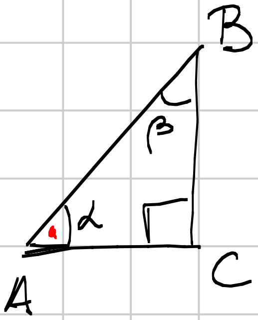
TRIGONOMETRIA (G1 basic)

Titolo nota 03/09/2012

G1 - Trigo

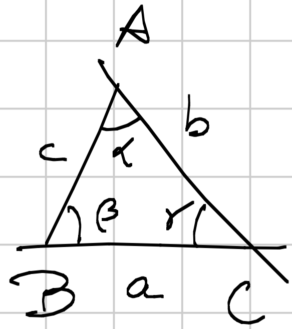
G2 - metodi analitici

G3 - sintetica



$\gamma = 90°$

$\alpha + \beta = 90°$



$$\cos\alpha = \frac{AC}{AB} = \frac{b}{c} = \frac{\text{CATETO ADIACENTE}}{\text{IPOTENUSA}}$$

$$\sin\alpha = \frac{BC}{AB} = \frac{a}{c} = \frac{\text{CATETO OPPOSTO}}{\text{IPOTENUSA}}$$

$$\cos\beta = \frac{\text{CAT. AD}}{\text{IP.}} = \frac{BC}{AB} \qquad \sin\beta = \frac{\text{CAT. OP}}{\text{IP}} = \frac{AC}{AB}$$

$$\boxed{\cos(90° - \theta) = \sin\theta \qquad \sin(90° - \theta) = \cos\theta}$$

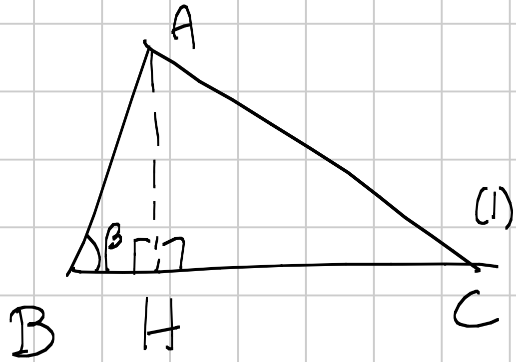
$$Tg\alpha = \frac{\text{CAT. OP}}{\text{CAT. AD}} = \frac{BC}{AC} = \frac{\sin\alpha}{\cos\alpha}$$

TANGENTE

$$Tg\beta = \frac{AC}{BC} = \frac{\sin\beta}{\cos\beta} = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{Tg\alpha} := \cot\alpha$$

COTANGENTE

$$Tg(90° - \alpha) = \cot\alpha$$



$AH = ?$

(I) Tri rett. $\overset{\triangle}{BHA}$ $\quad B\hat{H}A = 90°$

$A\hat{B}H = \beta$

$B\hat{A}H = 90° - \beta$

$$\frac{AH}{BA} = \sin\beta \implies AH = c \cdot \sin\beta$$

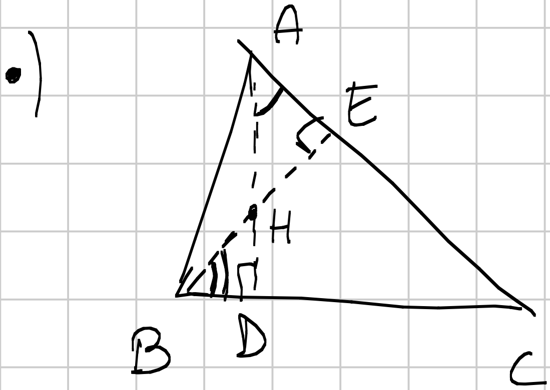
(II) Tri rett. $\overset{\triangle}{AHC}$

$$AH = b \cdot \sin\gamma$$

$BH = c \cdot \cos\beta$

$CH = b \cdot \cos\gamma$

•)



AD, BE altezze, H ortocentro

$AH = ?$

in $\overset{\triangle}{AHE}$, $\quad A\hat{E}H = 90°$

$H\hat{A}E = D\hat{A}C = 90° - A\hat{C}D =$

$= 90° - \gamma$

$$\cos(H\hat{A}E) = \frac{AE}{AH} = \frac{c \cdot \cos\alpha}{AH}$$

$$\cos(H\hat{A}E) = \cos(90° - \gamma) = \sin\gamma$$

$$AH = \frac{c \cdot \cos\alpha}{\sin\gamma}$$

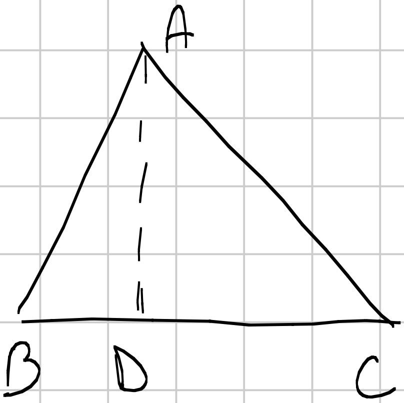
$$\boxed{HD = AD - AH = b \cdot \sin\gamma - c \cdot \frac{\cos\alpha}{\sin\gamma}}$$

in $\overset{\triangle}{BDH}$ $\quad B\hat{D}H = 90°$

$$\frac{HD}{BD} = Tg\, H\hat{B}C = Tg\, (90° - \gamma) = \cot\gamma$$

$BD = c \cdot \cos\beta$

$HD = \cot\gamma \cdot c \cdot \cos\beta$

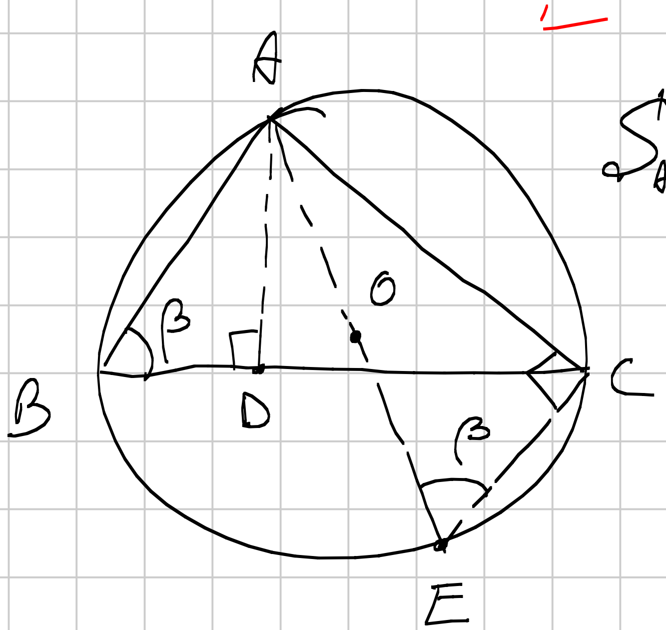


$$S_{ABC} = \frac{1}{2} BC \cdot AD = \boxed{\frac{1}{2} ac \cdot \sin\beta}$$

$$\boxed{= \frac{1}{2} ab \cdot \sin\gamma}$$

$$= \frac{1}{2} bc \cdot \sin\alpha$$

$$S_{ABC} = \frac{abc}{4R}$$



$$S_{ABC} = \frac{1}{2} BC \cdot AD$$

$A\hat{C}E = 90^\circ$

$A\hat{E}C = \beta$

$\overset{\triangle}{AB\hat{D}} \simeq \overset{\triangle}{A\hat{E}C}$

simili

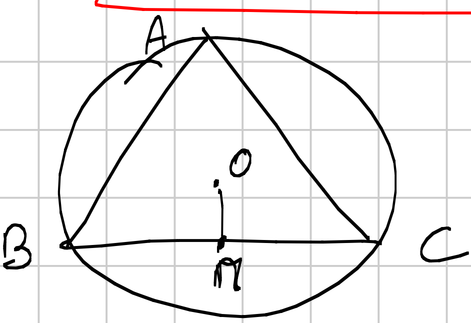
$$\frac{AD}{AC} = \frac{AB}{AE} \qquad AD = \frac{AB \cdot AC}{AE} = \frac{c \cdot b}{2R}$$

$$S = \frac{1}{2}\, a \cdot \frac{c \cdot b}{2R} = \frac{abc}{4R}$$

Teo dei Seni : $S_{ABC} = \cancel{\frac{1}{2}}\, \cancel{ab} \sin\gamma = \frac{\cancel{ab}c}{\cancel{4}R}$ 2

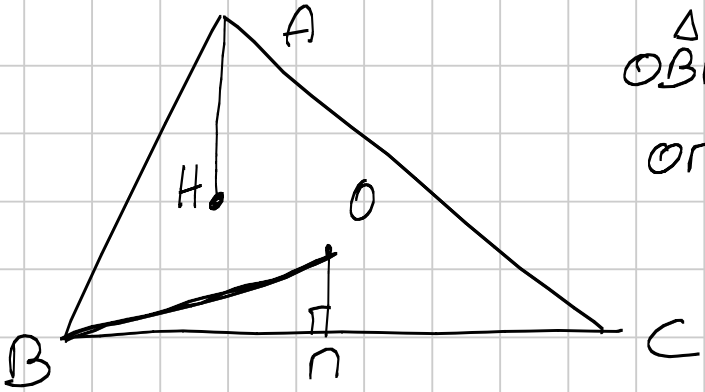
$$\Rightarrow \boxed{\sin\gamma = \frac{c}{2R}}$$

o)



M = pt. medio di BC

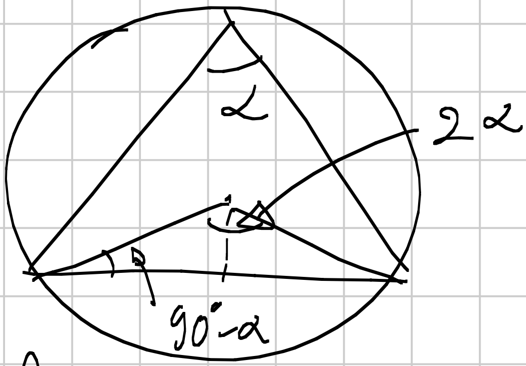
$OM = ?$



$\overset{\triangle}{OBM}$

$OM = OB \cdot \sin O\hat{B}M =$

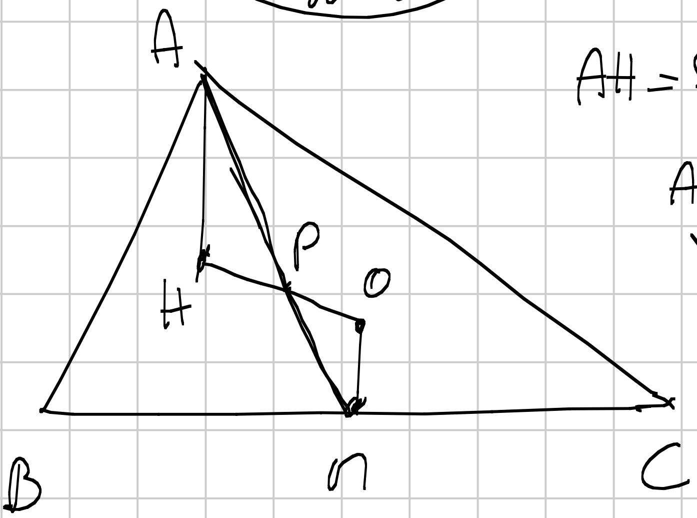
$= R \cdot \sin(90-\alpha) = R\cos\alpha$



$$AH = \frac{c \cdot \cos\alpha}{\sin\gamma} = 2R\cos\alpha$$

$\frac{c}{\sin\gamma} = 2R$ Teo del seno

$AH = 2OM$



$AH = 2OM$

$\overset{\triangle}{AHP} \sim \overset{\triangle}{POM}$

$\Rightarrow AP = 2PM$

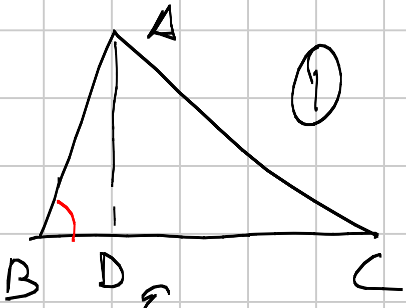
$\Rightarrow$

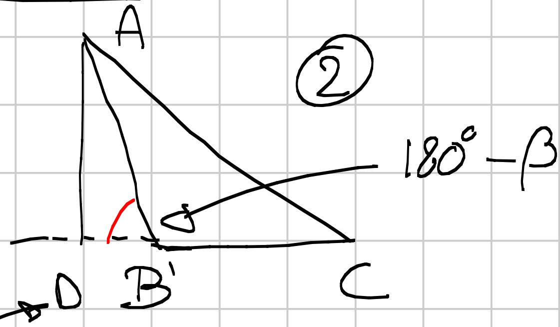
$P = G$ baricentro

$\Rightarrow H, G, O$ allineati in quest'ordine

con $HG = 2 \cdot GO$

Digressione sugli angoli ottusi





$\overline{BD} = \overline{B'D}$

$AD = AD$

$DC = DC$

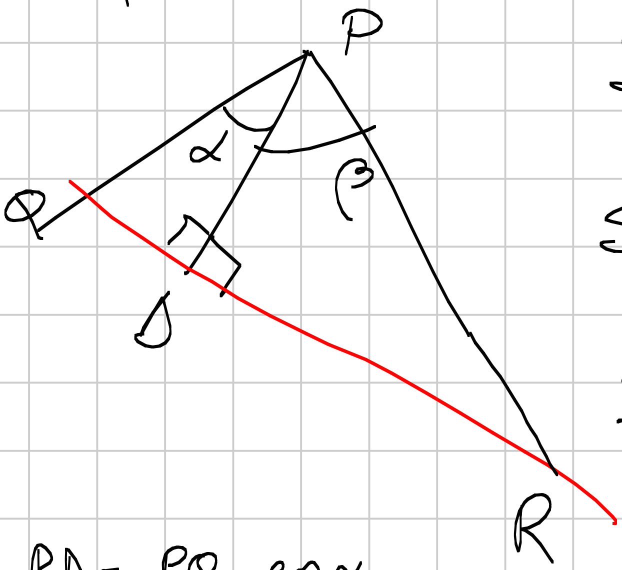
① $BD + DC = BC$

② $DC - B'D = B'C$

$\sin(180^\circ-\beta) = \sin\beta$ ① $c\cdot\cos\beta + b\cdot\cos\gamma = a$

$\cos(180^\circ-\beta) = -\cos\beta$

•) α, β $\alpha+\beta < 180^\circ$



$$S_{PQR} = \frac{1}{2} PQ\cdot PR\cdot \sin(\alpha+\beta)$$

$$\parallel$$

$$S_{PQD} + S_{PDR}$$

$$S_{PDR} = \frac{1}{2} PR\cdot PD\cdot \sin\beta = \frac{1}{2} PR\cdot PQ\cdot \cos\alpha \sin\beta$$

$$S_{PQD} = \frac{1}{2} PQ\cdot PD\cdot \sin\alpha = \frac{1}{2} PQ\cdot PR\cdot \cos\beta\cdot\sin\alpha$$

$PD = PQ\cdot\cos\alpha$

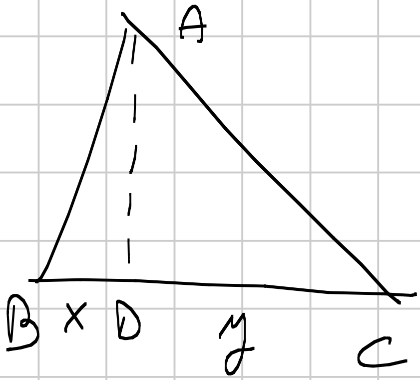
$PD = PR\cdot\cos\beta$

$$\cancel{\frac{1}{2} PQ\,PR}\cdot\sin(\alpha+\beta) = \cancel{\frac{1}{2} PQ\cdot PR}\cos\alpha\sin\beta + \cancel{\frac{1}{2} PQ\,PR}\sin\alpha\cos\beta$$

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

•)



$AD^2 = AB^2 - BD^2$

$AD^2 = AC^2 - DC^2$

$$\begin{cases} c^2 - b^2 = x^2 - y^2 \\ x + y = a \end{cases} \qquad \begin{cases} c^2 - b^2 = a(x-y) \\ a = x + y \end{cases}$$

$$\begin{cases} x + y = a \\ x - y = \dfrac{c^2-b^2}{a} \end{cases} \qquad \begin{cases} x = \dfrac{1}{2}\left(a + \dfrac{c^2-b^2}{a}\right) \\ y = \dfrac{1}{2}\left(a - \dfrac{c^2-b^2}{a}\right) \end{cases}$$

$$\begin{cases} x = \dfrac{a^2+c^2-b^2}{2a} \\[2ex] y = \dfrac{a^2-c^2+b^2}{2a} \end{cases} \qquad \begin{aligned} x &= c \cdot \cos\beta \\ y &= b \cdot \cos\gamma \end{aligned}$$

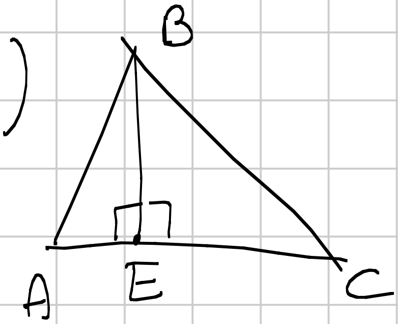
$$c \cdot \cos\beta = \frac{a^2+c^2-b^2}{2a}$$

Teo del Coseno : $\boxed{\cos\beta = \dfrac{a^2+c^2-b^2}{2ac}}$

(di Carnot)

$b^2 = a^2+c^2-2ac \cdot \cos\beta$ PITAGORA generalizzato.

a)



$$AB^2+BC^2 = AE^2+EB^2+EB^2+EC^2 =$$

$$= AE^2+EC^2+2EB^2 = AC^2-2AE \cdot EC+2EB^2 =$$

$$= AC^2-2AB \cdot \cos\alpha \cdot BC\cos\gamma+2AB \cdot BC \cdot \sin\alpha \cdot \sin\gamma$$

$$= AC^2-2AB \cdot BC\,(\cos\alpha\cos\gamma-\sin\alpha\sin\gamma)$$

$EB = AB \cdot \sin\alpha$

$EB = BC \cdot \sin\gamma$

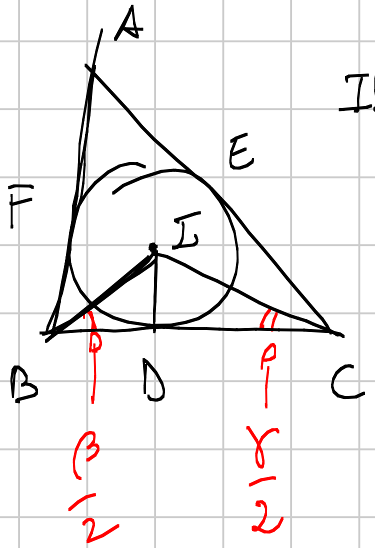
$$c^2+a^2+2ac\,(\cos\alpha\cos\gamma-\sin\alpha\sin\gamma) = b^2$$

$$c^2+a^2-2ac\cos\beta = b^2$$

$$-\cos\beta = \cos\alpha\cos\gamma-\sin\alpha\sin\gamma$$

$$\begin{aligned} &-\cos\beta \\ &\quad \| \\ &-\cos(180°-(\alpha+\gamma)) \\ &\quad \| \\ &\cos(\alpha+\gamma) \end{aligned}$$

$$\boxed{\begin{aligned} \cos(\alpha+\gamma) &= \cos\alpha\cos\gamma-\sin\alpha\sin\gamma \\ \cos(\alpha-\gamma) &= \cos\alpha\cos\gamma+\sin\alpha\sin\gamma \end{aligned}}$$



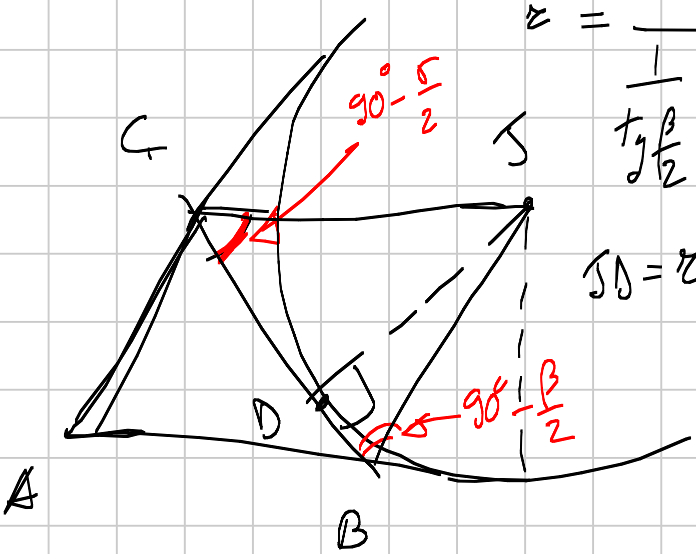
$ID = r = ?$

$Tg\frac{\beta}{2} = \frac{ID}{BD}$ $\quad Tg\frac{\gamma}{2} = \frac{ID}{DC}$

$BD = \frac{ID}{Tg\frac{\beta}{2}}$ $\quad DC = \frac{ID}{Tg\frac{\gamma}{2}}$

$a = \frac{r}{Tg\frac{\beta}{2}} + \frac{r}{Tg\frac{\gamma}{2}}$

$r = \frac{a}{\frac{1}{tg\frac{\beta}{2}} + \frac{1}{tg\frac{\gamma}{2}}} = \frac{a}{\cot\frac{\beta}{2} + \cot\frac{\gamma}{2}}$



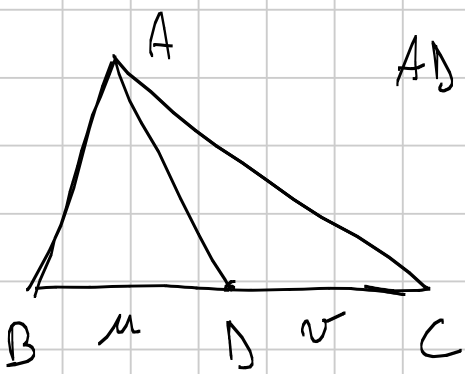
$JD = r_a$

$a = \frac{r_a}{Tg(90° - \frac{\gamma}{2})} + \frac{r_a}{tg(90° - \frac{\beta}{2})} =$

$= \frac{r_a}{\cot\frac{\gamma}{2}} + \frac{r_a}{\cot\frac{\beta}{2}}$

$r_a = \frac{a}{\frac{1}{\cot\frac{\gamma}{2}} + \frac{1}{\cot\frac{\beta}{2}}} = \frac{a}{tg\frac{\beta}{2} + tg\frac{\gamma}{2}}$

•)



AD = ~~retta~~ a CASO ceviana

$BD = u$

$DC = v$

$AD = ?$

$u + v = a$

in $\overset{\triangle}{ABD}$, $A\hat{B}D = \beta$, $AB = c$, $BD = u$

$$AD^2 = c^2 + u^2 - 2c \cdot u \cdot \cos\beta \quad \text{(Teo di Carnot)}$$

In $\overset{\Delta}{ABC}$ $\quad \cos\beta = \dfrac{c^2+a^2-b^2}{2ac}$ (Teo di Carnot)

$$AD^2 = c^2 + u^2 - \cancel{2}\cancel{c} \cdot u \left(\frac{c^2+a^2-b^2}{\cancel{2}a\cancel{c}} \right) = \qquad a = u + v$$

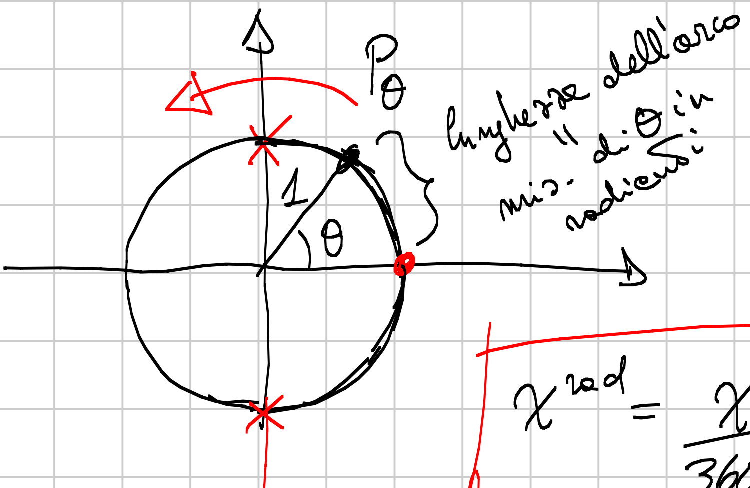
$$= c^2 + u^2 - u \left(\frac{c^2 + u^2 + v^2 + 2uv - b^2}{u+v} \right) =$$

$$= \frac{\cancel{uc^2} + vc^2 + \cancel{u^3} + \cancel{u^2v} - \cancel{uc^2} - \cancel{u^3} - uv^2 - \cancel{2}u^2v + ub^2}{u+v} =$$

$$AD^2 = \boxed{\frac{ub^2 + vc^2 - uv(u+v)}{u+v}} = \boxed{\frac{ub^2+vc^2}{u+v} - uv}$$

Teo di STEWART

$$\boxed{= \frac{u(b^2-uv)}{u+v} + \frac{v(c^2-uv)}{u+v}}$$



360°	2π
0°	0

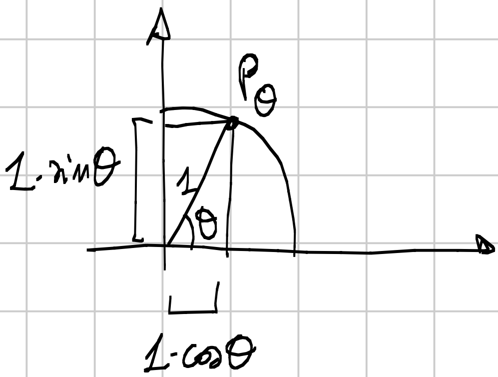
$$\frac{x^\circ}{360^\circ} = \frac{x^{rad}}{2\pi}$$

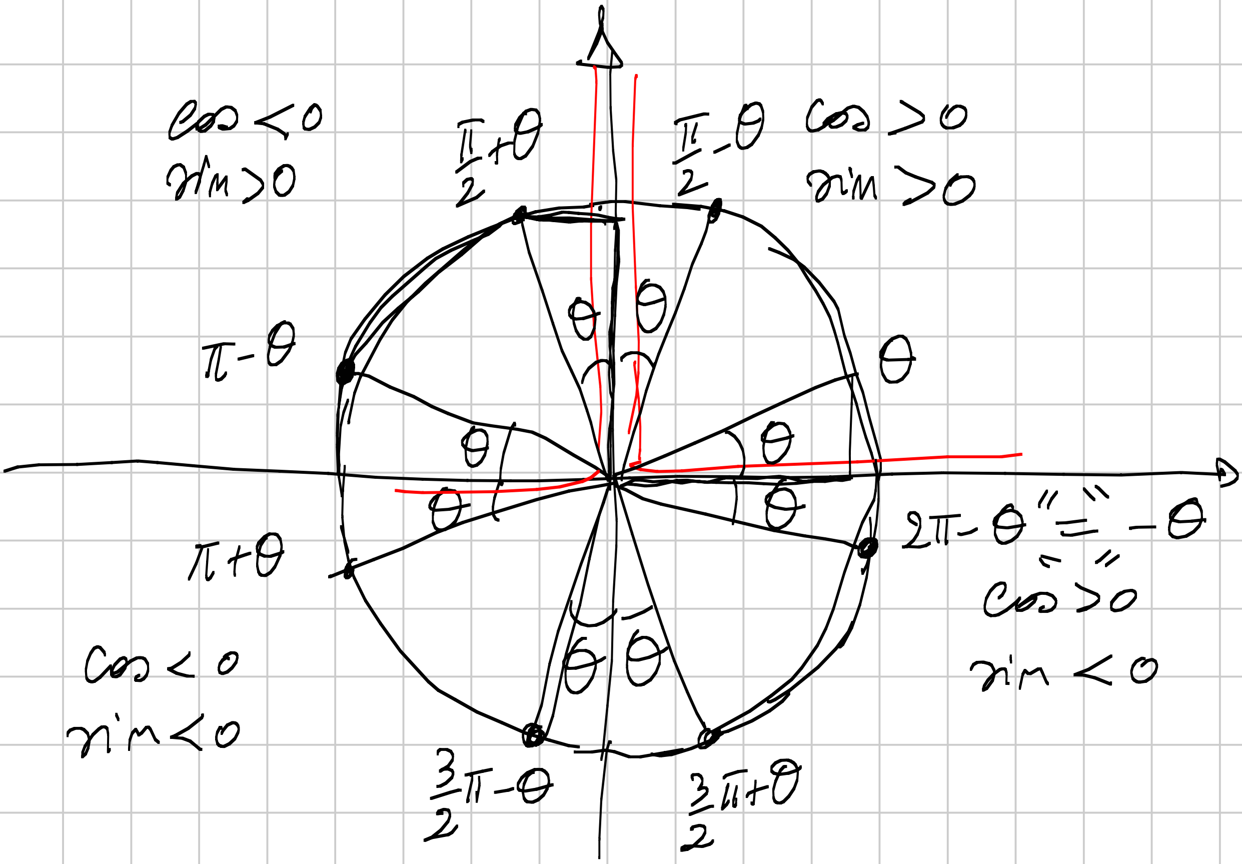
$$\boxed{x^{rad} = \frac{x^\circ}{360^\circ} \cdot 2\pi}$$

Es:

rad	deg
0	$0°$
$\pi/6$	$30°$
$\pi/4$	$45°$
$\pi/3$	$60°$
$\pi/2$	$90°$

$$P_\theta = (\cos\theta, \sin\theta)$$





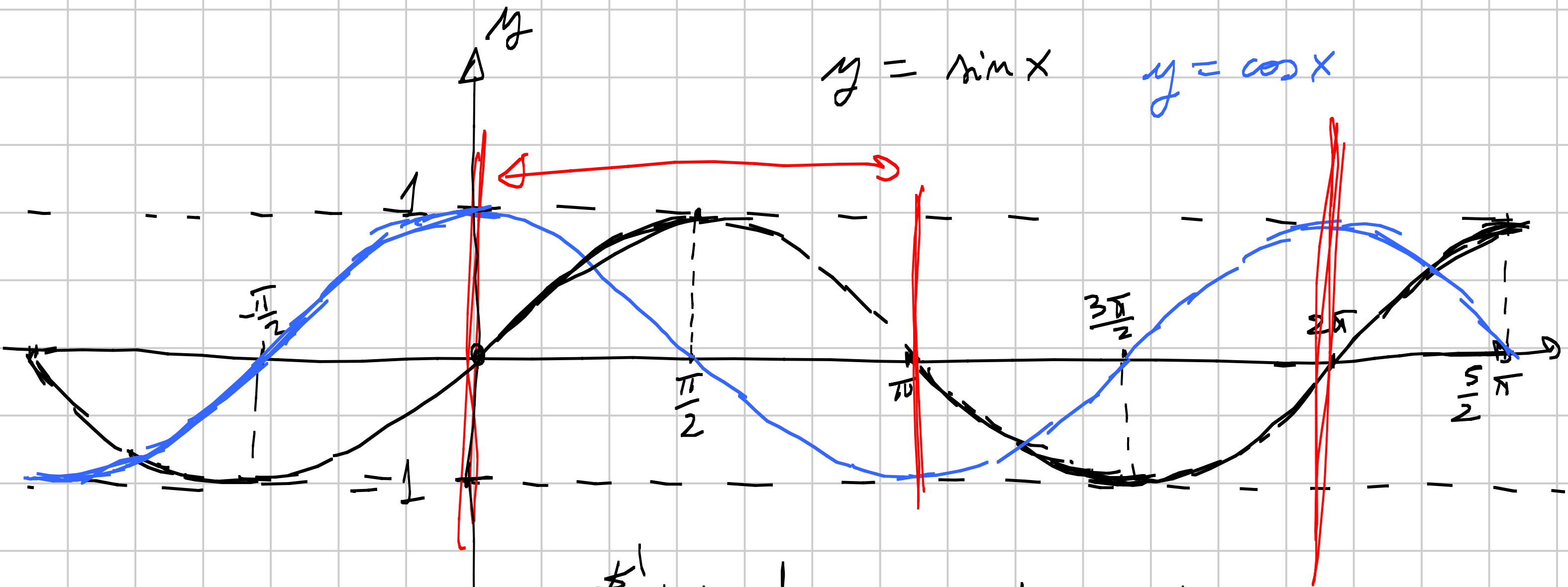
$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{3}{2}\pi - \theta\right) = -\cos\left(\frac{3}{2}\pi + \theta\right)$$

$$\boxed{\sin(x) = \sin(x+2\pi) \quad \forall x \in \mathbb{R}}$$

$$\cos(x) = \cos(x+2\pi) \quad \forall x \in \mathbb{R}$$

SENO e COSENO hanno <u>periodo</u> 2π



È vero che se $\sin x = \sin y$ allora $x = y$?

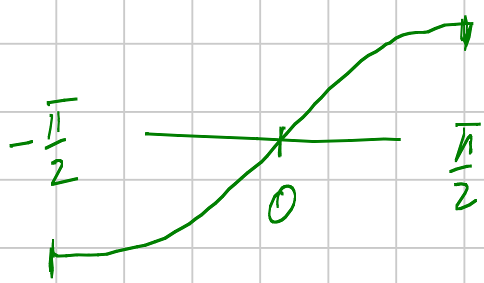
no, nemmeno <u>come</u> <u>angoli</u>

E se $0 \le x, y \le \pi$? Nemmeno.

Se $0 \le x, y \le \frac{\pi}{2}$ e $\sin x = \sin y \implies x = y$

Se $-\frac{\pi}{2} \le x, y \le \frac{\pi}{2}$ e $\sin x = \sin y \implies x = y$

$\left(\frac{\pi}{2} \le x, y \le \frac{3}{2}\pi\right)$



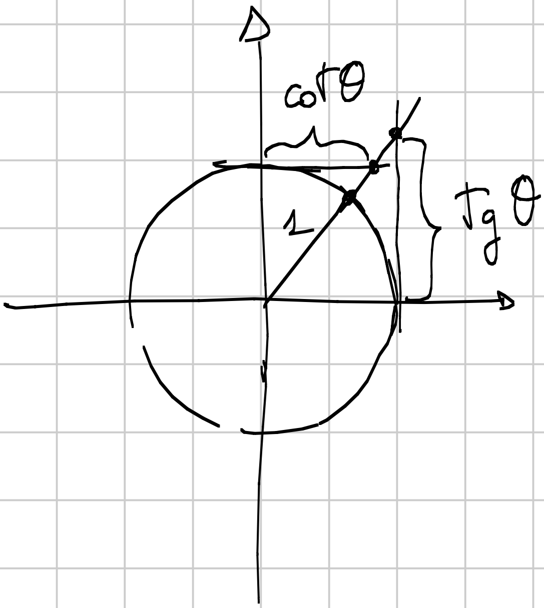
Su $\left[-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right]$ $\sin(x)$ è monotona crescente

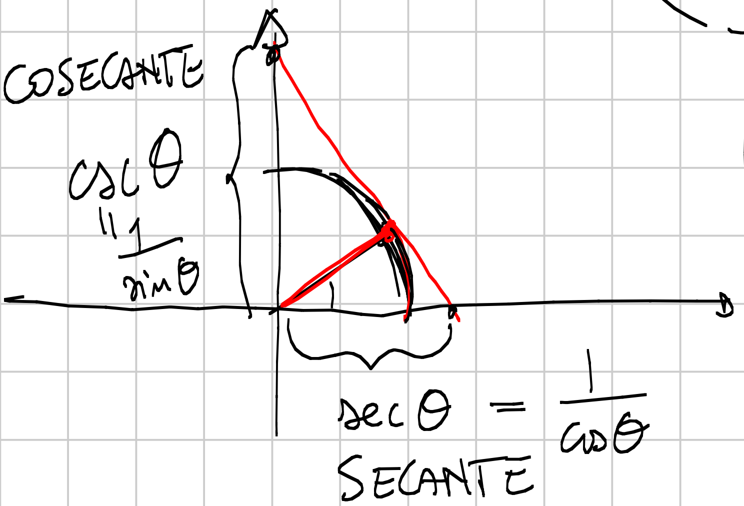
$\implies$ <u>iniettiva</u>

Su $\left[\frac{\pi}{2}+2k\pi,\ \frac{3}{2}\pi+2k\pi\right]$ $\sin(x)$ è monotona decrescente

$\Rightarrow$ iniettiva.

$$\mathrm{Tg}\,\theta = \frac{\sin\theta}{\cos\theta}$$

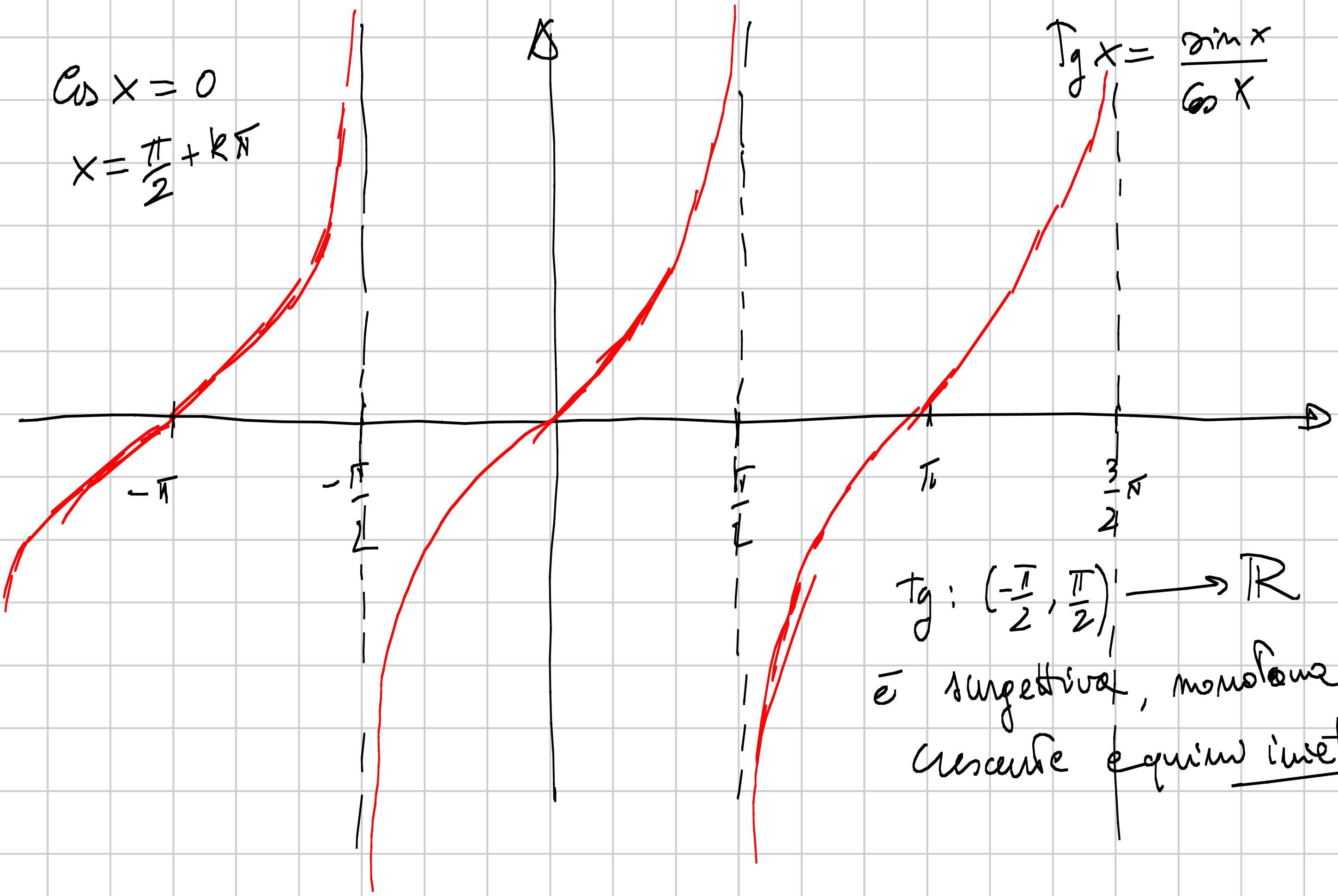




$\cos x = 0$

$x = \frac{\pi}{2} + k\pi$

$$\mathrm{Tg}\,x = \frac{\sin x}{\cos x}$$



$\mathrm{Tg}: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$

è surgettiva, monotona crescente quindi iniettiva

" Tg è iniettiva su $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Formule di ADDIZIONE e SOTTRAZIONE

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$Tg(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} =$$

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{Tg\,x + Tg\,y}{1 - Tg\,x\,Tg\,y}$$

$$Tg(x) = -Tg(-x)$$

$$Tg(x-y) = \frac{Tg\,x - Tg\,y}{1 + Tg\,x\,Tg\,y}$$

Formule di DUPLICAZIONE $(x=y)$

$$\cos 2x = \cos^2 x - \sin^2 x \qquad 1 = \cos 0 = \cos^2 x + \sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$
$$Tg\,2x = \frac{2\,Tg\,x}{1 - Tg^2 x}$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$

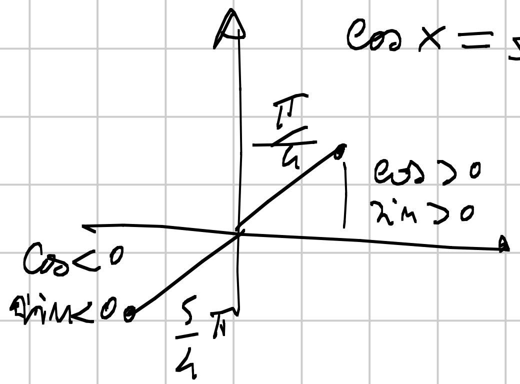
b) $\sin x = \pm\sqrt{\dfrac{1-\cos 2x}{2}}$

$\cos x = \pm\sqrt{\dfrac{\cos 2x + 1}{2}}$

$2\theta = \dfrac{\pi}{2} + 2k\pi \qquad k \in \mathbb{Z}$

$\theta = \dfrac{\pi}{4} + k\pi$

$\theta = \dfrac{\pi}{4}, \quad \theta = \dfrac{5}{4}\pi$



c) $\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x =$

$= (\cos^2 x - \sin^2 x)\cos x - 2\sin x \cos x \sin x =$

$= \cos^3 x - \cos x \sin^2 x - 2\sin^2 x \cos x =$

$= \cos^3 x - 3\underbrace{\sin^2 x}_{(1-\cos^2 x)} \cos x = 4\cos^3 x - 3\cos x$

$\operatorname{tg} 2x = \dfrac{2\operatorname{tg} x}{1 - \operatorname{tg}^2 x}$

$\operatorname{tg} x = \pm\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

$\operatorname{tg}^2 x + 1 = \dfrac{\sin^2 x}{\cos^2 x} + 1 = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$

$\cos^2 x = \dfrac{1}{\operatorname{tg}^2 x + 1}$

$\cos 2x = 2\cos^2 x - 1 =$

$= \dfrac{2}{\operatorname{tg}^2 x + 1} - 1 =$

$= \dfrac{1 - \operatorname{tg}^2 x}{1 + \operatorname{tg}^2 x}$

$$\sin 2x = \operatorname{tg} 2x \cdot \cos 2x = \frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}\cdot\frac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x}$$

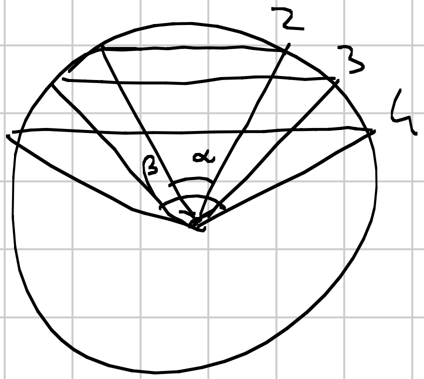
$$t = \operatorname{tg}\frac{x}{2}$$

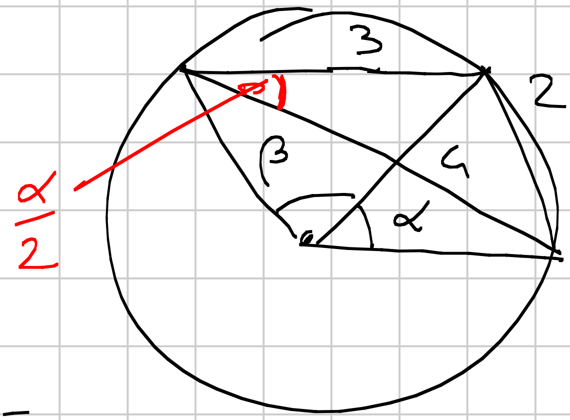
$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\operatorname{tg} x = \frac{2t}{1-t^2}$$

Ex: In una cfr. ci sono tre corde lunghe 2, 3, 4 che insistono su archi ampi α, β, $\alpha+\beta$. Quanto vale $\cos\alpha$?

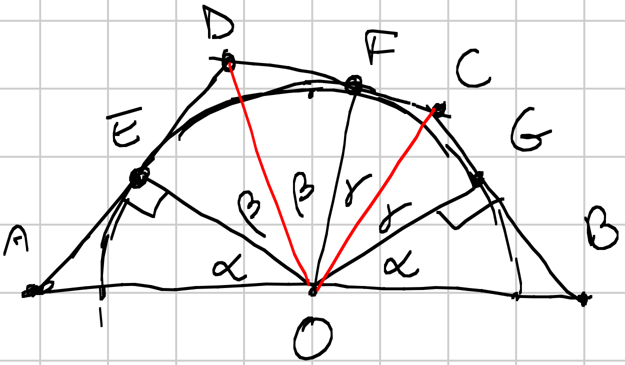




$$\cos\frac{\alpha}{2} = \frac{3^2+4^2-2^2}{2\cdot 3\cdot 4} = \frac{9+16-4}{24} = \cancel{\frac{21}{24}}\ \frac{7}{8}$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 2\,\frac{7^2}{8^2} - 1 = \frac{2\cdot 7^2 - 8^2}{8^2} = \frac{2\cdot 49 - 64}{64} = \frac{98-64}{64} =$$

$$= \frac{34}{64} = \frac{17}{32}$$



$AO = OB$

$\Rightarrow AB^2 = 4\, BC \cdot AD$

Dim: $\overset{\triangle}{OAE} \cong \overset{\triangle}{OBG} \Rightarrow O\hat{A}E = O\hat{B}G$

R = raggio

$$2\alpha + 2\beta + 2\gamma = \pi$$

$$\alpha + \beta + \gamma = \frac{\pi}{2}$$

$$BC = BG + GC = R\,Tg\,\alpha + R\,Tg\,\gamma$$

$$AB = 2 \cdot AO = 2\,\frac{R}{\cos\alpha}$$

$$AD = R\,Tg\,\alpha + R\,Tg\,\beta$$

$\boxed{R = 1}$

$$AB^2 = \frac{\cancel{4}}{\cos^2\alpha} \overset{?}{=} 4\, BC \cdot AD = \cancel{4}(Tg\,\alpha + Tg\,\gamma)(Tg\,\alpha + Tg\,\beta)$$

$$\frac{1}{\cos^2\alpha} = \cancel{Tg^2\alpha} + Tg\,\alpha\, Tg\,\gamma + Tg\,\alpha\, Tg\,\beta + Tg\,\gamma\, Tg\,\beta$$

$\|$

$\cancel{Tg^2}\alpha + 1$

$$\gamma = \frac{\pi}{2} - (\alpha + \beta)$$

$$Tg\,\gamma = Tg\left(\frac{\pi}{2} - (\alpha+\beta)\right) = \frac{1}{Tg(\alpha+\beta)} = \frac{1 - Tg\,\alpha\, Tg\,\beta}{Tg\,\alpha + Tg\,\beta}$$

$$1 \overset{?}{=} (Tg\,\alpha + Tg\,\beta)\, Tg\,\gamma + Tg\,\alpha\, Tg\,\beta$$

$$\cancel{(Tg\,\alpha + Tg\,\beta)}\, \frac{(1 - Tg\,\alpha\, Tg\,\beta)}{\cancel{Tg\,\alpha + Tg\,\beta}} + Tg\,\alpha\, Tg\,\beta = 1 \quad \boxed{\text{OK}}$$

•) $5\cos x + 2\sin x = 1.$ $\qquad t = Tg\,\frac{x}{2}$

$$5\,\frac{1-t^2}{1+t^2} + 2\,\frac{2t}{1+t^2} = 1 \qquad 5 - 5t^2 + 4t = 1 + t^2$$

$$6t^2 - 4t - 4 = 0$$

$$t = \frac{2 \pm \sqrt{4+24}}{6} = \frac{2 \pm \sqrt{28}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

$$tg\frac{x}{2} = \frac{1+\sqrt{7}}{3} \qquad tg\frac{x}{2} = \frac{1-\sqrt{7}}{3}$$

$$\frac{x}{2} = \theta_1 + k\pi \qquad \frac{x}{2} = \theta_2 + k\pi$$

$$x = 2\theta_1 + 2k\pi \qquad x = 2\theta_2 + 2k\pi$$

<u>Ex</u>: $0 < \alpha, \beta, \gamma < \pi$ $\qquad \alpha+\beta+\gamma = \pi$

$$\Updownarrow$$

$$tg\frac{\alpha}{2} tg\frac{\beta}{2} + tg\frac{\beta}{2} tg\frac{\gamma}{2} + tg\frac{\gamma}{2} tg\frac{\alpha}{2} = 1$$

Sappiamo che vale $\boxed{\Longrightarrow}$

Facciamo $\boxed{\Longleftarrow}$: Sappiamo che

1) per ipotesi $tg\frac{\alpha}{2} tg\frac{\beta}{2} + tg\frac{\beta}{2} tg\frac{\gamma}{2} + tg\frac{\gamma}{2} tg\frac{\alpha}{2} = 1$

2) per conticci $tg\frac{\alpha}{2} tg\frac{\beta}{2} + tg\frac{\beta}{2} tg\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}\right) + tg\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}\right) tg\frac{\alpha}{2} = 1.$

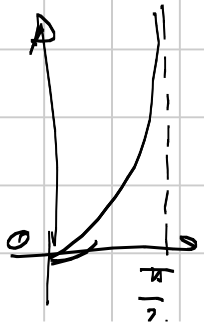
$$tg\frac{\gamma}{2} = \frac{1 - tg\frac{\alpha}{2} tg\frac{\beta}{2}}{tg\frac{\alpha}{2} + tg\frac{\beta}{2}} \qquad tg\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}\right) = \frac{1 - tg\frac{\alpha}{2} tg\frac{\beta}{2}}{tg\frac{\alpha}{2} + tg\frac{\beta}{2}}$$

$$\Downarrow$$

$$tg\frac{\gamma}{2} = tg\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\beta}{2}\right)$$

$0 < \gamma < \pi$

$0 < \frac{\gamma}{2} < \frac{\pi}{2}$



tg è iniettiva tra 0 e $\frac{\pi}{2}$

$$\Rightarrow \frac{\gamma}{2} = \frac{\pi}{2} - \frac{\alpha}{2} - \frac{\beta}{2}$$

Es x voi: $\alpha+\beta+\gamma = \pi$

$\Downarrow$

- $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \tan\beta \tan\gamma$
- $\sin\alpha + \sin\beta + \sin\gamma = 4 \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2}$

- $$\frac{\cos 2\alpha - \cos 4\alpha}{\sin 2\alpha + \sin 4\alpha} = \operatorname{Tg}\alpha$$

Formule brutte (Somma/Prodotto)

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\cos x \cos y = \frac{\cos(x+y) + \cos(x-y)}{2}$$

$$\alpha = \frac{x+y}{2}$$
$$\beta = \frac{x-y}{2}$$

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$$

$$2\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos\alpha + \cos\beta$$

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$$

$$\sin y \cos x = \frac{\sin(x+y) - \sin(x-y)}{2}$$

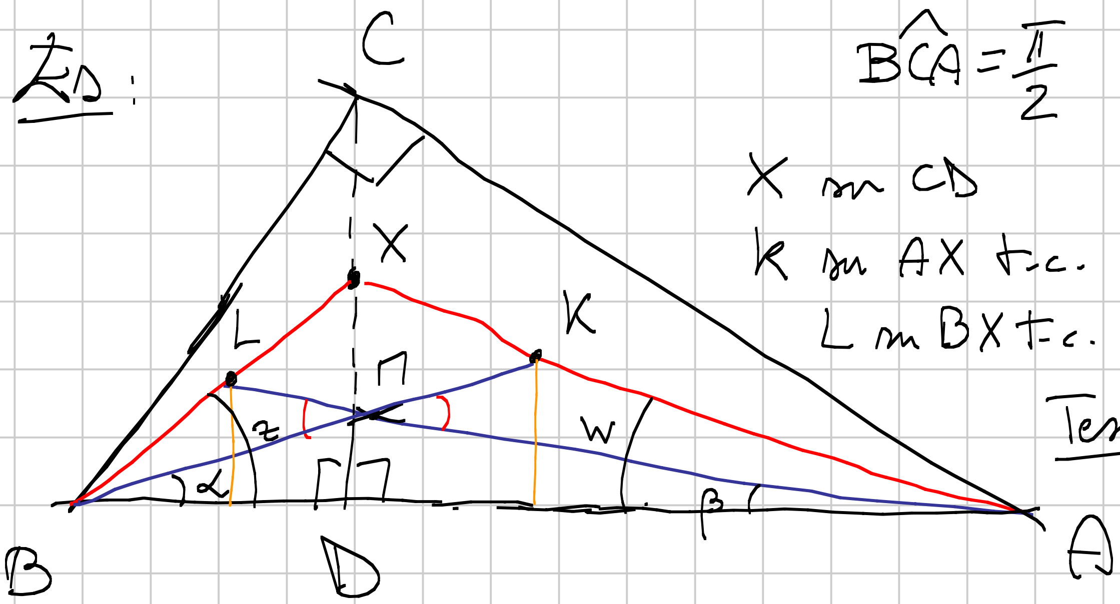
Es: $\cos x + \cos\left(x + \frac{\pi}{3}\right) = 0$

$$2\cos\left(2x + \frac{\pi}{3}\right)\cos\left(-\frac{\pi}{3}\right) = 0 \Longrightarrow \cos\left(2x+\frac{\pi}{3}\right) = 0$$

$$2x + \frac{\pi}{3} = \frac{\pi}{2} + k\pi$$

$$x = \left(\frac{\pi}{6} + k\pi\right)\frac{1}{2} = \frac{\pi}{12} + k\frac{\pi}{2}$$

Es:



$\widehat{BCA} = \frac{\pi}{2}$

X su CD

K su AX t.c. $BK = BC$

L su BX t.c. $AL = AC$

Tesi: $ML = MK$

$\frac{MK}{ML}$ $\qquad$ MK in $\overset{\triangle}{AMK}$ $\qquad$ ML in $\overset{\triangle}{BML}$

$$\frac{MK}{\sin KAM} = \frac{AK}{\sin KMA} \qquad \frac{ML}{\sin LBM} = \frac{BL}{\sin LMB}$$

$$\frac{MK}{ML} = \frac{AK \cdot \sin KAM}{BL \sin LBM}$$

•) AK in AKB

$$\frac{AK}{\sin ABK} = \frac{AB}{\sin AKB}$$

•) BL in ALB

$$\frac{BL}{\sin BAL} = \frac{AB}{\sin ALB}$$

$$\frac{AK}{BL} = \frac{\sin ABK \cdot \sin ALB}{\sin BAL \cdot \sin AKB}$$

$$\frac{MK}{ML} = \frac{\sin ABK \cdot \sin ALB \cdot \sin KAM}{\sin BAL \cdot \sin AKB \cdot \sin LBM} = \frac{\sin\alpha \cdot \sin(z+\beta)\sin(w-\beta)}{\sin\beta \cdot \sin(w+\alpha)\sin(z-\alpha)}$$

$$\sin\alpha\left[\sin z\cos\beta + \cos z\sin\beta\right]\left[\sin w\cos\beta - \sin\beta\cos w\right] =$$

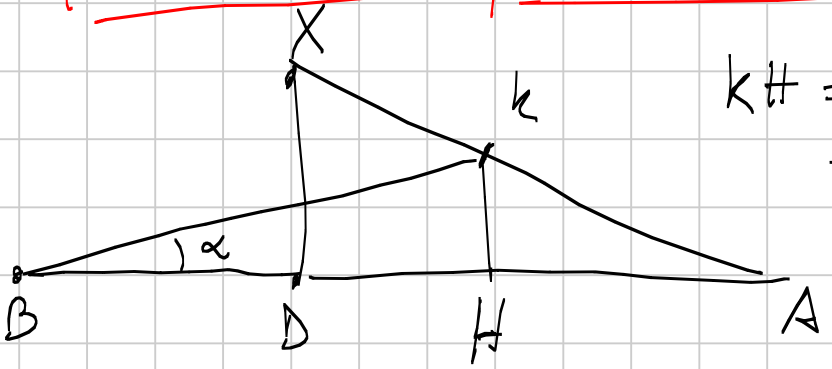
$$= \left[\cos z\cos w\right]^2 \sin\alpha\left[\operatorname{tg} z\,\frac{\cos\beta}{\cos w} + \frac{\sin\beta}{\cos w}\right]\left[\operatorname{tg} w \cdot \frac{\cos\beta}{\cos z} - \frac{\sin\beta}{\cos z}\right] =$$

$$= \left[\cos z\cos w\right]^{\cancel{2}} \sin\alpha\left[\operatorname{tg} z\operatorname{tg} w\,\frac{\cos^2\beta}{\cancel{\cos z\cos w}} - \operatorname{tg} z\,\frac{\cos\beta\sin\beta}{\cancel{\cos w\cos z}} + \operatorname{tg} w\,\frac{\cos\beta\sin\beta}{\cancel{\cos w\cos z}} - \right.$$

$$-\frac{\sin^2\beta}{\cancel{\cos w\cos z}}\Bigg] = \cos z\cos w\sin\alpha\Big[\tan z\tan w\cos^2\beta +$$

$$+\,(\tan w-\tan z)\left(\frac{\sin 2\beta}{2}\right) - \sin^2\beta\Big]$$

$$\boxed{\tan z = \frac{c\cdot XD}{a^2}} \qquad \boxed{\tan w = \frac{c\cdot XD}{b^2}}$$



$KH = BK\cdot\sin\alpha =$

$= BC\sin\alpha = a\cdot\sin\alpha$

$AH = c - a\cdot\cos\alpha$

$$\frac{a\sin\alpha}{c-a\cos\alpha} = \frac{KH}{AH} = \frac{XD}{DA} = \frac{XD\cdot c}{b^2}$$

$$\frac{b\sin\beta}{c-b\cos\beta} = \frac{XD\cdot c}{a^2} \to \boxed{\sin\beta = \frac{c^2 XD}{a^2 b} - \frac{cXD}{a^2}\cos\beta}$$

$$\Big[\tan z\tan w\cos^2\beta - (\tan z-\tan w)\cos\beta\sin\beta - \sin^2\beta\Big] =$$

$$= \left[\frac{XD^2 c}{b^2 a^4}\cos\beta - \frac{c^5 XD^2}{b^3 a^4}\right] = \frac{c^3 XD}{b^2 a^2}\big[-\sin\beta\big]$$

$$-\frac{c^3 XD}{b^2 a^2}\cos z\cos w\sin\alpha\sin\beta.$$

$ML = MK.$

(IMO'12 / 5)