ALGEBRA 3 - BASIC

Titolo nota 07/09/2012

→ 1) Somme tipo $\underbrace{\sum_{k=1}^{n} p(k)\, a^k}$ con k polinomio

2) Succ. definite per ricorrenza lineari.

3) espressioni funzionali

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\boxed{(n+1)^3 - n^3 = 3n^2 + 3n + 1}$$

$$(n+1)^3 - \cancel{n^3} = 3n^2 + 3n + 1$$

$$\cancel{n^3} - (n-\cancel{1})^3 = 3(n-1)^2 + 3(n-1) + 1$$

$$(n-\cancel{1})^3 - (n-\cancel{2})^3 = 3(n-2)^2 + 3(n-2) + 1$$

$$\vdots$$

$$\cancel{2^3} - 1^3 = 3\cdot 1^2 + 3\cdot 1 + 1$$

$$S_n = \sum_{k=1}^{n} k^2$$

$$\boxed{(n+1)^3 - 1 = 3 S_n + 3\frac{n(n+1)}{2} + n}$$

$$S_n = \frac{1}{3}\Big(n^3 + 3n^2 + \underbrace{3n - n}_{2n} - \frac{3}{2} n(n+1)\Big) =$$

$$= \frac{2n^3 + 6n^2 + \cancel{6}4n - \cancel{3}3n(n+1)}{6} =$$

$$2n^2 + 6n + 4$$
$$2(n^2 + 3n + 2)$$
$$2(n+2)(n+1)$$

$$= \frac{n\big(2n^2 + 6n + \cancel{2}4 - 3(n+1)\big)}{6} =$$

$$= \frac{n\big(2(n+2)(n+1) - 3(n+1)\big)}{6} =$$

$$= \frac{n(n+1)(2n+4-3)}{6} =$$

$$= \frac{n(n+1)(2n+1)}{6} =$$

$$= \frac{n(n+1)\left(n+\frac{1}{2}\right)}{3}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$S_n = \sum_{1}^{n} k^3$$

$$(n+1)^4 - \cancel{n^4} = 4n^3 + 6n^2 + 4n + 1$$

$$\cancel{n^4} - \cancel{(n-1)^4} = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$

$$\vdots$$

$$\vdots$$

$$\cancel{2^4} - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$$

$$(n+1)^4 - 1 = 4 S_n + 6 \cdot \frac{n(n+1)(n+\frac{1}{2})}{3} + 4 \frac{n(n+1)}{2} + n$$

$$\vdots$$

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(n+\frac{1}{2})(n^2+n-\frac{1}{3})}{5}$$

$$\sum_{k=1}^{n} k^5 = \ldots\ldots$$

$$\sum_{k=1}^{n} k^p = \quad n, p$$

$$\sum_{k=1}^{n} P(k) \qquad \sum k^p$$

$$\underbrace{\sum_{k=1}^{n} (3k^2 + 5k + 11)} = 3 \sum_{k=1}^{n} k^2 + 5 \sum_{k=1}^{n} k + 11 \sum_{k=1}^{n} 1$$

$$\sum_{k=0}^{50} (k+4)(k+3)(k+2)(k+1) =$$

$$= \sum_{k=0}^{50} 4! \boxed{\frac{(k+4)(k+3)(k+2)(k+1)}{4!}} =$$

$$= 4! \sum_{k=0}^{50} \binom{k+4}{4} = 24 \binom{55}{5}$$

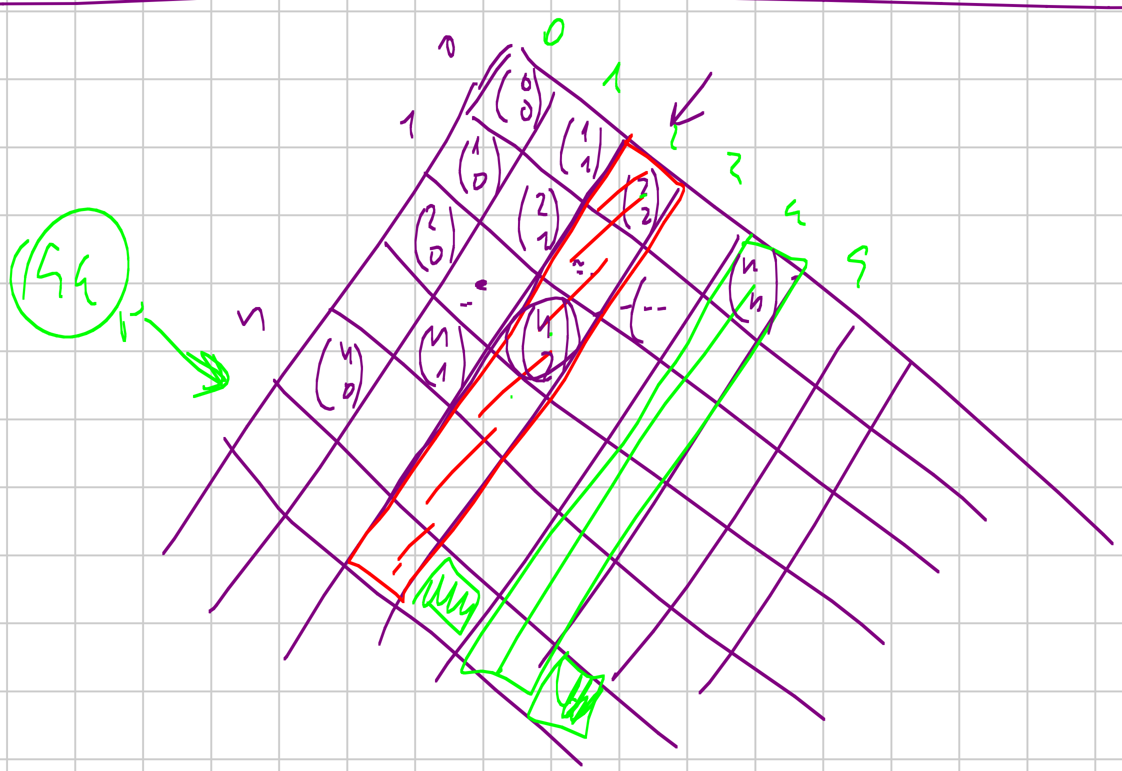
$$= \dots$$

$$\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \dots + \binom{54}{4} =$$

$$= \binom{55}{5}$$

$$= \sum_{k=1}^{n} k^p = \bigcirc$$

$$\sum_{k=0}^{n} \binom{k+p}{p} = \boxed{\binom{n+p+1}{p+1}}$$



$$\sum_{k=0}^{400} k^2 + 7k + 5 =$$

$$\binom{k+2}{2} \leftarrow$$

$$\binom{k+1}{1} \leftarrow$$

$$\binom{k}{0} \leftarrow$$

$$k^2+7k+5 = \underbrace{k^2+3k+2}_{(k+2)(k+1)} + 4k+3 =$$

$$= \frac{2(k+2)(k+1)}{2} + 4\frac{(k+1)}{1} - 1 =$$

$$= \underbrace{2\binom{k+2}{2} + 4\binom{k+1}{1} - \binom{k}{0}}$$

$$\sum_{k=0}^{100} k^2+7k+5 = \sum_{k=0}^{100} 2\binom{k+2}{2} + 4\binom{k+1}{1} - \binom{k}{0} =$$

$$= 2\binom{103}{3} + 4\binom{102}{2} - \binom{101}{1} = \dots$$

$$\sum_{k=0}^{100} \underbrace{(k+1)^2(k+3)^3}$$

$$\boxed{6} \qquad \underbrace{\alpha_1\binom{k+5}{5} + \alpha_2\binom{k+4}{4} + \dots + \alpha_6\binom{k}{0}}$$

$$\sum_{k=0}^{n} p(k)\, a^k \qquad \sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1} \qquad a \neq 1$$

$$\boxed{\sum_{k=1}^{100} k\,2^k} = 1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 2^3 + 4\cdot 2^4 + \dots + 100\cdot 2^{100}$$

$$S_n = \sum_{k=1}^{n} k 2^k$$

$$\boxed{S_n - S_{n-1} = n 2^n}$$

$$S_1 = 2$$

$$\boxed{\begin{cases} S_n - S_{n-1} = n 2^n \\ S_1 = 2 \end{cases}}$$

$$S_n = \quad n$$

$$\sum_{k=1}^{n} k^5 2^k = S_n$$

$$\begin{cases} S_n - S_{n-1} = \boxed{n^5 2^5} \\ S_1 = 2 \end{cases}$$

$$2^{n^2}$$

EQ. DIFF. FINITE LINEARI

(OMOGENEE)

non omogenee

$$F_n - \alpha_1 F_{n-1} - \alpha_2 F_{n-2} - \dots - \alpha_k F_{n-k} = 0 + \cancel{R(n)}$$

$$\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$$

k ordine

Es. Fibonacci $F_n = F_{n-1} + F_{n-2}$

$$F_{n+2} = F_{n+1} + F_n$$

Polinomio caratteristico

$$\lambda^k - \alpha_1 \lambda^{k-1} - \alpha_2 \lambda^{k-2} - \dots - \alpha_{k-1} \lambda - \alpha_k = 0$$

Teorema

Data $F_{n+2} + a F_{n+1} + b F_n = 0$

tale che le radici del pol. caratteristico

$\lambda^2 + a\lambda + b$ siano λ_1 e λ_2 reali e distinte, allora tutte e sole le sue soluzioni sono quelle della forma

$$\longrightarrow F_n = \alpha \lambda_1^n + \beta \lambda_2^n \quad \text{per ogni } \alpha, \beta \in \mathbb{R}$$

Esempio

$$\longrightarrow F_{n+2} = 5 F_{n+1} - 6 F_n$$

$$\lambda^2 - 5\lambda + 6 =$$

$\cancel{F_1 = 2.}$

$\cancel{F_2 = 4.}$

$\lambda = 2$

$\lambda = 3$

$$\boxed{F_n = 2^n} \qquad \boxed{F = 3^n}$$

$$2^{n+2} \overset{?}{=} 5 \cdot 2^{n+1} - 6 \cdot 2^n$$

$$4 \cdot 2^n \overset{?}{=} \underbrace{10 \cdot 2^n - 6 \cdot 2^n}_{4 \cdot 2^n}$$

$$\underbrace{3^{n+2}}_{9\cdot 3^n} \overset{?}{=} \underbrace{\underbrace{5\cdot 3^{n+1} - 6\cdot 3^n}_{15\cdot 3^n - 6\cdot 3^n}}$$

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_1 = 1 \\ F_0 = 0 \end{cases} \qquad \longleftarrow \qquad F_{n+2} - F_{n+1} - F_n = 0$$

$$\underbrace{\lambda^2 - \lambda - 1 = 0}$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2} \quad \begin{matrix} \nearrow \dfrac{1+\sqrt{5}}{2} \\ \searrow \dfrac{1-\sqrt{5}}{2} \end{matrix}$$

$$\boxed{F_n = \underset{\uparrow\, \frac{1}{\sqrt{5}}}{\alpha}\left(\frac{1+\sqrt{5}}{2}\right)^n + \underset{\uparrow\, -\frac{1}{\sqrt{5}}}{\beta}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

$$F_1 = 1 \quad \longleftrightarrow$$

$$F_0 = 0 \quad \longleftrightarrow \quad \begin{cases} \alpha + \beta = 0 \\ \underbrace{\alpha \dfrac{1+\sqrt{5}}{\cancel{2}} + \beta \dfrac{1-\sqrt{5}}{\cancel{2}}}_{} = \cancel{1}^{\,2} \end{cases}$$

$$\alpha + \alpha\sqrt{5} + \beta - \beta\sqrt{5} = 2$$

$$\begin{cases} \alpha + \beta = 0 \iff \alpha = -\beta \\ \alpha\sqrt{5} - \beta\sqrt{5} = 2 \end{cases}$$

$$\cancel{2}\alpha\sqrt{5} = \cancel{2}$$

$$\alpha = \frac{1}{\sqrt{5}}$$

$$\beta = -\frac{1}{\sqrt{5}}$$

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

I^a parte: tutte le succ. del tipo

$\alpha \cdot \lambda_1^n + \beta \cdot \lambda_2^n$ vanno bene

1) [se A_n e B_n sono succ. che vanno bene anche $\alpha A_n + \beta B_n$ va bene.

$\longrightarrow$ $A_{n+2} + a A_{n+1} + b A_n = 0$

$\longrightarrow$ $B_{n+2} + a B_{n+1} + b B_n = 0$

$\forall \alpha, \beta \in \mathbb{R}$

$$\underbrace{(\alpha A_{n+2} + \beta B_{n+2})} + a\underbrace{(\alpha A_{n+1} + \beta B_{n+1})} + b\underbrace{(\alpha A_n + \beta B_n)} = 0$$

2) Se λ_1 è una radice del pol. caratteristico allora λ_1^n è una succ. buona.

$$(1, \lambda_1, \lambda_1^2, \lambda_1^3, \lambda_1^4, \dots)$$

$$? \quad F_{n+2} + a F_{n+1} + b F_n = 0$$

$$? \quad \lambda_1^{n+2} + a\lambda_1^{n+1} + b\lambda_1^n \stackrel{?}{=} 0$$

$$\lambda_1^2 + a\lambda_1 + b = 0$$

(1) + (2) $\Rightarrow$ Vanno bene tutte queste:

$$(*) \quad \boxed{\alpha\lambda_1^n + \beta\lambda_2^n} \leftarrow$$

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(3) (*) sono tutte

$$F_{n+2} = -aF_{n+1} - bF_n$$

$\forall$ numeri F_0 e F_1

$$\begin{cases} \alpha + \beta = F_0 \\ \alpha\lambda_1 + \beta\lambda_2 = F_1 \end{cases}$$

$$\begin{cases} \alpha\lambda_1 + \beta\lambda_1 = \lambda_1 F_0 \\ \alpha\lambda_1 + \beta\lambda_2 = F_1 \end{cases}$$

$$(\lambda_1 - \lambda_2)\beta = \lambda_1 F_0 - F_1$$

$$\boxed{\beta = \frac{\lambda_1 F_0 - F_1}{\lambda_1 - \lambda_2}}$$

$\boxed{\text{OSS.}}$ Se c'era una sola radice reale con molt. 2.

$$(\lambda - \lambda_0)^2$$

$$\boxed{\lambda_0^n} \qquad \boxed{(\alpha + \beta n)\lambda_0^n}$$

$$\boxed{n\lambda_0^n}$$

$$\varsigma$$

$$1 \quad \varsigma \quad \varsigma^2 \quad \varsigma^3 \quad - -$$

$$0 \quad 1\cdot\varsigma \quad 2\cdot\varsigma^2 \quad 3\cdot\varsigma^3 \quad - - -$$

$$\boxed{F_{n+2} + aF_{n+1} + bF_n = 0}$$

$$\boxed{F_{n+2} + aF_{n+1} + bF_n = f(n)}$$

$$\boxed{ax + by = c}$$

$$(x_1, y_1)$$

$$\boxed{ax + by = 0}$$

A_n e B_n risoluzioni non omogenee

$$A_{n+2} + aA_{n+1} + bA_n = \boxed{f(n)}$$

$$B_{n+2} + aB_{n+1} + bB_n = f(n)$$

$$\underbrace{(A_{n+2} - B_{n+2})} + a\underbrace{(A_{n+1} - B_{n+1})} + b\underbrace{(A_n - B_n)} = 0$$

$$\underbrace{F_{n+2} + aF_{n+1} + bF_n} = \underbrace{f(n)}_{\boxed{p(n) \cdot a^n}}$$

$$q(n)\, a^n$$

$$\begin{cases} S_n - S_{n-1} = n2^n \\ S_1 = 2 \end{cases}$$

$$H_n = (an+b)2^n$$

2 $\quad$ -2

$$\boxed{S_n = \sum_{k=1}^{n} k2^k}$$

$$S_n - S_{n-1} = 0$$

$$(an+b)2^n - (a(n-1)+b)2^{n-1} \overset{?}{=} n2^n$$

$$2(an+b) - a(n-1) - b \overset{?}{=} 2n$$

$$2an + 2b - an + a - b = 2n$$

$$an + b + a = 2n$$

$$a = 2 \qquad b = -2$$

$$\boxed{S_n = (2n-2)2^n + C}$$

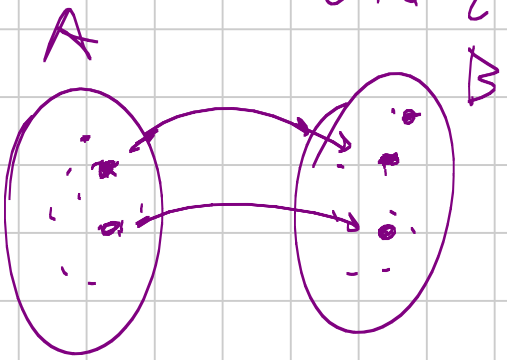
$$S_1 = 2$$

$$S_n - S_{n-1} = 0$$

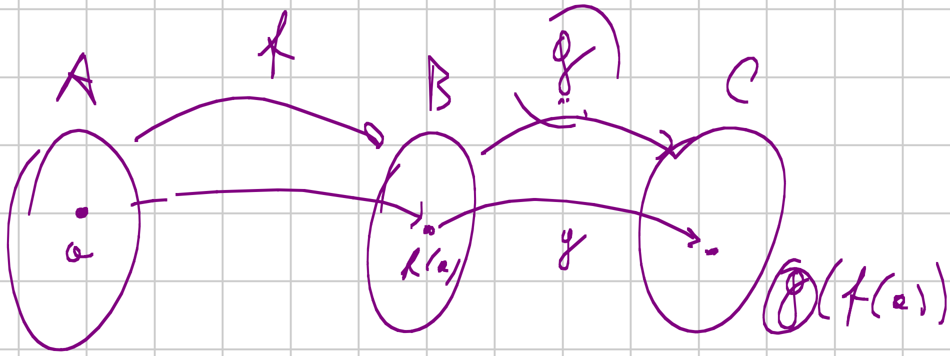
$$\boxed{S_n = (2n-2)2^n + 2}$$

$$S_n = S_{n-1}$$

EQUAZIONI FUNZIONALI



iniettiva
suriettiva
biiettiva



$\cancel{g \circ f} : a \longrightarrow \boxed{g(f(a))}$

$g(f(a))$ è iniettiva $\Rightarrow$ f è iniettiva

$g(f(a))$ è suriettiva $\Rightarrow$ g è suriettiva

CAUCHY

$\boxed{f: \mathbb{Q} \to \mathbb{Q}}$ $\boxed{f(x+y) = f(x) + f(y)}$ $\boxed{\forall x, y \in \mathbb{Q}}$

$\mathbb{R}$

$\boxed{f(x) = ax}$

1) f continua
2) f monotona
3) $\exists\, (a,b)$ e $\exists\, K$ t.c. $|f(x)| \leq K$ $\forall x \in (a,b)$

$a(x+y) = ax + ay$

1) $f(0+0) = f(0) + f(0)$

$f(0) = 2f(0)$

$\Downarrow$

$\boxed{f(0) = 0}$

$\underbrace{f(x+y) = f(x) + f(y)}$

$f((x+y)+z) = f(x+y) + f(z)$

$= f(x) + f(y) + f(z)$

$f((x_1 + x_2 + \dots) + x_{n+1}) =$

$= f(x_1 + x_2 + \dots + x_n) + f(x_{n+1}) =$

$= f(x_1) + f(x_2) + \dots + f(x_n) + f(x_{n+1})$

2) $f(1+1) = f(1) + f(1)$

$f(2) = 2f(1)$

3) $\boxed{f(n) = n f(1)}$?

$f(\underbrace{1+1+\dots+1}_{n}) = \underbrace{f(1) + f(1) + \dots + f(1)}_{n} = n f(1)$

4) $f\left(\frac{1}{n}\right) = \frac{1}{n} f(1)$??

$\boxed{f(1)} = f(\underbrace{\tfrac{1}{n} + \tfrac{1}{n} + \dots + \tfrac{1}{n}}_{n}) = \overbrace{f(\tfrac{1}{n}) + f(\tfrac{1}{n}) + \dots + f(\tfrac{1}{n})}^{n}$

$\boxed{= n f\left(\frac{1}{n}\right)}$

5) $\boxed{f\left(\frac{m}{n}\right) = \frac{m}{n} f(1)} \leftarrow \boxed{\forall x \in \mathbb{Q} \quad f(x) = x f(1)}$

$f\left(\frac{m}{n}\right) = f(\overbrace{\tfrac{1}{n} + \tfrac{1}{n} + \dots + \tfrac{1}{n}}^{m}) =$

$$= \underbrace{f\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{1}{n}\right)}_{m} =$$

$$\stackrel{\text{i}}{=} m\, f\left(\frac{1}{n}\right) = m \cdot \frac{1}{n} f(1) =$$

$$= \frac{m}{n} f(1)$$

$$\boxed{f(1) = a} \qquad \boxed{\forall x \in \mathbb{Q}} \quad f(x) = x \cdot f(1) = \boxed{a x}$$

$$\underline{f(x+y) = f(x) + f(y)}$$

$$\underline{f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n)}$$

$$f((x_1 + x_2 + \cdots + x_n) + x_{n+1}) = f(\underbrace{x_1 + \cdots + x_n}) + f(x_{n+1})$$

$$= f(x_1) + f(x_2) + \cdots + f(x_n) + f(x_{n+1})$$

$f : \mathbb{R} \to \mathbb{R}$

$$f(x + f(y)) = f(x) + f(y) \qquad \underline{\text{iniettiva}}$$

$$f(y + f(x)) = f(y) + f(x)$$

$\forall x, y \in \mathbb{R}$ $\quad f(x + f(y)) = f(y + f(x))$

$\Downarrow \Leftarrow$ in.

$x + f(y) = y + f(x)$

$f(y) - y = f(x) - x$

$f(x) - x =$ costante

$\boxed{f(x) = \text{costante} + x}$ $\quad f(x) = c + x$

$f(x) = 2011 + x$

$f(x) = \lfloor x \rfloor$ $\quad f(x + f(y))$ $\quad f(x) + f(y)$

$\lfloor x + \lfloor y \rfloor \rfloor = \lfloor y \rfloor + \lfloor x \rfloor$

$f(x + f$

$\boxed{f(x) = x + \lfloor x \rfloor}$

$f(x) \equiv 0$

$f(y) = a \neq 0$

$$f(x+f(y)) = f(x)+f(y)$$

$$f(x+a) = f(x)+a \qquad \forall x$$

$$f(x+2a) = f((x+a)+a) = f(x+a)+a = f(x)+2a$$

$$f(x+3a) = \;\ldots\; = f(x)+3a$$

$$\vdots$$

$$f(x+na) = f(x)+na$$