-1} A_n (b_{n+1} - b_n)$$

Sommazione per parti

$$A_n \doteq \sum_{j=1}^{n} a_j$$

$$a_n = e^{in^2} \qquad b_n = \frac{1}{n}$$

$$(|A_N| = o(n).).$$

$$b_{n+1} - b_n \sim \frac{1}{n^2}.$$

Disuguaglianza di Weyl

$$\left| \sum_{n=M+1}^{M+N} \chi(n) \right| \ll \sqrt{N} \cdot \log N$$