G1 - MEDIUM TRIGONOMETRIA

Titolo nota — Maria — 03/09/2012

- Numeri complessi
- Triangoli
- Ex centri
- Teorema di Casey e Feuerbach.

<u>Numeri complessi</u>

$$a+ib = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta)$$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = e^{i\alpha}\cdot e^{i\beta} = e^{i(\alpha+\beta)}$$

$$\| \qquad\qquad = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$$

<u>Es:</u> $\cos 4\alpha$

$$\cos 4\alpha + i\sin 4\alpha = e^{i4\alpha} = (e^{i\alpha})^4 = (\cos\alpha + i\sin\alpha)^4$$

$$\cos 4\alpha = \mathrm{Re}(\cos\alpha + i\sin\alpha)^4 = \cos^4\alpha - 6\sin^2\alpha\cos^2\alpha + \sin^4\alpha$$

<u>Es:</u>

$$\sum_{k=0}^{n} \sin^2 kx = \frac{1}{2}\sum_{k=0}^{n}[1 - \cos 2kx]$$

$$= \frac{n+1}{2} - \frac{1}{2}\underbrace{\sum_{k=0}^{n}\cos 2kx}_{= S}$$

$$S = \sum_{k=0}^{n} \mathrm{Re}\, e^{i2kx} = \mathrm{Re}\sum_{k=0}^{n} e^{i2kx}$$

$$= \mathrm{Re}\left\{\frac{e^{i2x(n+1)} - 1}{e^{i2x} - 1}\right\}$$

$$= \mathrm{Re}\left\{\frac{(e^{i2x(n+1)} - 1)(e^{-i2x} - 1)}{2 - 2\cos 2x}\right\}$$

$$= \mathrm{Re}\left\{\frac{e^{i2xn} - e^{i2x} - e^{i2x(n+1)} + 1}{2 - 2\cos 2x}\right\}$$

$$= \frac{\boxed{\cos 2xn} - \cos 2x \boxed{- \cos 2x(n+1)} + 1}{2 - 2\cos 2x}$$

$$= \frac{\cancel{2}\sin (2n+1)x \cdot \sin x + \cancel{2} \sin^2 x}{2 \cancel{4} \sin^2 x}$$

$$= \frac{\sin (2n+1)x}{2 \sin x} + \frac{1}{2} .$$

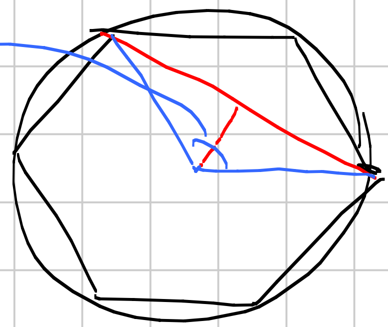
$1 - \cos 2x = 2 \sin^2 x$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

Es: n-agono regolare. Quanto vale il prodotto di lati e diagonali uscenti da un vertice?

$k = 1, \dots, n-1$.

$\frac{2k\pi}{n}$



$$\text{diagonale} = 2 \sin \frac{k\pi}{n}$$

$$\prod_{k=1}^{n-1} 2 \sin \frac{k\pi}{n} = 2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$$

$$= \cancel{2^{n-1}} \prod_{k=1}^{n-1} \left| \frac{e^{\frac{ik\pi}{n}} - e^{-\frac{ik\pi}{n}}}{\cancel{2}} \right|$$

$|ab| = |a| \cdot |b|$

$$= \prod_{k=1}^{n-1} \left| e^{\frac{2ik\pi}{n}} - 1 \right| \cdot \underbrace{\left| e^{-\frac{ik\pi}{n}} \right|}_{=1}$$

$$= \left| \prod_{k=1}^{n-1} \left(e^{i\frac{2k\pi}{n}} - 1 \right) \right|$$

$$= \left| \prod_{k=1}^{n-1} (\zeta_k - 1) \right| = |p(1)| = n .$$

$$p(x) = \prod_{k=1}^{n-1} (\zeta_k - x) = \pm \prod_{k=1}^{n-1} (x - \zeta_k) \cdot \frac{(x-1)}{x-1}$$

$$= \pm \frac{x^n - 1}{x - 1} = \pm \left(x^{n-1} + x^{n-2} + \dots + x + 1 \right)$$

$$|p(1)| = n$$

Es: $x_1, \dots, x_5$ numeri reali. No esistono due tali che

$$|x_i x_j + 1| > |x_i - x_j|.$$

$$x_i = \operatorname{tg} \alpha_i \qquad \alpha_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$1 > \frac{|\operatorname{tg}\alpha_i - \operatorname{tg}\alpha_j|}{|1 + \operatorname{tg}\alpha_i \operatorname{tg}\alpha_j|} = |\operatorname{tg}(\alpha_i - \alpha_j)|$$

$$\iff |\alpha_i - \alpha_j| < \pi/4 .$$

ABC triangolo $\quad A, B, C > 0 \quad A+B+C = \pi$

① $$\tan\frac{A}{2} \cdot \tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2} \cdot \tan\frac{A}{2} = 1$$

$C = \pi - A - B$

$$\tan\frac{C}{2} = \tan\left(\frac{\pi}{2} - \frac{A}{2} - \frac{B}{2}\right) = \frac{1}{\operatorname{tg}\left(\frac{A}{2} + \frac{B}{2}\right)} = \frac{1 - \operatorname{tg}\frac{A}{2} \cdot \operatorname{tg} B/2}{\operatorname{tg}\frac{A}{2} + \operatorname{tg}\frac{B}{2}}.$$

(1.1) $$\prod_{A,B,C} \tan\frac{A}{2} \le \frac{1}{3\sqrt{3}}$$

$$\left(\prod_{A,B,C} \tan\frac{A}{2}\right)^{2/3} \overset{\text{AM-GM}}{\le} \frac{\sum \tan\frac{A}{2}\tan B/2}{3} = \frac{1}{3}$$

(1.2) $$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Divido ① per $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

② $$\operatorname{tg} A + \operatorname{tg} B + \operatorname{tg} C = \operatorname{tg} A \cdot \operatorname{tg} B \cdot \operatorname{tg} C$$

$$\operatorname{tg} C = \operatorname{tg}(\pi - A - B) = -\operatorname{tg}(A+B) = -\frac{\operatorname{tg} A + \operatorname{tg} B}{1 - \operatorname{tg} A \operatorname{tg} B}$$

(2.1) $$\prod \operatorname{tg} A \ge 3\sqrt{3}$$

$$\frac{\prod \operatorname{tg} A}{3} = \frac{\sum \operatorname{tg} A}{3} \ge \left(\prod \operatorname{tg} A\right)^{1/3}$$

(2.2) $$\cot A \cot B + \cot B \cdot \cot C + \cot C \cdot \cot A = 1$$

③ $$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1$$

(ex)

Valgono anche: viceversa.

$x, y, z > 0$ verificano $x^2+y^2+z^2+2xyz=1$

$\Rightarrow$ $\exists$ ABC triangolo t.c. $x = \sin\frac{A}{2}$ $\quad y = \sin\frac{B}{2}$ $\quad z = \sin\frac{C}{2}$.

$\exists\, A, B \in (0, \pi)$

$x = \sin\frac{A}{2}$, $\quad y = \sin\frac{B}{2}$. $\quad$ Consideriamo $C = \pi - A - B$.

Fissati x, y, z è determinato: infatti

$$z = -xy \pm \sqrt{x^2y^2 - x^2 - y^2 + 1}$$

e per avere $z > 0$ devo prendere $+$.

Bisogna vedere $A+B<\pi$, poi sappiamo che $z = \sin\frac{\pi - A - B}{2}$ risolve.

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} = x^2+y^2 < 1$$

$$1-\cos^2\frac{A}{2} + \sin^2\frac{B}{2} < 1 \iff \sin\frac{B}{2} < \cos\frac{A}{2}$$

$$\iff \frac{A}{2} + \frac{B}{2} < \frac{\pi}{2}.$$

Es: $x, y, z > 1$ $\quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \overset{?}{\leq} \sqrt{x+y+z}$$

$(x, y, z) \rightsquigarrow (a+1, b+1, c+1)$ $\quad a, b, c > 0$

con $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$

$$(b+1)(c+1) + (a+1)(c+1) + (a+1)(b+1) = 2(a+1)(b+1)(c+1)$$

$$\sum_c ab + \cancel{2\sum_c a} + 3 = 2\left(abc + \sum_c ab + \cancel{\sum_c a} + 1\right)$$

$$1 = 2abc + ab + bc + ca$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \overset{?}{\leq} \sqrt{a+b+c+3}$$

$$\cancel{a+b+c} + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) \overset{?}{\leq} \cancel{a+b+c} + 3$$

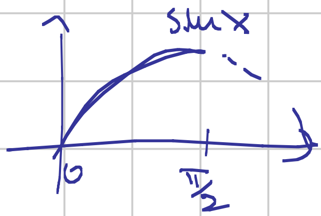
Poniamo (grazie al fatto precedente)

$\sqrt{ab} = \sin\frac{A}{2}$ $\quad \sqrt{bc} = \sin\frac{B}{2}$ $\quad \sqrt{ac} = \sin\frac{C}{2}$

Tesi: $\iff$ $\underbrace{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}_{S} \overset{?}{\leq} \frac{3}{2}$

(Jensen)

$$\frac{S}{3} \leq \sin\left(\frac{A+B+C}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$



Es: $a, b, c \in (0,1)$ t.c. $ab + bc + ca = 1$. Allora
$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4}\left(\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c}\right).$$

(ex).

Es: (4) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$

(5) $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \geq 3/4$

(6) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \geq 9/4$

(7) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{8}$

Es: $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$.

ABC triangolo, r = raggio inscritto, R = r. circoscr., S = area, s = semiperimetro

(1) $S = \frac{abc}{4R}$

$S = \frac{1}{2} ab \sin C = \frac{1}{2} ab \frac{c}{2R}$.

(2) $S = 2R^2 \sin A \sin B \sin C$

In (1) metto $a = 2R\sin A$, ...

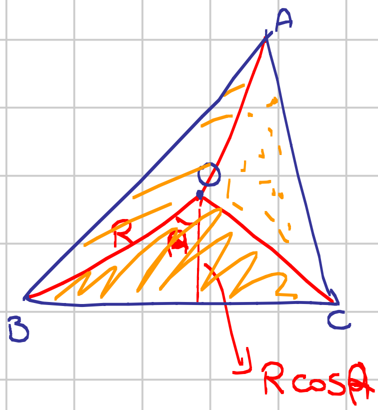
(3) $2R \sin A \cdot \sin B \cdot \sin C = r(\sin A + \sin B + \sin C)$

$$2R^2 \sin A \sin B \sin C \overset{(2)}{=} S = r\frac{(a+b+c)}{2} = $$
$$= rR(\sin A + \sin B + \sin C)$$

(4) $a\cos A + b\cos B + c\cos C = \frac{abc}{2R^2}$

(ex: dimostrarla "algebricamente")

$$\frac{aR\cos A}{2} + \frac{bR\cos B}{2} + \frac{cR\cos C}{2} \overset{?}{=} S$$



⑤ $\frac{r}{R} = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2} \overset{\text{carnot}}{=} \frac{1 - \frac{-a^2+b^2+c^2}{2bc}}{2} = \frac{1}{4bc}(2bc-b^2-c^2+a^2)$$

$$= \frac{1}{4bc}[a^2-(b-c)^2] = \frac{1}{4bc}(a+b-c)(a-b+c)$$

$$= \frac{(s-c)(s-b)}{bc} \qquad \text{[formule di Briggs]}$$

$$\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = \frac{s(s-a)(s-b)(s-c)}{abc\, s} = \frac{S^{\cancel{2}}}{4R\cancel{S}\, s} = \frac{r}{4R}$$

⑥ $4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = s$

$$4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \cancel{4}R\ \frac{\sin A}{\cancel{2}\sin A/2} \cdot \frac{\sin B}{\cancel{2}\sin B/2} \cdot \frac{\sin C}{2\sin C/2}$$

$$\overset{②}{=} \frac{\cancel{R}}{\cancel{2}} \frac{S}{\cancel{2}\cancel{R}^{\cancel{2}}} \cdot \frac{\cancel{4R}}{r} = s$$

⑦ $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1 + \frac{r}{R}$

Dim 1 di:

$$\cos A + \cos B \overset{\text{ex: perché?}}{=} 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 2\sin\frac{C}{2}\cos\frac{A-B}{2}$$

$$\cos C - 1 = -2\sin^2\frac{C}{2}$$

$$\cos A + \cos B + \cos C - 1 = 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]$$

$$= 2\sin\frac{C}{2}\, 2\sin\frac{A}{2} \cdot \sin\frac{B}{2}.$$

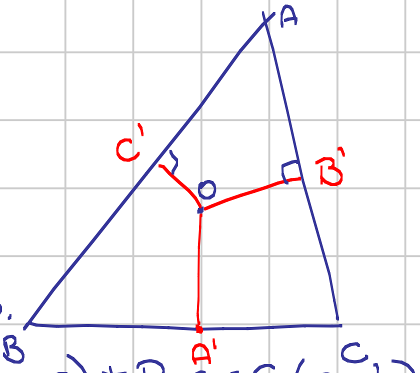
Dim 2:

$$R\cos A + R\cos B + R\cos C \overset{?}{=} R + r$$

Tolomeo su $AB'OC'$

$$AO\cdot B'C' = OB'\cdot AC' + OC'\cdot AB'$$

$$R\cdot \frac{a}{\cancel{2}} = R\cos B\,\frac{c}{\cancel{2}} + R\cos C\,\frac{b}{\cancel{2}}$$



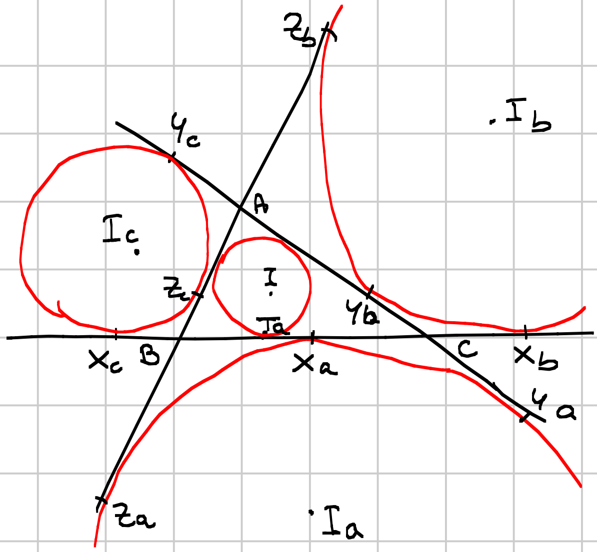
Faccio relazioni cicliche e sommo.

$$R(a+b+c) = R\cos B(a+c) + R\cos A(b+c) + R\cos C(a+b)$$

$$R\cdot \cancel{2s} = (R\cos B + R\cos A + R\cos C)\cancel{2s} - R(a\cos A + b\cos B + c\cos C)\frac{1}{2s}$$

$$= (R\cos B + R\cos A + R\cos C)\cancel{2s} - \frac{\cancel{2}S}{s}$$

In - excentri



(1) B, I, I_b allineati:
sulla bisettrice di $\hat{B}$

(2) I_a, B, I_c allineati:
sulla bis esterna di $\hat{B}$

(3) $AZ_a = AY_a = \frac{a+b+c}{2}$

$AZ_a = AY_a$

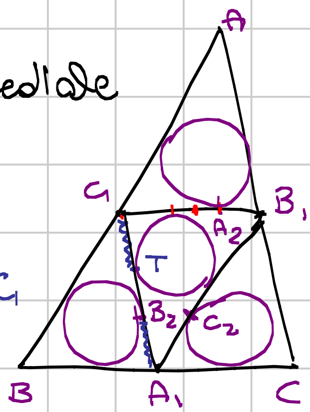
$$AZ_a + AY_a = AB + BZ_a + AC + CY_a$$
$$= c + BX_a + b + CX_a = a+b+c.$$

(4) $BX_a = CT_a = \frac{a+b-c}{2}$

$$BX_a = BZ_a = AZ_a - AB \overset{(3)}{=} \frac{a+b+c}{2} - c$$

(5) AX_a, BY_b, CZ_c concorrono nel pto di Nagel.
Per Ceva + (4).

Esercizio: ABC triangolo, $A_1B_1C_1$ triangolo mediale
Allora AA_2, B_1B_2, C_1C_2 concorrono.

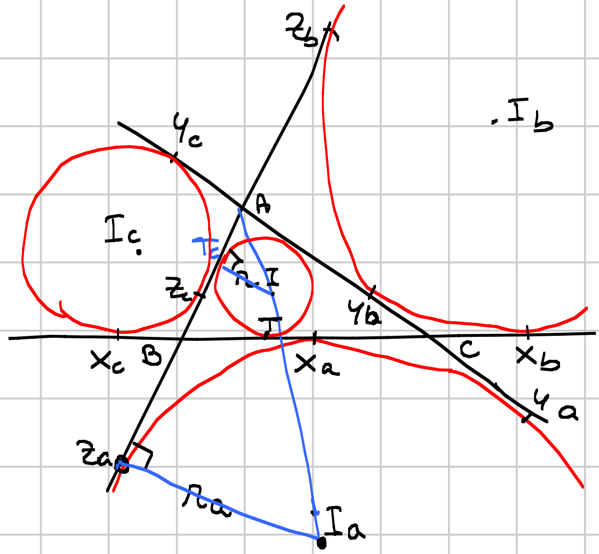


$C_1B_2 = A_1T$
↑ A_1BC_1 è simmetrico (centrale) di $A_1B_1C_1$

B_2 è il pto di tangenza della circo exscritta a $A_1B_1C_1$

Concorrono nel pto di Nagel di $A_1B_1C_1$.

(6) $r_a = \frac{2S}{b+c-a} = \frac{S}{s-a}$



Similitudine AIT_c e AI_aZ_a

$$\frac{r}{r_a} = \frac{AT_c}{AZ_a} \overset{(3)}{=} \frac{\frac{-a+b+c}{2}}{\frac{a+b+c}{2}}$$

$$S = r\left(\frac{a+b+c}{2}\right) = r_a\left(\frac{-a+b+c}{2}\right)$$

(7) $r\, r_a r_b r_c = S^2$

$$r\, r_a r_b r_c = \frac{r s S^3}{s(s-a)(s-b)(s-c)} = \frac{S^4}{S^2}$$

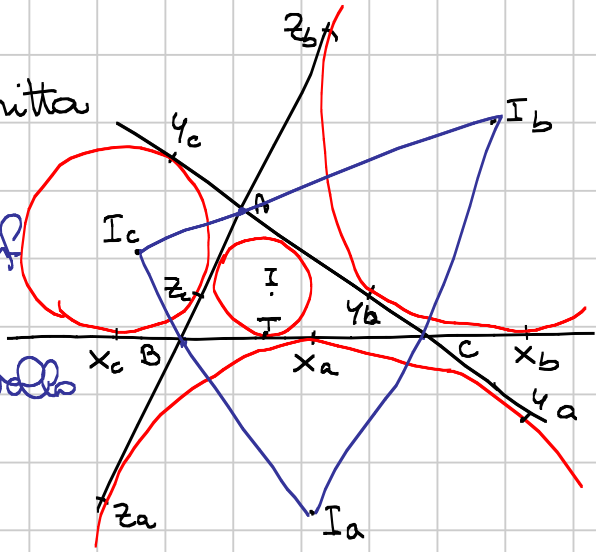
(8) $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{1}{2S}\left[\cancel{b}+c-\cancel{a}+\cancel{a}+\cancel{c}-\cancel{b}+a+b-\cancel{c}\right] = \frac{a+b+c}{2S} = \frac{1}{r}$

(9) $\frac{a^2}{r_a}+\frac{b^2}{r_b}+\frac{c^2}{r_c} = 4(R+r)$

(ex).

(10) raggio della circonferenza circoscritta a $I_aI_bI_c$ = $2R$

Uso (11) e Γ_{ABC} è la circonf di Feuerbach di $I_aI_bI_c$, che ha raggio = metà di quello del triangolo iniziale



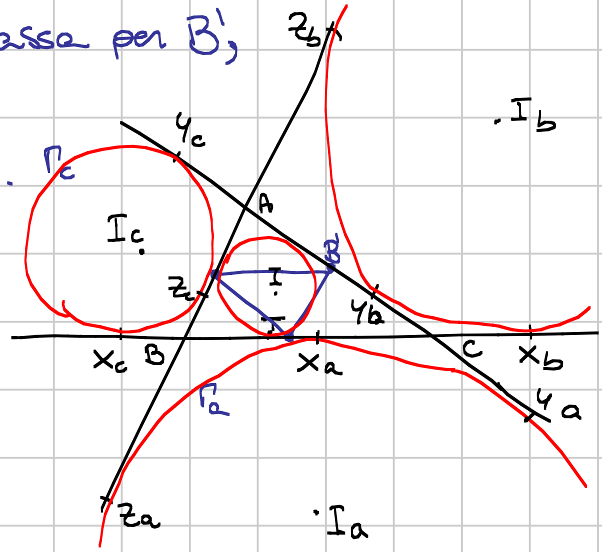
(11) ABC è il triangolo ortico di $I_aI_bI_c$

(12) centro radicale dei 3 excerchi è l'incentro del triangolo mediale di ABC.

l'asse radicale tra Γ_a, Γ_c è $\perp I_aI_c$
e quindi // bisettrice di $\hat{B}$ // bisettrice di $\hat{B}'$.

Vogliamo che l'asse radicale passa per B', cioè $B'Y_c^2 = B'Y_a^2$

$B'Y_c = s - \frac{b}{2} = B'Y_a$. ok.



(13) $BICI_a$ è ciclico con centro M.

BI bisettrice interna $\perp$ BI_a bis est

$\Rightarrow I\hat{B}I_a = 90°$

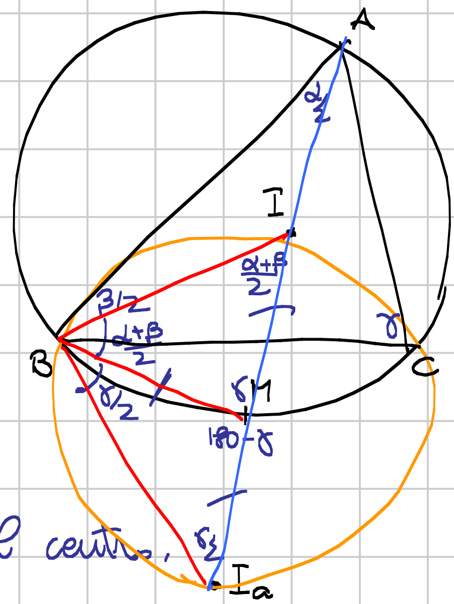
$\Rightarrow BICI_a$ è ciclico, il centro è pto medio di II_a.

Se ABC è scaleno, considero l'asse di BC. L'asse passa per M.

La bisettr di $\hat{A}$ passa per M.

Asse e bis passano anche per il centro.

$\Rightarrow$ M è il centro

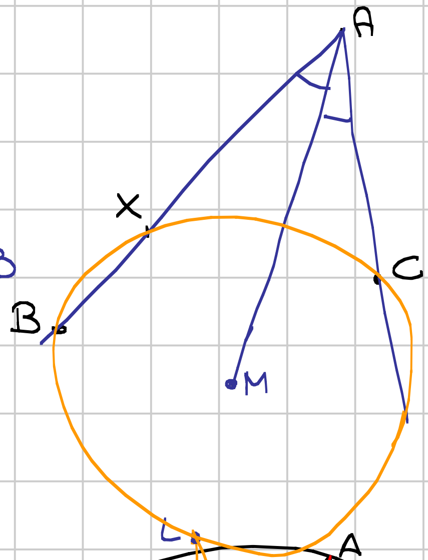


(14) $AI \cdot AI_a = AB \cdot AC$

$AC = AX$ per simmetria

$AI \cdot AI_a = \text{pow}\, A = AX \cdot AB = AC \cdot AB$

(se $AB = AC \Rightarrow B$ e C sono pti di tangenza a ... ex)



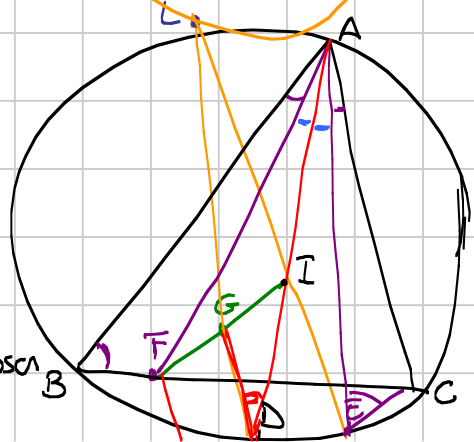
Es: IMO 2010 - 2

ABC triangolo, D pto medio $\widehat{BC}$

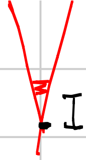
$F \in BC$ $E \in \widehat{BC}$ t.c. $B\hat{A}F = E\hat{A}C$

G pto medio di FI

Allora $DG \cap IE$ sta sulla circonf circoscr



Tesi $\iff$ LDEA è ciclico $\iff G\hat{D}I = I\hat{E}A$



$\iff$ (siccome $FI_a \parallel GD$) $I\hat{E}A = FI_aI$

$\iff F\hat{A}I_a$ è simile a $I\hat{A}E$

$\iff \frac{AF}{AI} = \frac{AI_a}{AE}$, cioè $AI \cdot AI_a = AF \cdot AE$

ABF e AEC sono simili $\Rightarrow \frac{AB}{AE} = \frac{AF}{AC} \iff AB \cdot AC = AF \cdot AE$

$\overset{(16)}{\parallel}$ $AI \cdot AI_a$ ok.

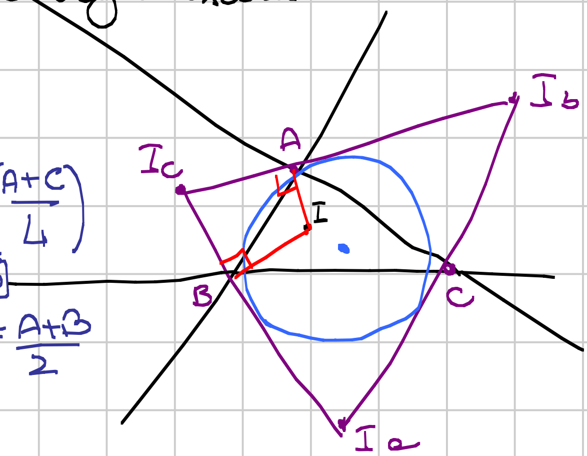
Ricordo: $2r \leq R$

<u>Dim 1</u>: $OI^2 = R^2 - 2Rr > 0$

<u>Dim 2</u>: $\frac{r}{R} = 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \underset{\text{ex precedente}}{\leq} 4 \cdot \frac{1}{8} = \frac{1}{2}$

<u>Es</u>: ABC triangolo, r_0 = raggio del cerchio iscritto nel triangolo degli excentri

Allora $r_0 \geq 2r$



$$\frac{r_0}{2R} \overset{(10)}{=} \frac{r_0}{R_0} \overset{\text{formula precedente}}{=} 4 \sin\left(\frac{A+B}{4}\right) \sin\left(\frac{B+C}{4}\right) \sin\left(\frac{A+C}{4}\right)$$

$$A\hat{I}_cB \overset{\boxed{AIBI_c \text{ ciclico}}}{=} 180 - A\hat{I}B \overset{\boxed{\text{triangolo } AIB}}{=} B\hat{A}I + A\hat{B}I = \frac{A+B}{2}$$

$$\frac{r}{R} \overset{\text{formula precedente}}{=} 4 \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}.$$

Vogliamo $\sin\frac{A+B}{4} \sin\frac{B+C}{4} \sin\frac{A+C}{4} \overset{?}{\geq} \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$.

Dim 1: $\sin^2\frac{A+B}{4} \geq \sin\frac{A}{2} \cdot \sin\frac{B}{2}$ (ex: mostrarlo a mano)

$$\ln \sin\frac{A+B}{4} \overset{?}{\geq} \frac{\ln \sin\frac{A}{2} + \ln \sin\frac{B}{2}}{2}$$

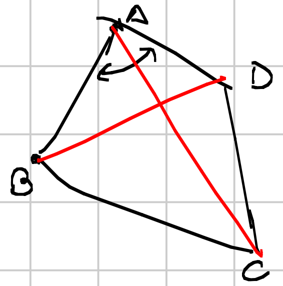
Se $\ln(\sin(x)) = f(x)$ è concava, la disug prec è Jensen.

$f'(x) = \frac{1}{\sin(x)} \cos(x) = \frac{1}{\operatorname{tg} x}$ è decrescente in $(0, \pi/2)$

$\iff f''(x) \leq 0$ in $(0, \pi/2) \iff f$ concava.

<u>Es</u>: $II_a + II_b + II_c = 4R + 2r_0$.

Teo Tolomeo: ABCD quadrilatero. $AC \cdot BD \leq AB \cdot CD + AD \cdot BC$



ABCD ciclico $\iff AC \cdot BD = AB \cdot CD + AD \cdot BC$

Dim: coi complessi:

$$LHS = (a-b)(c-d) + (a+d)(c-b)$$
$$= \cancel{ac} - bc - ad + \cancel{bd} - \cancel{ac} + cd + ab - \cancel{bd}$$
$$= (b-d)(a-c) = RHS$$

Prendo i moduli:

$$|RHS| = BD \cdot AC = |LHS| \leq |(a-b)(c-d)| + |(d-a)(c-b)|$$
$$= AB \cdot CD + AD \cdot BC.$$

C'è $= \iff (a-b)(c-d)$ e $(d-a)(c-b)$ sono allineati

$$\iff \arg \frac{(a-b)(c-d)}{(d-a)(c-b)} = \begin{smallmatrix} 0 \\ \pi \end{smallmatrix}$$

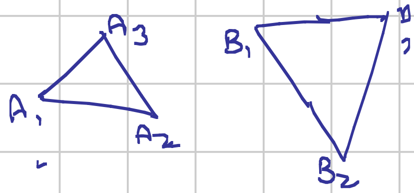
$$\iff \arg \frac{a-b}{d-a} + \arg \frac{c-d}{c-b} = \begin{smallmatrix} 0 \\ \pi \end{smallmatrix}$$

$\iff$ ABCD ciclico (ex)

Parentesi: • $z_1, z_2, z_3, z_4 \in \mathbb{C}$ sono conciclici

$$\iff \frac{\frac{z_3 - z_2}{z_1 - z_2}}{\frac{z_3 - z_4}{z_1 - z_4}} \in \mathbb{R} \setminus \{0\}$$ (ex)

• $a_1 a_2 a_3$, $b_1 b_2 b_3$ triangoli. Sono simili (con la stessa orientazione) $\iff \frac{a_2 - a_1}{a_3 - a_1} = \frac{b_2 - b_1}{b_3 - b_1}$

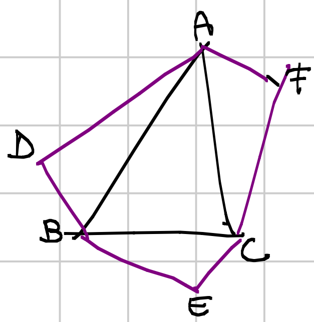


Dim: $\frac{A_2A_1}{A_3A_1} = \frac{B_2B_1}{B_3B_1}$ e $A_1\hat{A}_2A_3 = B_1\hat{B}_2B_3$

Se non hanno la stessa orientazione

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{\bar{b}_2 - \bar{b}_1}{\bar{b}_3 - \bar{b}_1},$$

Es: ABC triangolo. ADB, BEC, CFA simili. Dimostrare che ABC e DEF hanno lo stesso baricentro.



$$\frac{d-a}{b-a} = \frac{e-b}{c-b} = \frac{f-c}{a-c} = z$$

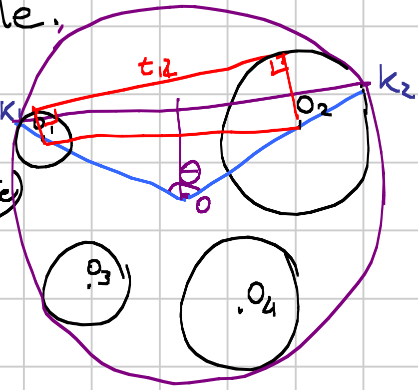
$\Rightarrow d = a + (b-a)z \qquad e = b + (c-b)z \qquad f = c + (a-c)z$

$$\frac{d+e+f}{3} = \frac{a+b+c}{3}.$$

<u>Teorema di Casey</u> $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ disgiunte, t_{ij} = lunghezza della tg esterna a $\Gamma_i\Gamma_j$.

Γ sia tangente esternamente (o internamente) a Γ_i. $\iff$

$$t_{13}t_{24} = t_{12}\cdot t_{34} + t_{14}t_{23}$$



Tolomeo è un corollario se $\Gamma_i = P_i$ pti.

$1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right) = \frac{K_1K_2^2}{2R^2}$

<u>Dim:</u>

<u>Oss:</u> il sist è determinato da r_i, R, $O_1\hat{O}O_2$, $O_2\hat{O}O_3$, $O_3\hat{O}O_4$

Pitagora

$$t_{12}^2 = O_1O_2^2 - (r_1-r_2)^2$$

Carnot su O_1O_2O

$$= (R-r_1)^2+(R-r_2)^2 - 2\cos\theta\,(R-r_1)(R-r_2) - (r_1-r_2)^2$$

$$= 2(R-r_1)(R-r_2)(1-\cos\theta)$$

$$= \frac{(R-r_1)(R-r_2)}{R^2}K_1K_2^2$$

$$t_{12}\cdot t_{34} = \sqrt{\frac{(R-r_1)(R-r_2)(R-r_3)(R-r_4)}{R^4}}\; K_1K_2\cdot K_3K_4$$

Tolomeo

$$= \sqrt{}\cdot(K_1K_3\cdot K_2K_4 - K_2K_3K_1K_2)$$

$$= t_{13}\cdot t_{24} - t_{23}\cdot t_{14}.$$