Senior 2012

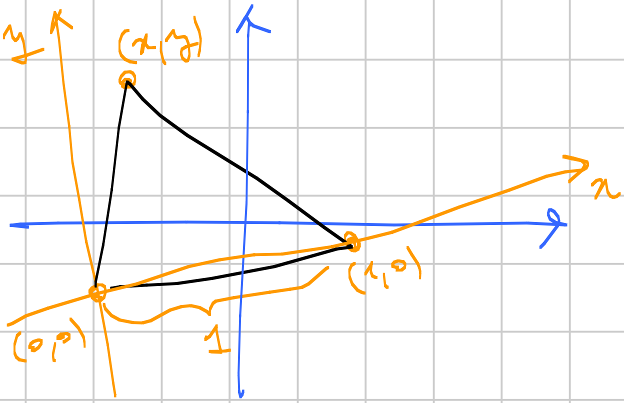
GEOMETRIA 2 BASIC

F. Morandin

Titolo nota 05/09/2012

Metodi algebrici : coordinate cartesiane
vettori
complessi

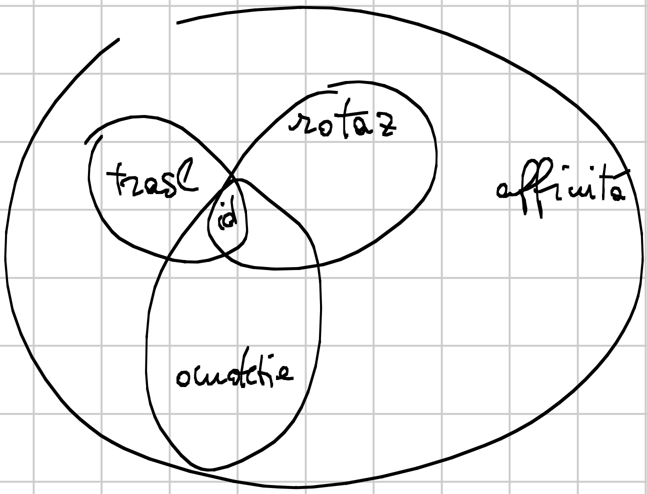
CARTESIANE

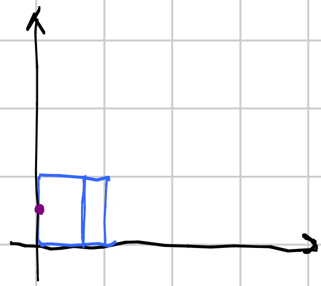


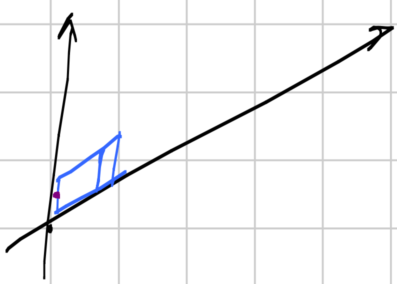
problemi invarianti per rotazioni e traslazioni
↳ scelgo O e la direzione di x in modo comodo

se non è fissata una scala, fisso anche quella
(inv. per omotetie)

AFFINITÀ e INVARIANZA







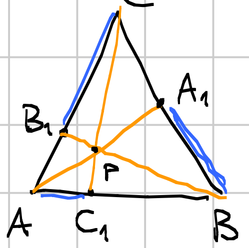
$$\begin{cases} x' = ax + by + c \\ y' = dx + ey + f \end{cases}$$

a, b, c, d, e, f determinano l'affinità

$$P' = MP + Q \qquad M = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \qquad Q = \begin{pmatrix} c \\ f \end{pmatrix}$$

Conservano : parallelismo tra rette (manda rette in rette)
rapporti tra aree
rapporti tra segmenti paralleli

Es 1. Thm di Ceva

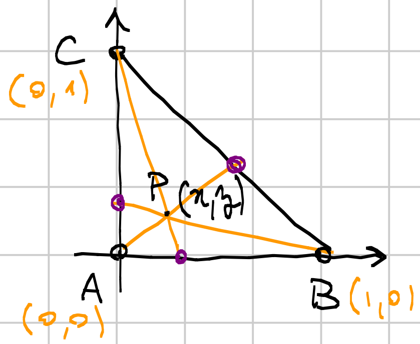


$$AC_1 \cdot BA_1 \cdot CB_1 = C_1B \cdot A_1C \cdot B_1A$$

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$$

↑ rapporto tra segmenti paralleli

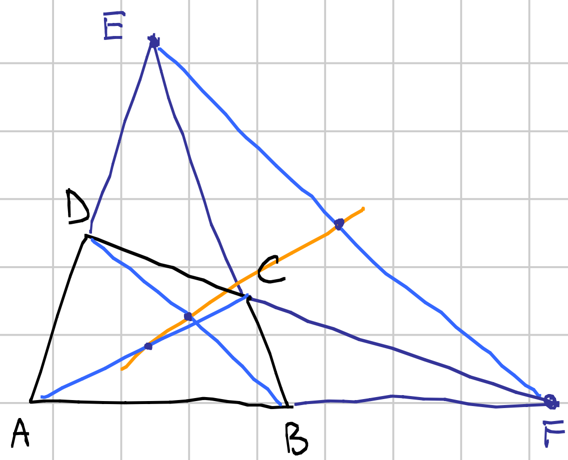
enunciato invariante per affinità



posso mandare 3 punti qualsiasi in 3 punti qualsiasi (a meno che non siano allineati)

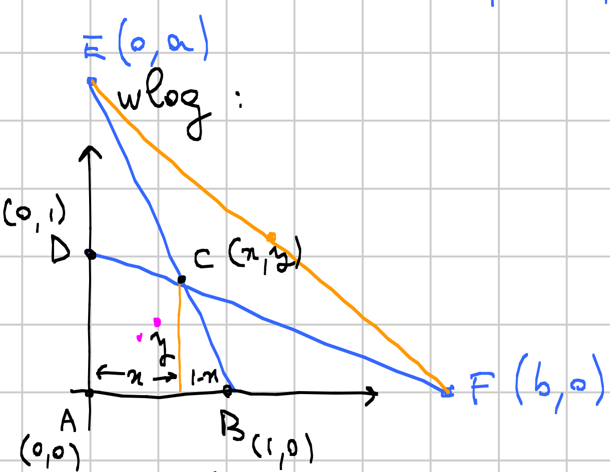
Es 2. Quadrilatero A, B, C, D $\quad E = AD \cap BC \quad F = AB \cap CD$

Tesi : i pti medi di EF, AC, BD sono allineati



enunciato invariante per affinità

wlog :



$a : y = 1 : 1-x \qquad a = \dfrac{y}{1-x} \qquad b = \dfrac{x}{1-y}$

pto medio di EF : $\left(\frac{b}{2}, \frac{a}{2}\right) = \left(\frac{x/2}{1-y}, \frac{y/2}{1-x}\right)$

pto medio di AC : $\left(\frac{x}{2}, \frac{y}{2}\right)$ Q

pto medio di BD : $\left(\frac{1}{2}, \frac{1}{2}\right)$ P

Per vedere se tre punti P, Q, R sono allineati, verifico se esiste $\lambda \in \mathbb{R}$ t.c. $\lambda P + (1-\lambda)Q = R$

$$\begin{cases} \lambda x_P + (1-\lambda)x_Q = x_R \\ \lambda y_P + (1-\lambda) y_Q = y_R \end{cases}$$

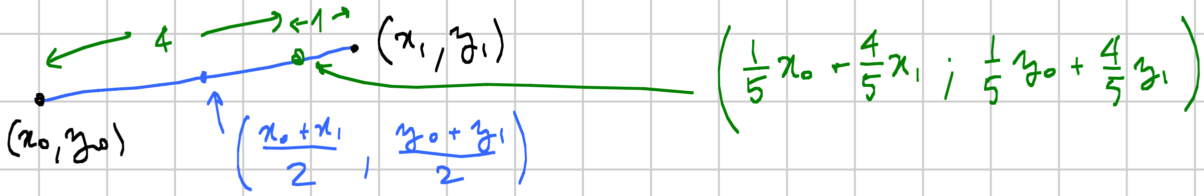
$$\begin{cases} \lambda + (1-\lambda)x = \frac{x}{1-y} \\ \lambda + (1-\lambda)y = \frac{y}{1-x} \end{cases} \qquad \begin{cases} \lambda(1-x) + x = \frac{x}{1-y} \\ \lambda(1-y) + y = \frac{y}{1-x} \end{cases}$$

$$\begin{cases} \lambda = \left(\frac{x}{1-y} - x\right)\frac{1}{1-x} = \frac{xy}{(1-x)(1-y)} \\ \lambda = \qquad = \frac{xy}{(1-x)(1-y)} \end{cases}$$

Un po' di teoria

$y = mx + q$ retta non verticale

$ax + by + c = 0$ retta qualunque

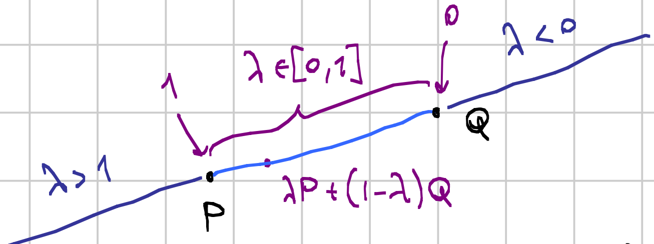


Punti del segmento : combinazioni lineari convesse degli estremi

$(a x_0 + b x_1 ; a y_0 + b y_1)$ $\downarrow$

$\lambda \quad (1-\lambda) \quad \lambda \quad (1-\lambda)$

$0 \le a, b \le 1, \quad a + b = 1$



$$\begin{cases} \lambda x_P + (1-\lambda) x_Q = x \\ \lambda y_P + (1-\lambda) y_Q = y \end{cases}$$ retta per PQ

$$\begin{cases} x = \lambda (x_P - x_Q) + x_Q \\ y = \lambda (y_P - y_Q) + y_Q \end{cases}$$ eq. parametrica

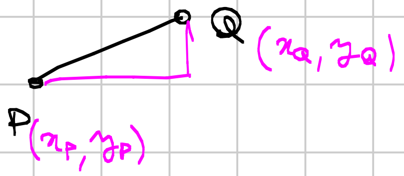
$$\lambda = \frac{x - x_Q}{x_P - x_Q}$$

$$y = \frac{y_P - y_Q}{x_P - x_Q}(x - x_Q) + y_Q$$ eq. implicita

---

DISTANZA e CIRCONFERENZA

$$d(P,Q) = \overline{PQ} = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$$ distanza Euclidea



$$\Gamma = \{ (x,y) : d((x,y),(x_0,y_0)) = r \}$$

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$x^2 + y^2 - 2x_0 x - 2y_0 y + \underbrace{x_0^2 + y_0^2 - r^2}_{c} = 0$$

$$\boxed{x^2 + y^2 + ax + by + c = 0}$$ non sempre questa è una circonf.

$$x_0 = -\frac{a}{2} \qquad y_0 = -\frac{b}{2} \qquad \frac{a^2}{4} + \frac{b^2}{4} - c \overset{?}{\leq} 0$$

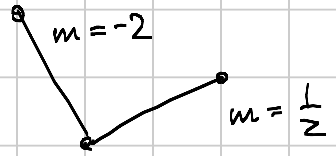
- Asse di un segmento

$$r = \{ (x,y) : d((x,y),(x_P,y_P)) = d((x,y),(x_Q,y_Q)) \}$$

$$(x - x_Q)^2 + (y - y_Q)^2 = (x - x_P)^2 + (y - y_P)^2$$

$$2x(x_P - x_Q) + 2y(y_P - y_Q) + x_Q^2 + y_Q^2 - x_P^2 - y_P^2 = 0$$

$$ax + by + c = 0$$



$$y = \boxed{m}x + q$$

$$y = \boxed{-\frac{1}{m}}x + \text{mostro}$$

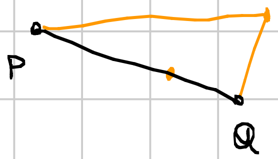
equazione retta ortogonale

- Circonferenza di Apollonio

$$r = \{(x,y) : d((x,y),(x_P,y_P)) = \alpha \cdot d((x,y),(x_Q,y_Q))\}$$

$\alpha \neq 1$

$\alpha > 0$



$\alpha = 2$

$$(x - x_Q)^2 + (y - y_Q)^2 = \alpha^2 (x - x_P)^2 + \alpha^2 (y - y_P)^2$$

$$(1-\alpha^2)x^2 + (1-\alpha^2)y^2 + \dots = 0$$ circonferenza

- Bisettrici, potenza, asse radicale .. alla fine, se riesco

VETTORI

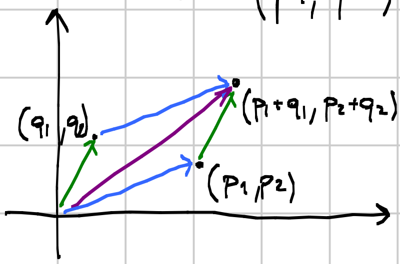
$\vec{X} = (x_1, x_2, x_3, \dots, x_n)$ $\quad n = 2, 3$ o più. $\quad x_i \in \mathbb{R} \quad i = 1, 2, \dots, n$

$\vec{X}, X, x \qquad \vec{Y} = (y_1, y_2, \dots, y_n)$

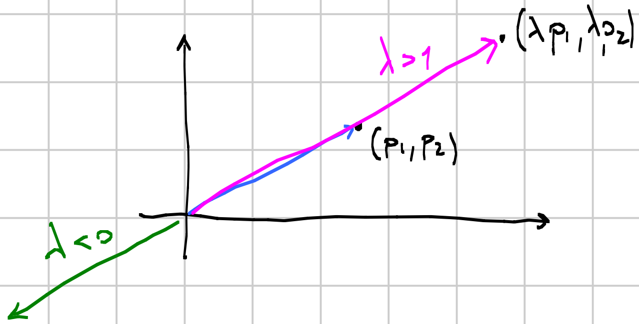
$\vec{X} + \vec{Y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ somma

$\lambda \in \mathbb{R} \qquad \lambda \vec{X} = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$ prod per scalare

$n=2$ $\qquad \vec{P}=(p_1,p_2)$ $\qquad \vec{Q}=(q_1,q_2)$ $\qquad \vec{P}+\vec{Q}=(p_1+q_1,\,p_2+q_2)$



$\vec{P}$ è il punto o la freccia a seconda di cosa ci fa comodo



- Prodotto scalare

$$\mathbb{R} \ni \vec{X}\cdot\vec{Y} = \langle \vec{X},\vec{Y}\rangle = (\vec{X},\vec{Y}) = x_1y_1 + \underset{\text{simmetrico}}{x_2y_2} + \dots + x_ny_n = \sum_{i=1}^{n} x_iy_i$$

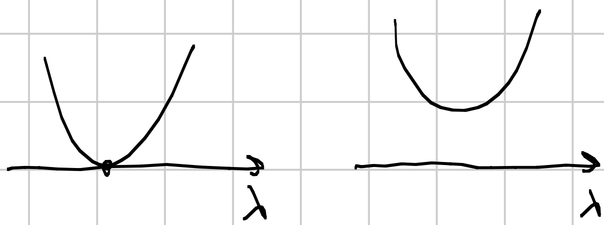
$$0 \le \vec{X}\cdot\vec{X} = \sum_{i=1}^{n} x_i^2 =: \|\vec{X}\|^2$$ norma (o lunghezza) del vettore al quadrato

$$(\lambda\vec{X})\cdot\vec{Y} = (\lambda x_1,\dots)\cdot(y_1,\dots) = \lambda x_1y_1 + \lambda x_2y_2 + \dots = \lambda(\vec{X}\cdot\vec{Y}) = \vec{X}\cdot\lambda\vec{Y}$$

$$(\vec{X}+\vec{Y})\cdot\vec{Z} = \dots = \vec{X}\cdot\vec{Z} + \vec{Y}\cdot\vec{Z}$$ distributiva

$$0 \le \|\vec{X}+\lambda\vec{Y}\|^2 = (\vec{X}+\lambda\vec{Y})\cdot(\vec{X}+\lambda\vec{Y}) = \vec{X}\cdot(\vec{X}+\lambda\vec{Y}) + \lambda\vec{Y}\cdot(\vec{X}+\lambda\vec{Y})$$

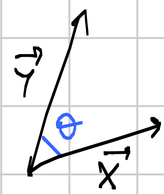
$$= \vec{X}\cdot\vec{X} + \lambda\vec{X}\cdot\vec{Y} + \lambda\vec{Y}\cdot\vec{X} + \lambda^2\vec{Y}\cdot\vec{Y} = \|\vec{X}\|^2 + 2(\vec{X}\cdot\vec{Y})\lambda + \|\vec{Y}\|^2\lambda^2$$



$$\frac{\Delta}{4} = (\vec{X}\cdot\vec{Y})^2 - \|\vec{X}\|^2\|\vec{Y}\|^2 \le 0$$

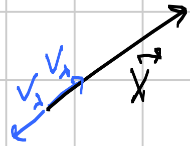
$$|\vec{X}\cdot\vec{Y}| \le \|\vec{X}\|\,\|\vec{Y}\| \qquad \text{C-S}$$

$n=2$ $\qquad \vec{X}\cdot\vec{Y} := x_1y_1 + x_2y_2 \overset{\text{thm}}{=} \|\vec{X}\|\,\|\vec{Y}\|\cos\theta$



in C-S vale = se e solo se $\vec{X} \parallel \vec{Y}$ $\qquad \vec{X}=\nu\vec{Y}$ $\quad \nu\in\mathbb{R}$

$$\frac{\vec{X}\cdot\vec{Y}}{\|\vec{X}\|\|\vec{Y}\|} = \cos\theta$$ lo dimostro



$$\frac{\vec{X}\cdot\vec{Y}}{\|\vec{X}\|\|\vec{Y}\|} = \frac{\vec{X}}{\|\vec{X}\|}\cdot\frac{\vec{Y}}{\|\vec{Y}\|} = (\cos\alpha, \operatorname{sen}\alpha)\cdot(\cos\beta, \operatorname{sen}\beta) = \cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta$$

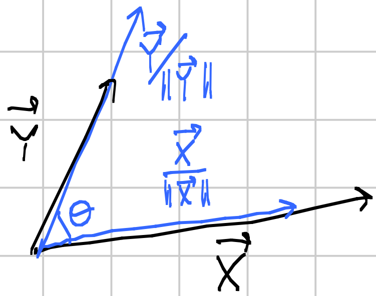
$$= \cos(\alpha-\beta) = \cos(\beta-\alpha) = \cos\theta$$

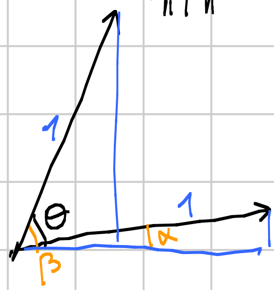
$\vec{V_\lambda} := \lambda\vec{X} \quad // \vec{X}$ $\qquad \|\vec{V_\lambda}\| = \|\lambda\vec{X}\| = \sqrt{\|\lambda\vec{X}\|^2} = \sqrt{\lambda\vec{X}\cdot\lambda\vec{X}} = |\lambda|\sqrt{\vec{X}\cdot\vec{X}} = |\lambda|\|\vec{X}\|$

$\|\lambda\vec{X}\| = |\lambda|\|\vec{X}\|$

$1 = \|\vec{V_\lambda}\| = |\lambda|\|\vec{X}\| \qquad \lambda = \pm\|\vec{X}\|^{-1}$

$\vec{Y}/\|\vec{Y}\| = (\cos\beta, \operatorname{sen}\beta)$



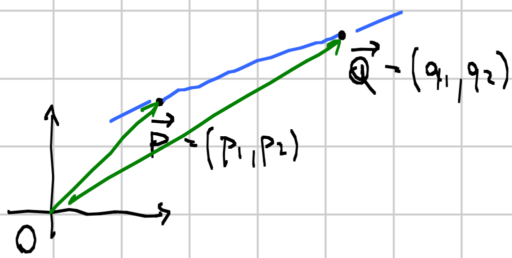


$\vec{X}/\|\vec{X}\| = (\cos\alpha, \operatorname{sen}\alpha)$

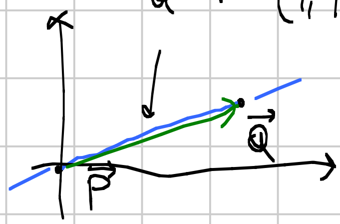
$\theta = \beta - \alpha$

- Retta per due punti

$\vec{Q} - \vec{P} = (q_1 - p_1, q_2 - p_2)$



traslo $\vec{P}$ nell'origine (sottraggo $\vec{P}$)

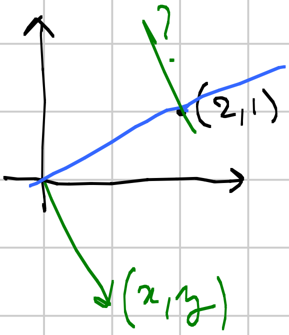


$\lambda \in \mathbb{R}$ $\quad \lambda(\vec{Q} - \vec{P})$ sono i punti della retta

sommo $\vec{P}$

$$\lambda(\vec{Q} - \vec{P}) + \vec{P}$$
$$\shortparallel$$
$$\lambda\vec{Q} + (1-\lambda)\vec{P}$$ di nuovo combinazione convessa di $\vec{P}$ e $\vec{Q}$

- Retta ortogonale



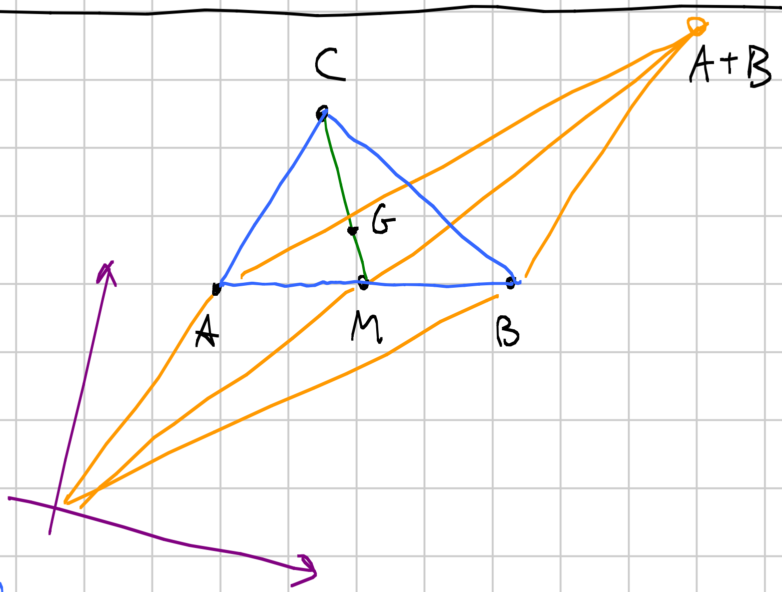
la direzione verde deve essere ortogonale a quella blu

$$0 = (2,1)\cdot(x,y) = 2x + y \quad \text{ad es} \quad x=1, \; y=-2$$

$$(1,-2) \text{ è ortogonale a } (2,1)$$

retta cercata: $(2,1) + \lambda(1,-2) = (2+\lambda,\, 1-2\lambda)$

Punti dei triangoli



- Baricentro

$$\vec{M} = \frac{\vec{A}+\vec{B}}{2}$$

comb. lin. conv.
⇒ non dipende da O

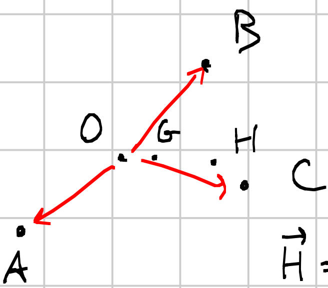
$$\vec{G} = \frac{2}{3}\vec{M} + \frac{1}{3}\vec{C} = \frac{2}{3}\left(\frac{\vec{A}+\vec{B}}{2}\right) + \frac{1}{3}\vec{C} = \frac{\vec{A}+\vec{B}+\vec{C}}{3}$$

baricentro della fisica

$$\boxed{\vec{G} = \frac{\vec{A}+\vec{B}+\vec{C}}{3}}$$

- Circocentro / ortocentro

spesso ci si mette l'origine



- $\|\vec{A}\| = \|\vec{B}\| = \|\vec{C}\|$
- $\boxed{\vec{H} = \vec{A}+\vec{B}+\vec{C}}$ solo se O è l'origine

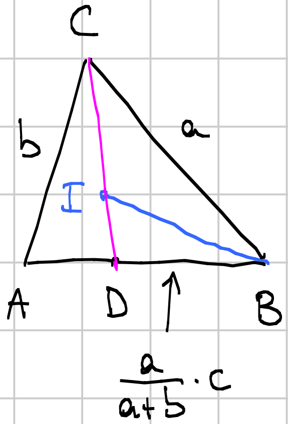
$\vec{H} = 3\vec{G}$

Più in generale

$$\boxed{\vec{H} = \vec{A}+\vec{B}+\vec{C}-2\vec{O}}$$

$$\vec{H} = \vec{O} + \underbrace{(\vec{H}-\vec{O})}_{\overrightarrow{OH}} = \vec{O} + 3(\vec{G}-\vec{O}) = \vec{O} + \vec{A}+\vec{B}+\vec{C}-3\vec{O}$$

- Incentro $\boxed{\vec{I} = \dfrac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}}$ non dipende da O



$$\vec{D} = \lambda\vec{A} + (1-\lambda)\vec{B}$$

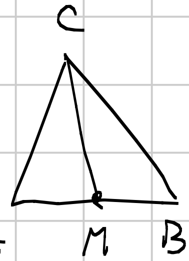
$$AD : DB = AC : CB = b : a$$

$$\vec{D} = \frac{a}{a+b}\vec{A} + \frac{b}{a+b}\vec{B}$$

$$\vec{I} = \frac{\frac{ac}{a+b}}{\frac{ac}{a+b}+a}\vec{C} + \frac{a}{\frac{ac}{a+b}+a}\vec{D}$$

$$\vec{I} = \frac{\cancel{a}c}{\cancel{a}c + a^2 + \cancel{a}b}\vec{C} + \frac{\cancel{a+b}}{a+b+c}\cdot\frac{1}{\cancel{a+b}}\left(a\vec{A} + b\vec{B}\right) = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}$$

- Lunghezza mediana



$$\overline{CM} = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

$$\vec{M} = \frac{\vec{A} + \vec{B}}{2} \qquad \overline{CM} = \|\vec{C} - \vec{M}\| = \left\|\frac{2\vec{C} - \vec{A} - \vec{B}}{2}\right\|$$

$$\overline{CM}^2 = \frac{1}{4}\left(2\vec{C} - \vec{A} - \vec{B}\right)\cdot\left(2\vec{C} - \vec{A} - \vec{B}\right)$$

$$= \frac{1}{4}\left(4\|\vec{C}\|^2 + \|\vec{A}\|^2 + \|\vec{B}\|^2 + 2\vec{A}\cdot\vec{B} - 4\vec{A}\cdot\vec{C} - 4\vec{B}\cdot\vec{C}\right)$$

$$a^2 = \overline{BC}^2 = \|\vec{B} - \vec{C}\|^2 = \|\vec{B}\|^2 + \|\vec{C}\|^2 - 2\vec{B}\cdot\vec{C}$$

$$b^2 = \|\vec{A}\|^2 + \|\vec{C}\|^2 - 2\vec{A}\cdot\vec{C}$$

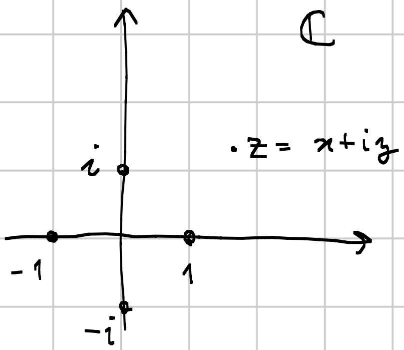
$$c^2 = \|\vec{A}\|^2 + \|\vec{B}\|^2 - 2\vec{A}\cdot\vec{B}$$

$$2a^2 + 2b^2 - c^2 = \|\vec{A}\|^2 + \|\vec{B}\|^2 + 4\|\vec{C}\|^2 + 2\vec{A}\cdot\vec{B} - 4\vec{B}\cdot\vec{C} - 4\vec{A}\cdot\vec{C}$$

$$\overline{CM}^2 = \frac{1}{4}\left(2a^2 + 2b^2 - c^2\right)$$

- altri esempi ... alla fine se riesco

COMPLESSI



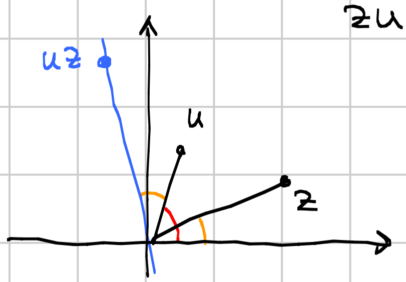
come i vettori, ma in più:

- coniugio (simmetrie vs rette)
- prodotto (rotazioni)

regola del parallelogramma vale sempre

$z = x+iy \qquad u = r+it \qquad \in \mathbb{C}$

$zu \in \mathbb{C} \qquad zu = (x+iy)(r+it) = xr - yt + i(yr + xt)$



$|z| = x^2+y^2 = z\bar{z}$

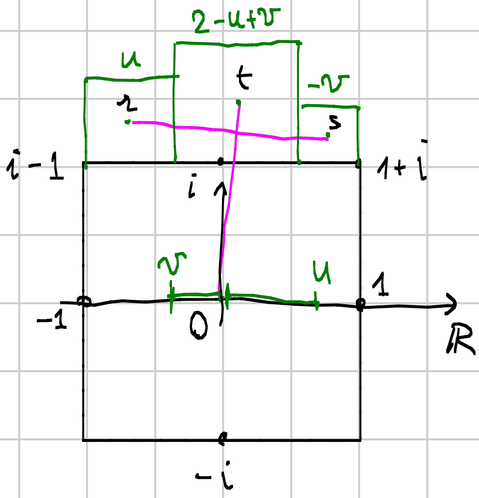
$|zu| = |z||u|$

$\arg(zu) = \arg z + \arg u$

se $|u| = 1$ moltiplicare per u esegue una rotazione

moltiplicare per i ↺ di $90°$

Es 1



$u, v \in \mathbb{R} \qquad u \in (0,1) \qquad v \in (-1,0)$

$r = i - 1 + \frac{u}{2} + i\frac{u}{2} = -1 + \frac{u}{2} + i\left(1 + \frac{u}{2}\right)$

$s = 1 + i + \frac{v}{2} - i\frac{v}{2} = 1 + \frac{v}{2} + i\left(1 - \frac{v}{2}\right)$

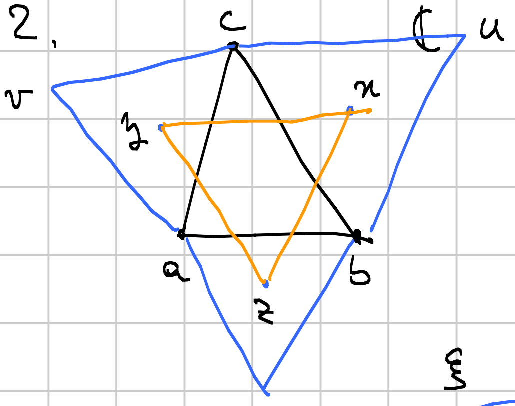
$t = i - \cancel{1} + u + \cancel{1} - \frac{u}{2} + \frac{v}{2} + i\left(1 - \frac{u}{2} + \frac{v}{2}\right)$

$= \frac{u+v}{2} + i\left(2 - \frac{u}{2} + \frac{v}{2}\right)$

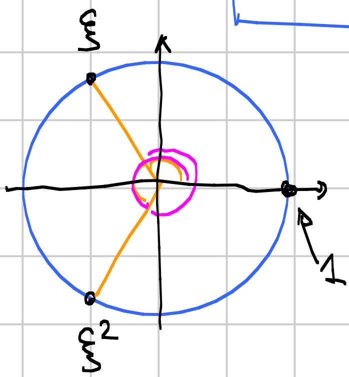
$s - r = 2 + \frac{v}{2} - \frac{u}{2} - i\left(\frac{u+v}{2}\right)$

$i(s-r) = t \qquad$ finito

Es 2.



> Lemma: u, v, w formano un triangolo equilatero se
> $$u+\xi v+\xi^2 w=0$$
> dove ξ è una delle radici terze dell'unità ($\xi \neq 1$)



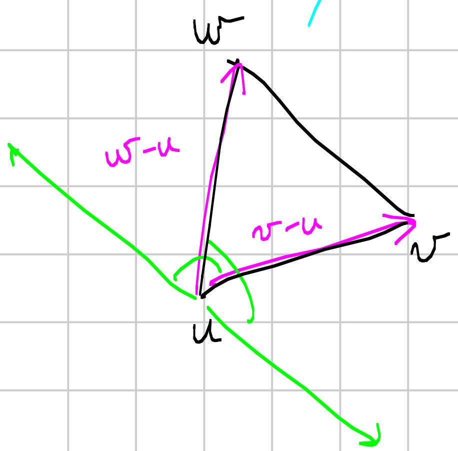
$$x^n-1=0$$

$$\xi = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$(1+\xi+\xi^2)(1-\xi)=0 \qquad 1+\xi+\xi^2=0$$

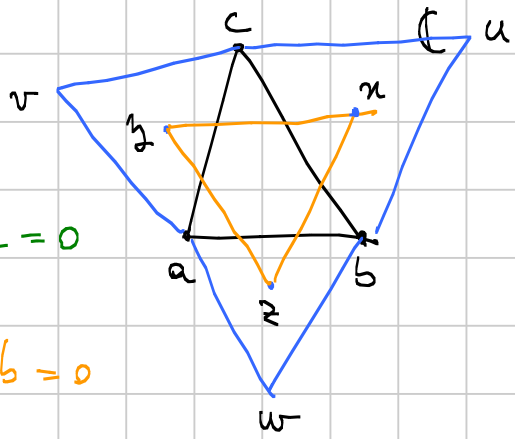
$$\cancel{u}+\xi v+\xi^2 w-\cancel{u}-\xi u-\xi^2 u=\xi(v-u)+\xi^2(w-u)=0$$

sse equil.



Occhio: se ξ è quello che ho scelto, u, v, w vanno presi in senso antiorario

torno all'esercizio



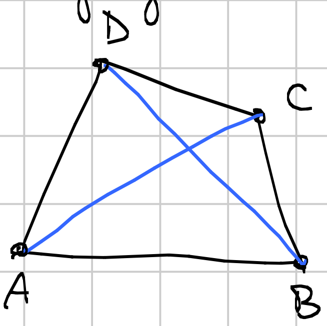
$$b+\xi u+\xi^2 c=0 \quad \cdot\xi^2 \quad \xi^2 b+u+\xi c=0$$
$$c+\xi v+\xi^2 a=0$$
$$a+\xi w+\xi^2 b=0 \quad \cdot\xi \quad \xi a+\xi^2 w+b=0$$

$x=\dfrac{b+u+c}{3}$ e analoghi

$$3(x+\xi y+\xi^2 z)=b+u+c+\xi(c+v+a)+\xi^2(a+w+b)=0$$

- Disuguaglianza di Tolomeo

$\mathbb{C}$



$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$

uguaglianze

$\overline{AB} = |a-b|$ e analoghi

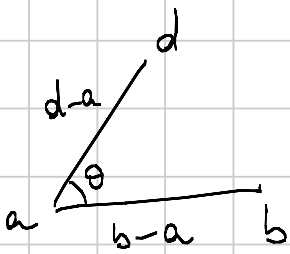
$|a-b||c-d| + |b-c||d-a| - |a-c||b-d| \overset{?}{\geq} 0$

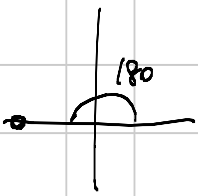
$|ac-ad-bc+bd| + |bd-ba-cd+ac| - |ab-ad-bc+cd| \geq$

$\geq |\cancel{a}c - ad - bc + \cancel{b}d - \cancel{b}d + ba + cd - \cancel{a}c| - |ab-ad-bc+cd| = 0$

vale = solo se

$\dfrac{(a-b)(c-d)}{(b-c)(a-d)} > 0$ reale e positivo

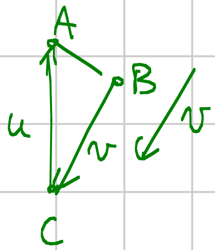




$\underbrace{\dfrac{b-a}{d-a} \cdot \dfrac{d-c}{b-c}}_{\arg = 180} < 0$

$\arg = 180 \iff \alpha + \gamma = 180$

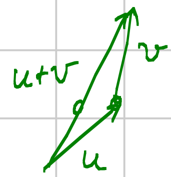
quindi ciclico o allineati



disuguaglianza triangolare

a, b, c

$\underbrace{|a-b|}_{u+v} \leq \underbrace{|a-c|}_{u} + \underbrace{|c-b|}_{v}$



$\forall u, v \in \mathbb{C}$

$|u+v| \leq |u| + |v|$

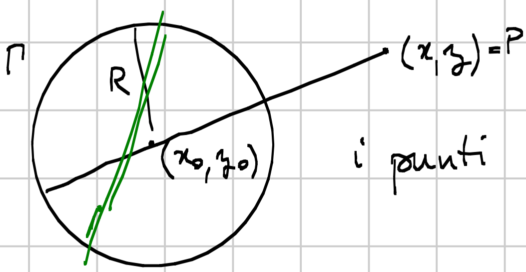
$\forall \vec{X}, \vec{Y}$

$\|\vec{X} + \vec{Y}\| \leq \|\vec{X}\| + \|\vec{Y}\|$

★ vale = se $u = \lambda v$ $\lambda > 0$

Torno alle cartesiane

- (Bisettrici), potenza, asse radicale ... alla fine, se riesco



$Pow_\Gamma(P) = d^2 - R^2 = (x - x_0)^2 + (y - y_0)^2 - R^2$

i punti della circonferenza sono quelli con $Pow_\Gamma(P) = 0$

Γ_1, Γ_2 due circonferenze $\quad \{P : \mathrm{Pow}_{\Gamma_1}(P) = \mathrm{Pow}_{\Gamma_2}(P)\}$ $\quad P=(x,y)$

$$(x-x_1)^2+(y-y_1)^2-R_1^2 = (x-x_2)^2+(y-y_2)^2-R_2^2$$

viene lineare :

$$\underbrace{2(x_2-x_1)}_{a}x + \underbrace{2(y_2-y_1)}_{b}y + \underbrace{x_1^2+y_1^2-x_2^2-y_2^2-R_1^2+R_2^2}_{c} = 0$$