G 2 Advanced – Sam

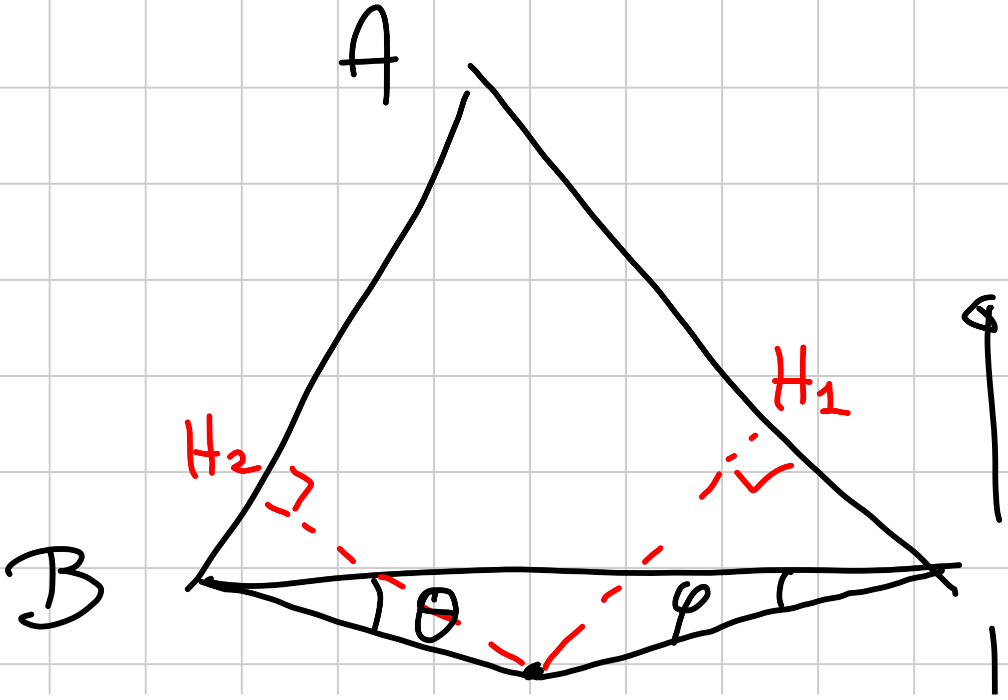
- Algebra lineare
  - $\mathbb{R}^n$
  - $T: \mathbb{R}^n \to \mathbb{R}^m$ lineare
  - Prodotto righe x colonne
  - Basi & $\underset{\perp}{\text{indip.}}$ lineare
  - Determinante
- Coord. baricentriche
  - Vari punti
  - Rette
  - Circonferenze

$$[-1 : -1 : -1] = [1 : 1 : 1]$$

$$[-1 : 1 : 1] = [1 : -1 : -1]$$

$$[ABP] + [APC] + [PBC] = [ABC]$$

$$\underline{\qquad\qquad} \cdot \underline{\qquad\qquad\qquad}$$



$\theta, \varphi < 0$

P

$\downarrow$ $\theta, \varphi > 0$

$$[ABP] = c \cdot \overbrace{BP \cdot \sin(\beta + \theta)}^{PH_2}$$

$$[APC] = b \cdot \underbrace{CP \cdot \sin(\gamma + \varphi)}_{PH_1}$$

$$\frac{BP}{CP} = \frac{\sin\varphi}{\sin\theta}$$

$$\frac{[ABP]}{[APC]} = \frac{c}{b} \cdot \frac{BP}{CP} \cdot \frac{\sin(\beta+\theta)}{\sin(\gamma+\varphi)} =$$

$$= \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin\varphi}{\sin\theta} \cdot \frac{\boxed{\sin\beta\cos\theta} + \boxed{\cos\beta\sin\theta}}{\boxed{\sin\gamma\cos\varphi} + \boxed{\cos\gamma\sin\varphi}} =$$

$$= \frac{\cot\theta + \cot\beta}{\cot\varphi + \cot\gamma}$$

$$\frac{[ABP]}{[PBC]} = \frac{c\cdot\cancel{BP}\cdot\sin(\beta+\theta)}{-a\cdot\cancel{BP}\cdot\sin\theta} = -\frac{\sin\gamma}{\sin\alpha}\cdot\frac{\sin\beta\cos\theta+\cos\beta\sin\theta}{\sin\theta} =$$

$$= -\frac{\sin\gamma\sin\beta}{\sin\alpha}\left(\cot\beta+\cot\theta\right)$$

$$\frac{[APC]}{[PBC]} = -\frac{\sin\gamma\sin\beta}{\sin\alpha}\left(\cot\gamma+\cot\varphi\right)$$

$$\left[\frac{-\sin\alpha}{\sin\gamma\sin\beta},\ \cot\gamma+\cot\varphi,\ \cot\beta+\cot\theta \right] = P$$

$$2\,[ABC]\,\frac{\sin\alpha}{\sin\gamma\sin\beta} = \cancel{2}\cdot\frac{1}{\cancel{2}}\,bc\sin\alpha\,\frac{\sin\alpha}{\sin\beta\sin\gamma} =$$

$$= 2R\cdot 2R\cdot\sin\alpha\cdot\sin\alpha = a^2$$

$S = 2[ABC]$ (Attenti!! 2 volte l'area)

$$P = [-a^2, S\cot\gamma + S\cot\varphi, S\cot\beta + S\cot\theta]$$

$$S\cot\gamma = ab\sin\gamma \cdot \cot\gamma = ab\cos\gamma = \frac{a^2+b^2-c^2}{2}$$

$S_\theta = S\cot\theta$ $\qquad$ $S_\alpha, S_\beta, S_\gamma$

$$P = [-a^2, S_\gamma + S_\varphi, S_\beta + S_\theta]$$

<u>Oss</u>: 1) $S_\alpha + S_\beta = c^2$

2) $S_\alpha S_\beta + S_\beta S_\gamma + S_\gamma S_\alpha = S^2$

$$0 = \operatorname{tg}(\alpha+\beta+\gamma) = \frac{\operatorname{tg}(\alpha) + \operatorname{tg}(\beta+\gamma)}{1 - \operatorname{tg}\alpha \operatorname{tg}(\beta+\gamma)} =$$

$$= \frac{\operatorname{tg}\alpha + \dfrac{\operatorname{tg}\beta + \operatorname{tg}\gamma}{1 - \operatorname{tg}\beta \operatorname{tg}\gamma}}{1 - \operatorname{tg}\alpha \dfrac{\operatorname{tg}\beta + \operatorname{tg}\gamma}{1 - \operatorname{tg}\beta \operatorname{tg}\gamma}} = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma - \operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma}{1 - \operatorname{tg}\beta \operatorname{tg}\gamma - \operatorname{tg}\alpha \operatorname{tg}\beta - \operatorname{tg}\alpha \operatorname{tg}\gamma}$$

$$\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma$$

$$\frac{1}{\operatorname{tg}\beta \operatorname{tg}\gamma} + \frac{1}{\operatorname{tg}\beta \operatorname{tg}\alpha} + \frac{1}{\operatorname{tg}\alpha \operatorname{tg}\gamma} = 1$$

$$\cot\beta\cot\gamma + \cot\beta\cot\alpha + \cot\alpha\cot\gamma = 1$$

$$\boxed{S_\beta S_\gamma + S_\beta S_\alpha + S_\alpha S_\gamma = S^2}$$

Punti più o meno notevoli

$$H = [\mathrm{tg}\,\alpha, \mathrm{tg}\,\beta, \mathrm{tg}\,\gamma] = \left[\frac{1}{S_\alpha}, \frac{1}{S_\beta}, \frac{1}{S_\gamma}\right] =$$

$$= [S_\beta S_\gamma, S_\alpha S_\gamma, S_\alpha S_\beta]$$

$$\mathrm{sen}\,2\alpha =$$

$$O = [a^2 S_\alpha, b^2 S_\beta, c^2 S_\gamma] =$$

$$= 2\,\mathrm{sen}\,\alpha\cos\alpha =$$

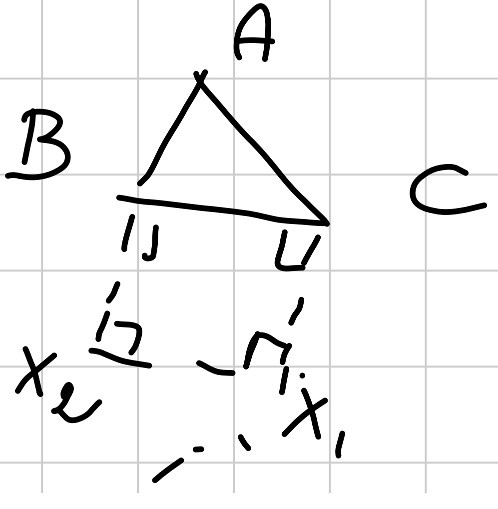
$$= \left[S_\alpha S_\beta + S_\alpha S_\gamma, \; \ldots, \ldots \right] = 2 \operatorname{sen}^2\alpha \cot\alpha$$

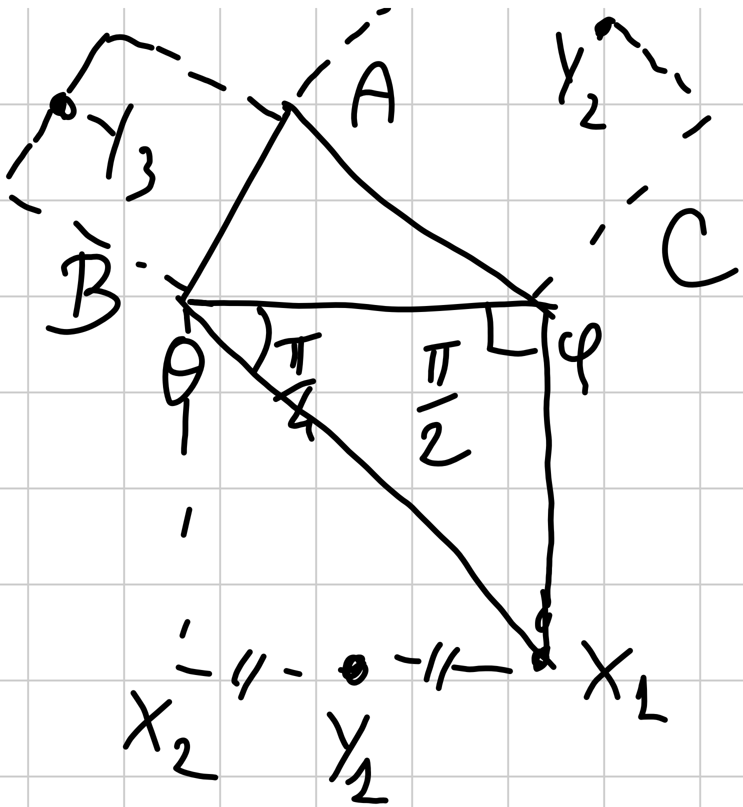
$$H = \frac{1}{2}\left(\vec{O} + \vec{H}\right) = \left[\frac{S_\alpha S_\beta + S_\alpha S_\gamma}{2} + S_\beta S_\gamma, \ldots, \ldots \right] =$$

$$= \left[\frac{S^2}{2} + \frac{S_\beta S_\gamma}{2}, \ldots, \ldots \right] = \left[S^2 + S_\beta S_\gamma, \ldots, \ldots \right]$$

* * *

Esempi:





$X_1 = [-a^2, S_\gamma + \cancel{S_\varphi}, S_\beta + \underset{S}{\underset{\shortparallel}{S_\theta}}]$

$X_1 = [-a^2, S_\gamma, S_\beta + S]$

$X_2 = [-a^2, S_\gamma + S, S_\beta]$

$Y_1 = [-a^2, S_\gamma + \frac{S}{2}, S_\beta + \frac{S}{2}]$

$Y_2 = [S_\gamma + \frac{S}{2}, -b^2, S_\alpha + \frac{S}{2}]$ $Y_3 = [S_\beta + \frac{S}{2}, S_\alpha + \frac{S}{2}, -c^2]$

<u>Id</u>: AY_1, BY_2, CY_3 concorrono

$$AY_1 = \left\{ \left(S_\beta + \frac{S}{2}\right) y = \left(S_\gamma + \frac{S}{2}\right) z \right\}$$

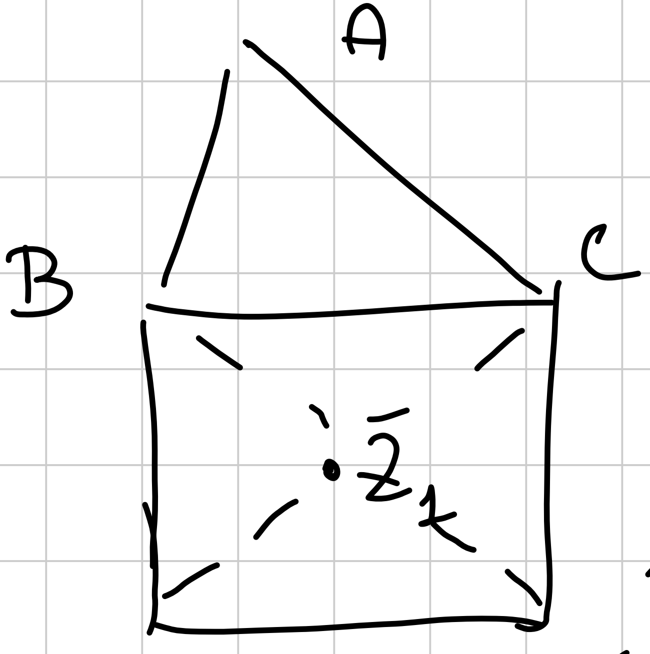
$[$[illegible]$, r, q] \longrightarrow yq = rz$

$[r,$ [illegible]$, p] \longrightarrow xp = rz$

$[q, p,$ [illegible]$] \longrightarrow xp = yq$

Le rette $\longrightarrow$ si incontrano in $\left[\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right]$

$$\left[\frac{1}{S_\alpha + \frac{S}{2}}, \frac{1}{S_\beta + \frac{S}{2}}, \frac{1}{S_\gamma + \frac{S}{2}}\right]$$



$z_1 = \left[-a^2, S_C+S, S_B+S\right]$

Az_1 Bz_2 Cz_3

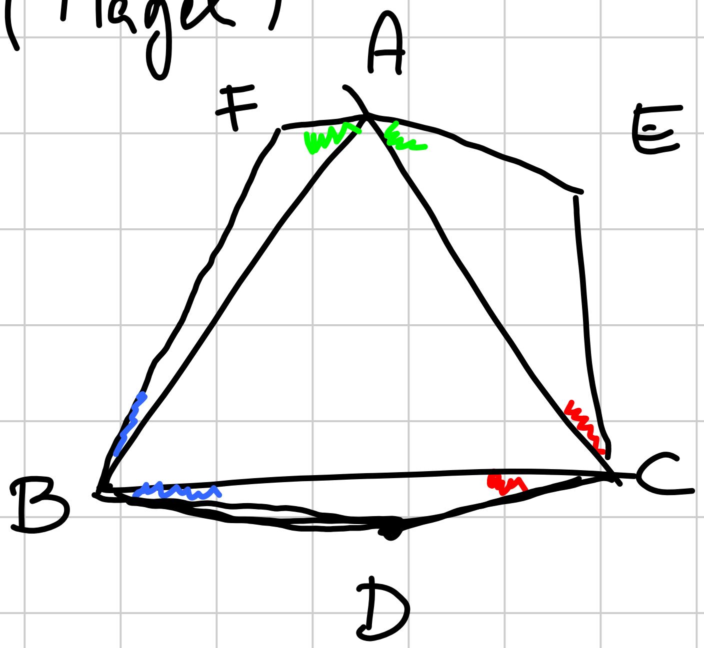
concorrono

in $V = \left[\frac{1}{S_A+S}, \frac{1}{S_B+S}, \frac{1}{S_C+S}\right]$

((primo) punto di Vecten (esterno))

$(Az_1 \perp z_2 z_3)$

Pensata (Nagel)



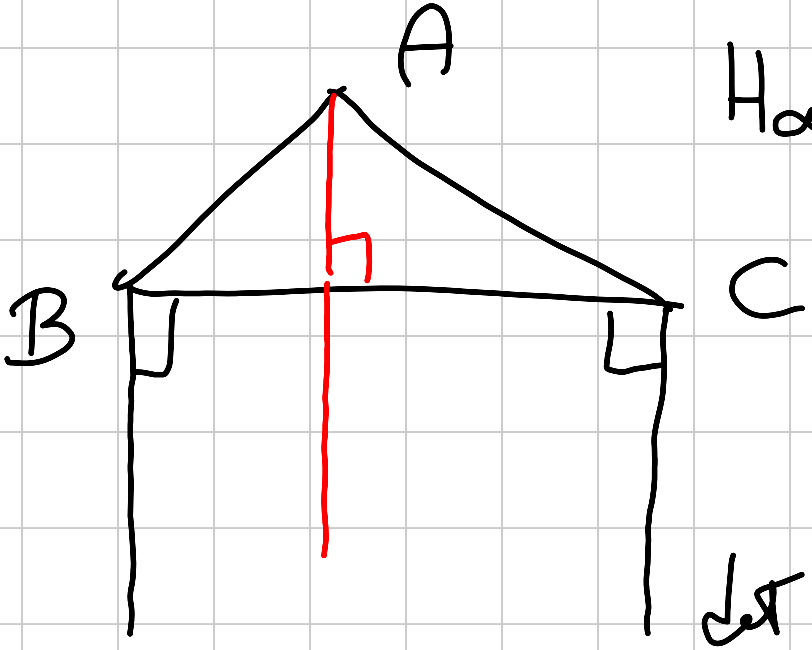
AD, BE, CF concorrono.

$$D = [-a^2, S_\gamma + S_{rosso}, S_\beta + S_{blu}]$$

$$E = [S_\gamma + S_{rosso}, -b^2, S_\alpha + S_{verde}]$$

$$F = [S_\beta + S_{blu}, S_\alpha + S_{verde}, -c^2]$$

Oss:



$H_\infty = [-a^2, S_\gamma, S_\beta]$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{S_\alpha} & \frac{1}{S_\beta} & \frac{1}{S_\gamma} \\ -a^2 & S_\gamma & S_\beta \end{pmatrix} =$$

H_∞

$$= 1 - 1 = 0.$$

$$[x,y,z] = \{ (kx, ky, kz) \in \mathbb{R}^3 \mid k \in \mathbb{R}^* \}$$

$$(x,y,z) \neq (0,0,0)$$

$$\mathbb{RP}^2 = \{ [x,y,z] \mid (x,y,z) \neq (0,0,0) \}$$

retta proiettiva : $\{ [x,y,z] \in \mathbb{RP}^2 \mid \ell x + my + nz = 0 \}$

per qualche $(\ell, m, n) \neq (0,0,0)$.

Passaggio da $\mathbb{RP}^2$ a $\mathbb{R}^2$

Basta togliere una retta

$$\{\ell x + my + nz = 0\} = r$$

$$\mathbb{RP}^2 \setminus r = \{[x,y,z] \in \mathbb{RP}^2 \mid \ell x + my + nz \neq 0\}$$

se $[x,y,z] \in \mathbb{RP}^2 \setminus r$

allora posso considerare $\left(\frac{x}{\ell x+my+nz}, \frac{y}{\ell x+my+nz}, \frac{z}{\cdots}\right)$

$$= (\tilde{x}, \tilde{y}, \tilde{z})$$

$$\tilde{x}\ell + \tilde{y}m + \tilde{z}n = 1$$

$$\mathbb{RP}^2 \setminus r = \{[x,y,z] \in \mathbb{RP}^2 \mid \ell x + my + nz = 1\}$$

$$[x,y,z] \in \mathbb{RP}^2 \setminus r \longrightarrow (\tilde{x},\tilde{y},\tilde{z}) \in \mathbb{R}^3$$

<u>bigezione</u>
che manda rette in
rette.

sta sul piano
$\{lx+my+nz=1\}$ in $\mathbb{R}^3$

Coord. baricentriche $\cong \{x+y+z=[ABC]\}$

<u>Es</u>: $r=\{z=0\}$ $\quad \mathbb{RP}^2 \setminus r = \{[x,y,z] \mid z=1\}$

$$[x, y, z] \longrightarrow \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

Proiettività

$$T : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$$

$$T([x, y, z]) = [a_{11}x + a_{12}y + a_{13}z,\ a_{21}x + a_{22}y + a_{23}z,\ a_{31}x + a_{32}y + a_{33}z] =$$

$$= \left[\underset{\overset{\shortparallel}{A}}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]$$

T è iniettiva

$\Updownarrow$

T è surgettiva

$\Uparrow$

Teo: Dati 4 punti A, B, C, D a 3 a 3 non allineati, $\exists\, T : \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ t.c.

$T([1,0,0]) = A \quad T([0,0,1]) = C$

$T([0,1,0]) = B \quad T([1,1,1]) = D.$

T è bijettiva

$\Updownarrow$

$\det A \neq 0.$

Dim: $A = [a_1, a_2, a_3]$ B, C, D simili

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$M \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad M \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad M \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\Pi_{hkj} = \begin{pmatrix} ha_1 & kb_1 & jc_1 \\ ha_2 & kb_2 & jc_2 \\ ha_3 & kb_3 & jc_3 \end{pmatrix} \qquad \Pi_{hkj}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} ha_1+kb_1+jc_1 \\ ha_2+kb_2+jc_2 \\ ha_3+kb_3+jc_3 \end{pmatrix}$$

$$\begin{cases} ha_1+kb_1+jc_1 = d_1 \\ ha_2+kb_2+jc_2 = d_2 \\ ha_3+kb_3+jc_3 = d_3 \end{cases} \qquad \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0.$$

$$\Downarrow$$

$\exists\, h, k, j$ che risolvono.

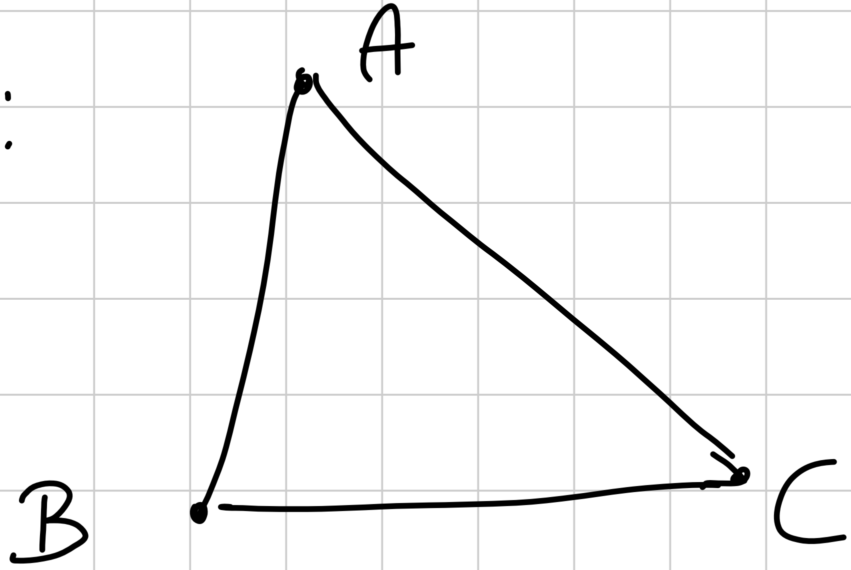
$\underline{\underline{\text{unica}}}$.

$$\Rightarrow T([x,y,z]) = \left[\Pi_{hkj}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]. \qquad \square$$

Oss: Inverso di una proiettività è una proiettività.

Oss 2: Date due quaterne di pti. a 3 a 3 non allineati: $\exists$ proiettività che manda l'una nell'altra

Es:



$\mathbb{RP}^1 = \{[x,y] \mid (x,y) \in \mathbb{R}^2 \setminus \{0\}\}$

$T : \mathbb{RP}^1 \to \mathbb{RP}^1$ proj.

$T([x,y]) = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right]$

Manda 3 qualsiasi pti. distinti in 3 altri qualsiasi punti distinti

$\mathcal{J} = \{\text{rette per } A\} = \left\{ \{\lambda y = \mu z\} \ , (\lambda, \mu) \neq (0,0) \right\}$

$f : \mathbb{RP}^1 \longrightarrow \mathcal{J}$ bigettiva.

$[\lambda, \mu] \longrightarrow \{\lambda y = \mu z\}$

$S : \mathcal{J} \longrightarrow \mathcal{J}$ simmetria risp. alla bisettrice interna.

$S(\{cy = bz\}) = \{cy = bz\}$

$\overset{\shortparallel}{\{cy = bz\}}$

$S(\{cy = -bz\}) = \{cy = -bz\}$

$S(\{y = 0\}) = \{z = 0\}$

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "S"] & \mathcal{Y} \arrow[d, "f^{-1}"] \\
\mathbb{RP}^1 \arrow[u, "f"] \arrow[r, dashed, "T"] & \mathbb{RP}^1
\end{tikzcd}$$

$T([\lambda, \mu]) = ?$

$T([c,b]) = [c,b]$

$T([c,-b]) = [c,-b]$

$T([1,0]) = [0,1]$

$T([0,1]) = [1,0]$

$$\begin{pmatrix} 0 & h \\ k & 0 \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} bh \\ ck \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}$$

$\Leftarrow$

$h = \frac{c}{b}$

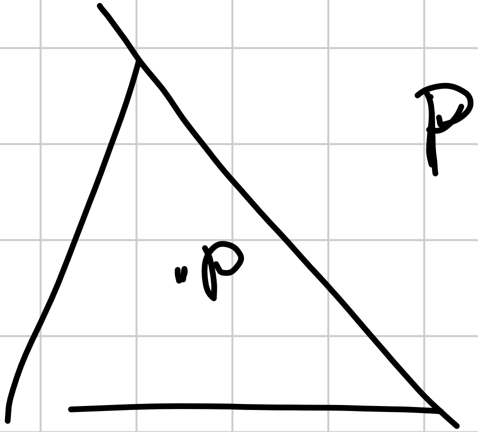
$k = \frac{b}{c}$

$$\begin{pmatrix} 0 & \frac{c}{b} \\ \frac{b}{c} & 0 \end{pmatrix} \begin{pmatrix} c \\ -b \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix}$$

Data la retta $\{\lambda y = \mu z\}$ la sua simm. risp. alle bisettrice è

$$\left\{\frac{c}{b}\mu y = \frac{b}{c}\lambda z\right\} = \left\{c^2 \mu y = b^2 \lambda z\right\} =$$

$$= \left\{\frac{c^2}{\lambda} y = \frac{b^2}{\mu} z\right\}$$



$P = [u, v, w]$

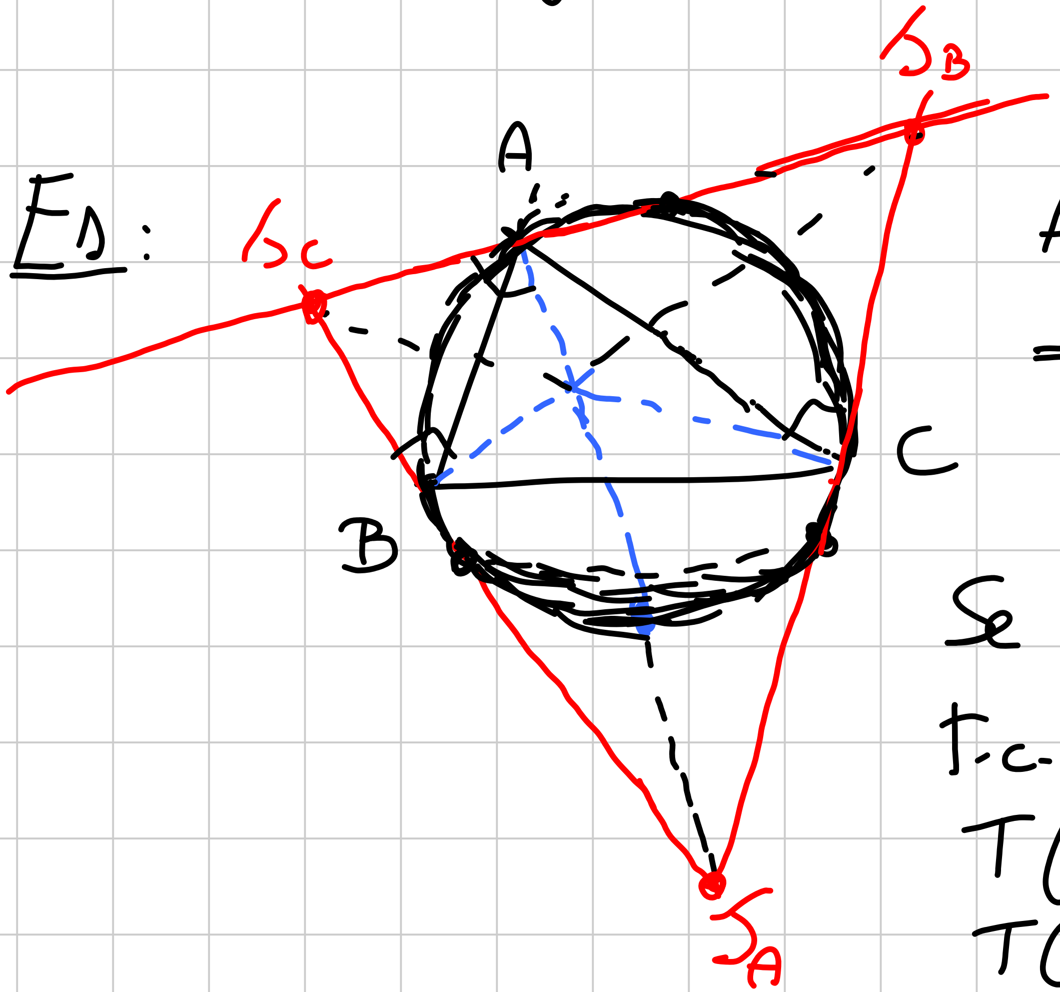
$wy = zv \longrightarrow \frac{c^2}{w} y = \frac{b^2}{v} z$

$wx = zu \longrightarrow \frac{c^2}{w} x = \frac{a^2}{u} z$

$vx = yu \longrightarrow \frac{b^2}{v} x = \frac{a^2}{u} y$

$\Rightarrow$ coniug. isog. di P è $\left[\frac{a^2}{u}; \frac{b^2}{v}, \frac{c^2}{w}\right]$.

Es:



ABC tri. ortico di $J_A J_B J_C$

$\Rightarrow$ O è centro della cf di Feuerbach di $J_A J_B J_C$.

Se ho una proiettività t.c.

$T([1,0,0]) = [-a, b, c]$

$T([0,1,0]) = [a, -b, c]$

$T([0,0,1]) = [a, b, -c]$

$$T(H) = I$$
$$T(O) \overset{?}{\doteq} N$$

Fatto: Le proiettività conservano il birapporto.

Es: $\mathcal{F} = \{\text{rette per } A\}$ $P: \mathcal{F} \to \mathcal{F}$

$r \to \text{retta} \perp r$

è una proiettività?

$[c, b] \to [c, -b]$

$[c, -b] \to [c, b]$

altezza da A $\left\{ \frac{y}{S_\gamma} = \frac{z}{S_\beta} \right\}$

Caso opposto $\{x = 0\}$

$[1,-1] \leftrightarrow \left[\frac{1}{S_\gamma}, \frac{1}{S_\beta}\right] \Rightarrow$ p.to all'∞ $\in [0,1,-1]$

$\parallel$ a BC per A $\{y=-z\}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} a_{11}c + a_{12}b \\ a_{21}c + a_{22}b \end{pmatrix} = \lambda\begin{pmatrix} c \\ -b \end{pmatrix}$$

$$\begin{pmatrix} a_{11}c - a_{12}b \\ a_{21}c - a_{22}b \end{pmatrix} = \mu\begin{pmatrix} c \\ b \end{pmatrix}$$

$$\begin{pmatrix} a_{11} - a_{12} \\ a_{21} - a_{22} \end{pmatrix} = \nu\begin{pmatrix} S_\beta \\ S_\gamma \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c^2S_\beta - b^2S_\gamma & a^2b^2 \\ c^2a^2 & c^2S_\beta - b^2S_\gamma \end{pmatrix}$$

$$\lambda y = \mu z \longrightarrow \left(\lambda c^2 S_\beta - \lambda b^2 S_\gamma + \mu a^2 b^2\right) y =$$

$$= \left(\lambda c^2 a^2 + \mu c^2 S_\beta - \mu b^2 S_\gamma\right) z$$

——— o ———

Rette.

- Asse di BC: passa per $[0,1,1]$ e $[-a^2, S_\gamma, S_\beta]$

$$\left\{ x(S_\beta - S_\gamma) - a^2 y + a^2 z = 0 \right\}$$

$O = [a^2 S_\alpha, \ldots,]$

$$a^2 S_\alpha S_\beta - a^2 S_\alpha S_\gamma - a^2 b^2 S_\beta + a^2 c^2 S_\gamma =$$

$$= a^2 \left(S_\alpha S_\beta - S_\alpha S_\gamma - (S_\alpha + S_\gamma) S_\beta + (S_\beta + S_\alpha) S_\gamma \right) = 0.$$

•) Retta di Eulero: passa per $[1,1,1]$ e $[a^2 S_\alpha, \dots, \dots]$

$$(c^2 S_\gamma - b^2 S_\beta) x + \dots + \dots = 0$$

$c^2 = S_\alpha + S_\beta$

$b^2 = S_\alpha + S_\gamma$

$$\searrow S_\alpha S_\gamma - S_\alpha S_\beta = S_\alpha (S_\gamma - S_\beta)$$

$$\sum_{cyc} S_\alpha (S_\gamma - S_\beta) x = 0$$

- Retta per I, O: $[a, b, c]$, $[a^2 S_\alpha, \dots]$

$$\sum_{cyc} (bc^2 S_\gamma - cb^2 S_\beta) x = 0$$

$$\sum_{cyc} \frac{cS_\gamma - bS_\beta}{a} x$$

Tri pedali

$P = [u, v, w]$ $\quad r // AH$, $P \in r$

$$r = \{ (S_\beta v - S_\gamma w) x - (S_\beta u + a^2 v) y + (S_\gamma u + a^2 w) z = 0 \}$$

$$\begin{pmatrix} u & v & w \\ -a^2 & S_\gamma & S_\beta \\ x & y & z \end{pmatrix} \qquad r \cap \{x=0\} = P_A$$

$$P_A = [0 : S_\gamma u + a^2 v : S_\beta u + a^2 w] \qquad a^2(v+w) + u(S_\beta + S_\gamma)$$

$$a^2 \sum u$$

$$P_B = [S_\gamma v + b^2 u : 0 : S_\alpha v + b^2 w] \qquad b^2 \sum u$$

$$P_C = [S_\beta w + c^2 u : S_\alpha w + c^2 v : 0] \qquad c^2 \sum u$$

Teo: Due rette con punto all'∞ $[f, g, h]$ e $[f', g', h']$ sono $\perp$

x e [illegible] x $\quad S_\alpha ff' + S_\beta gg' + S_\gamma hh' = 0.$

ED: $A0 \quad [1,0,0] \quad [aS_\alpha, \dots]$

$$\| \qquad \{c^2 S_\gamma y = b^2 S_\beta z\}$$

$$[-b^2 S_\beta - c^2 S_\gamma,\ b^2 S_\beta,\ c^2 S_\gamma] \qquad \begin{cases} S_\alpha(b^2 S_\beta + c^2 S_\gamma)\ell - b^2 S_\beta^2 m - c^2 S_\gamma^2 n = 0 \\ \ell + m + n = 0 \end{cases}$$

$$-m(b^2 S_\alpha S_\beta + c^2 S_\alpha S_\gamma - b^2 S_\beta^2) = n(b^2 S_\alpha S_\beta + c^2 S_\alpha S_\gamma + c^2 S_\gamma^2)$$

$$-m(b^2 S_\beta c^2 + c^2 S_\alpha S_\gamma) = n(b^2 S_\alpha S_\beta + c^2 S_\gamma b^2)$$

$$m = -b^2\left(S_\alpha S_\beta + c^2 S_\gamma\right)$$

$$n = c^2\left(S_\alpha S_\gamma + b^2 S_\beta\right)$$

$$\ell = -m - n = b^2 S_\alpha S_\beta + b^2 c^2 S_\gamma - c^2 S_\alpha S_\gamma - c^2 b^2 S_\beta =$$

$$= S_\alpha\left(b^2 S_\beta - c^2 S_\gamma\right) + c^2 b^2\left(S_\gamma - S_\beta\right) =$$

$$= S_\alpha^2\left(S_\beta - S_\gamma\right) + c^2 b^2\left(S_\gamma - S_\beta\right)$$

$[\ell, m, n]$ $[1, 0, 0]$

$my = mz$

Circonferenze

$$a^2yz + b^2xz + c^2xy + (x+y+z)(px+qy+rz) = 0.$$

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = 0.$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \underbrace{\begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}}_{M} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Per ogni punto $P \in \mathbb{RP}^2$ considero la retta

$$\{ {}^tP\, M \cdot X = 0 \}$$

$P = [u, v, w]$ $\quad {}^tP = (u, v, w)$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$${}^tP \cdot M = (u \;\; v \;\; w) \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} =$$

$$= \left(Au + Dv + Ew,\; Du + Bv + Fw,\; Eu + Fv + Cw \right)$$

Tale retta si chiama <u>POLARE</u> di P rispetto alla conica e si indica con pol(P)

$P \in \mathrm{pol}(P) \iff {}^tP \cdot M \cdot P = 0 \iff P \in$ conica

$P \in \mathrm{pol}(Q) \iff {}^tQ \cdot M \cdot P = 0 = {}^tP \cdot M \cdot Q \iff Q \in \mathrm{pol}(P)$

$P \in$ conica $\iff \mathrm{pol}(P)$ è tangente alla conica.

[Conica è "vera" $\iff \det M \neq 0$
(indicabile)]

$P \in$ conica $\iff P \in \mathrm{pol}(P)$ se $\exists\, Q \neq P$, $Q \in$ conica
t.c. $Q \in \mathrm{pol}(P) \implies \mathrm{pol}(P) = PQ$

$Q \in$ conica $\implies Q \in \mathrm{pol}(Q)$

$Q \in pol(P) \Rightarrow P \in pol(Q) \Rightarrow pol(Q) = PQ$

${}^tP \cdot \Pi \cdot X = 0$ sono lo stesso

${}^tQ \cdot \Pi \cdot X = 0$

$${}^tP \cdot \Pi = {}^tQ \cdot \Pi$$

$$({}^tP - {}^tQ) \cdot \Pi = 0$$

$$\Pi \cdot (P - Q) = 0$$

$\exists (a, b, c) \in \mathbb{R}^3$ t.c. $\Pi \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$

$\neq (0,0,0)$

$\det \Pi \neq 0$ Assurdo.

$\Rightarrow$ conica $\cap$ pol$(P) = \{P\}$

$\Rightarrow$ pol(P) é Tg. alla conica.

<u>ED</u>: $a^2yz + b^2xz + c^2xy = 0$

$$\begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} = M$$

$A = [1,0,0]$

$${}^tA \cdot M \cdot X = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & c^2 & b^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c^2y + b^2z$$

Oss: La polarità è una **DUALITÀ**.

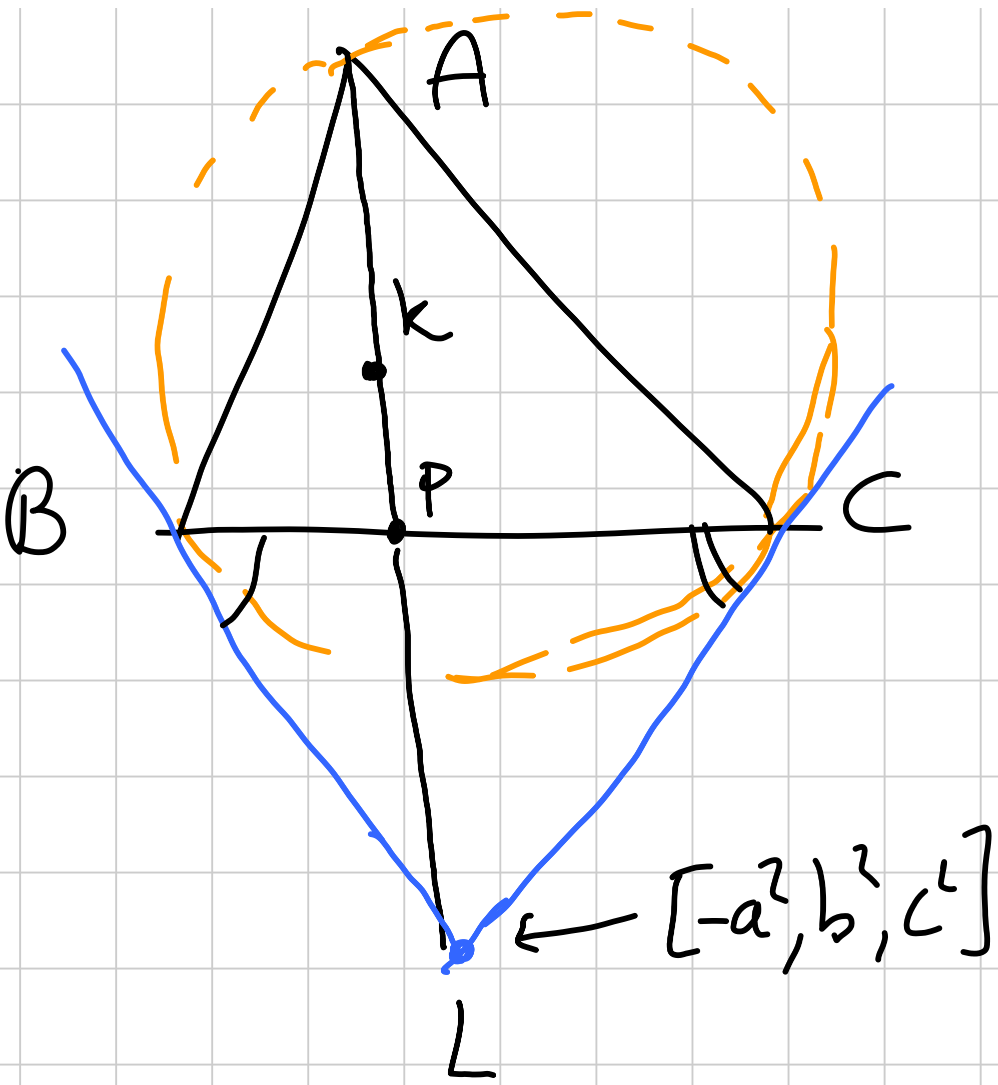
Se A, B, C sono allineati, $\mathrm{pol}(A), \mathrm{pol}(B), \mathrm{pol}(C)$ concorrono.

$\mathrm{pol}(r) = P \iff \mathrm{pol}(P) = r$

Es: $c^2y + b^2z = 0$ $\quad r_y$ in A

$c^2x + a^2z = 0$ $\quad r_y$ in B

si intersecano in $[a^2, b^2, -c^2]$.



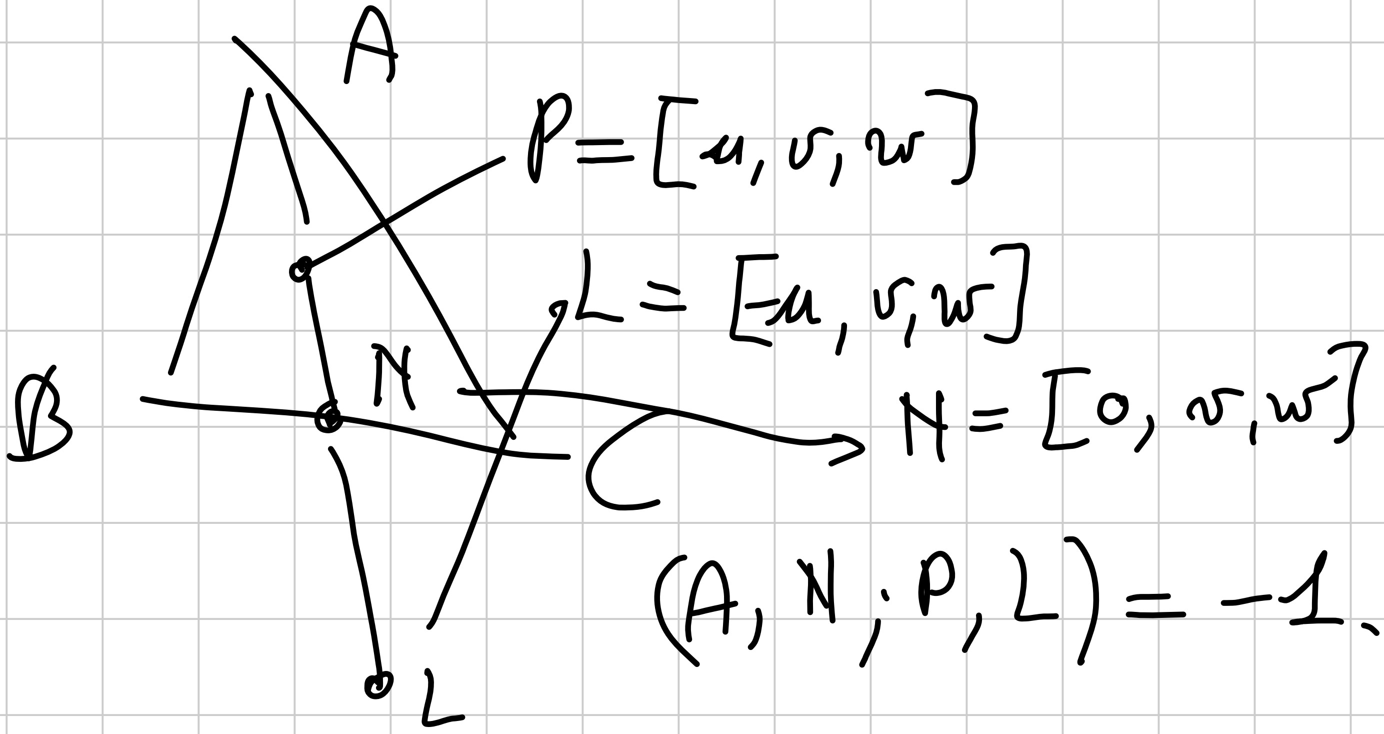
$K = [a^2, b^2, c^2]$

$b^2 z = c^2 y$

$$\frac{AK}{KP} = -\frac{AL}{LP}$$

$(A, P; K, L) = -1$

Fatto generale (e inutile)



$P = [u, v, w]$

$L = [-u, v, w]$

$N = [0, v, w]$

$(A, N; P, L) = -1.$