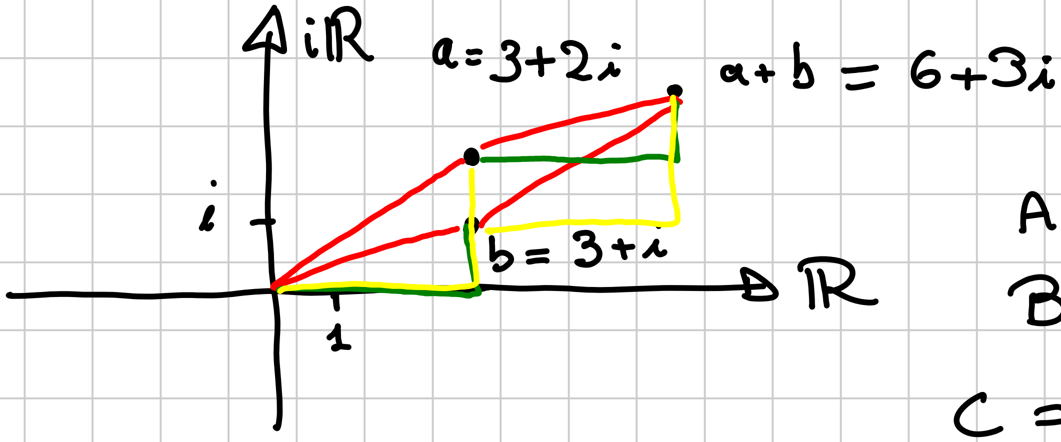
G 2 - Basic

Sam

Titolo nota 04/09/2013

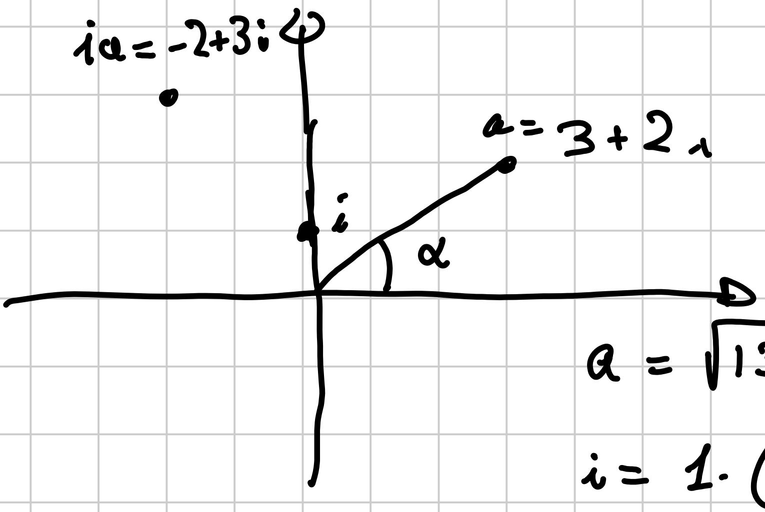


$A = (3, 2)$

$B = (3, 1)$

$C = (6, 3)$

t.c. OACB è un parallelogramma



$ia = -2+3i$

$a = \sqrt{13}\,(\cos\alpha + i \operatorname{sen}\alpha)$

$i = 1\cdot\left(\cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2}\right)$

$a = x+iy$

$b = u+iv$

$ab = xu + ixv + iyu + i^2 yv =$

$= (xu - yv) + i(xv + yu)$

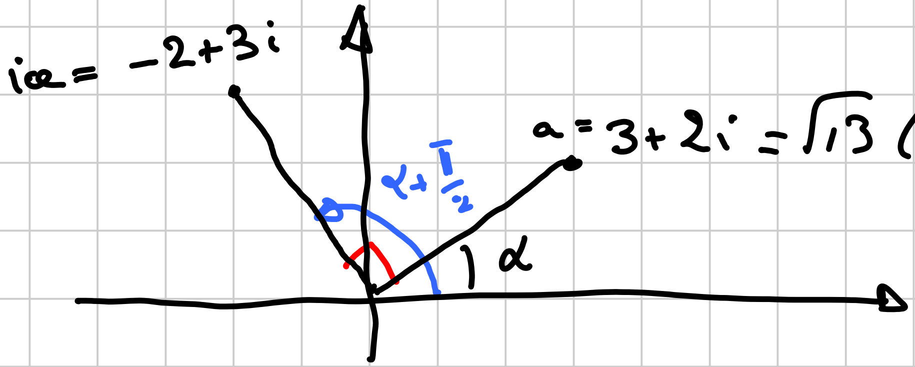
$a = r\cdot(\cos\alpha + i \operatorname{sen}\alpha)$

$b = R\cdot(\cos\beta + i \operatorname{sen}\beta)$

$ab = rR(\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta +$

$+ i(\cos\alpha\operatorname{sen}\beta + \operatorname{sen}\alpha\cos\beta)) =$

$= rR(\cos(\alpha+\beta) + i\operatorname{sen}(\alpha+\beta))$



$a = 3+2i = \sqrt{13}\,(\cos\alpha + i \,\mathrm{sen}\,\alpha)$

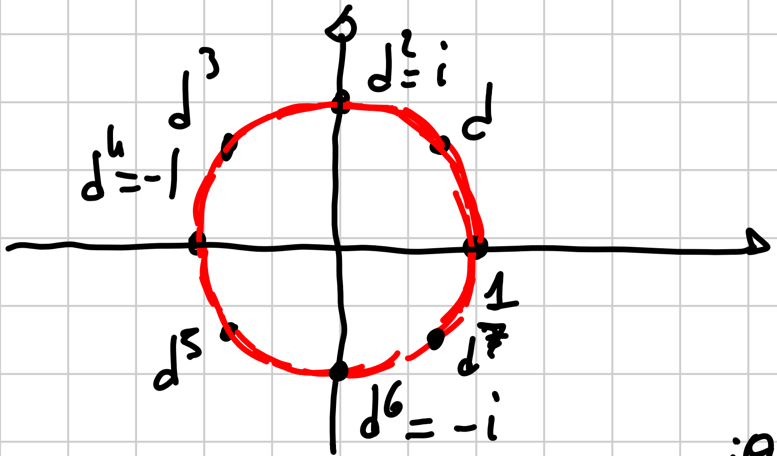
$i = 1\left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\,\frac{\pi}{2}\right)$

$$ia = \sqrt{13}\left(\cos\left(\alpha+\frac{\pi}{2}\right) + i\,\mathrm{sen}\left(\alpha+\frac{\pi}{2}\right)\right)$$

dist da 0   angolo formato con il semiasse reale positivo.

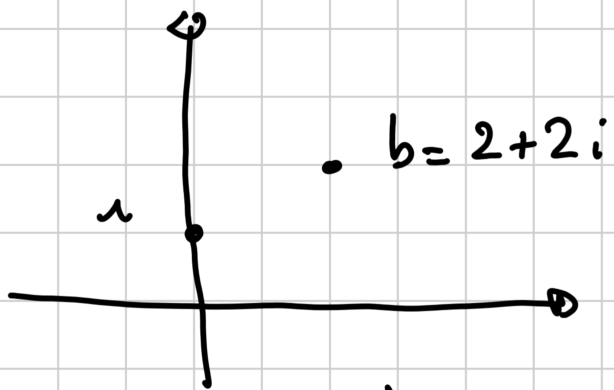
$$d = \frac{1}{\sqrt2} + \frac{i}{\sqrt2} = 1\left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\,\frac{\pi}{4}\right)$$

$1, d, d^2, d^3, d^4, d^5, \dots \quad d^8 = 1$



$c \longrightarrow c e^{i\theta}$ rot. in senso antiorario attorno all'origine di un angolo θ.

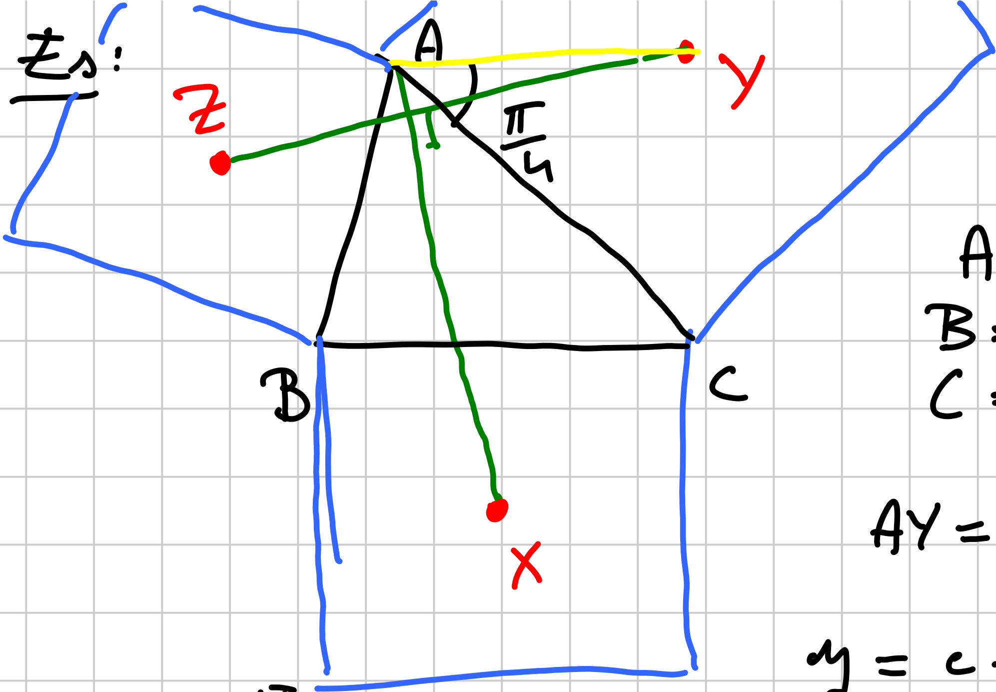
$$e^{i\theta} = \cos\theta + i\,\mathrm{sen}\,\theta$$



ruotare b attorno a i di $\frac{\pi}{4}$.

$$b \xrightarrow[\text{Traslazione di } i \text{ nell'origine}]{} b-i \xrightarrow[\text{rot.}]{} (b-i)e^{i\frac{\pi}{4}} \xrightarrow[\text{Trasl. indietro}]{} (b-i)e^{i\frac{\pi}{4}}+i$$

Es:



$AX \perp ZY$

$A = 0$
$B = b$
$C = c$

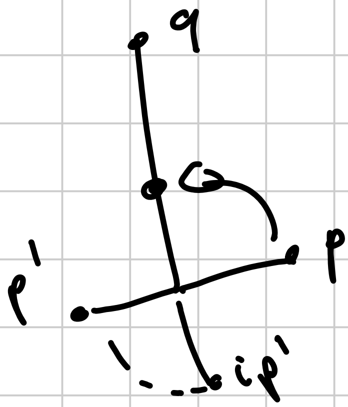
$$AY = \frac{AC}{2}\sqrt{2}$$

$$y = c \cdot e^{i\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}$$

$$z = b \cdot e^{-i\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}$$

$$(b-c)\,e^{i\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2} + c = x$$

Per verificare la $\perp$, portiamo uno tra Z e Y in A



$$Op \perp Oq \iff ip = kq$$
$$\Updownarrow$$
$$\frac{p}{q} = \frac{k}{i} = -ki$$
$$\Updownarrow$$
$\frac{p}{q}$ è immaginario puro.

Concludendo $z - y = b e^{-i\frac{\pi}{4}} \frac{\sqrt{2}}{2} - c e^{i\frac{\pi}{4}} \frac{\sqrt{2}}{2}$

$$x = (b-c)\,e^{i\frac{\pi}{4}} \frac{\sqrt{2}}{2} + c$$

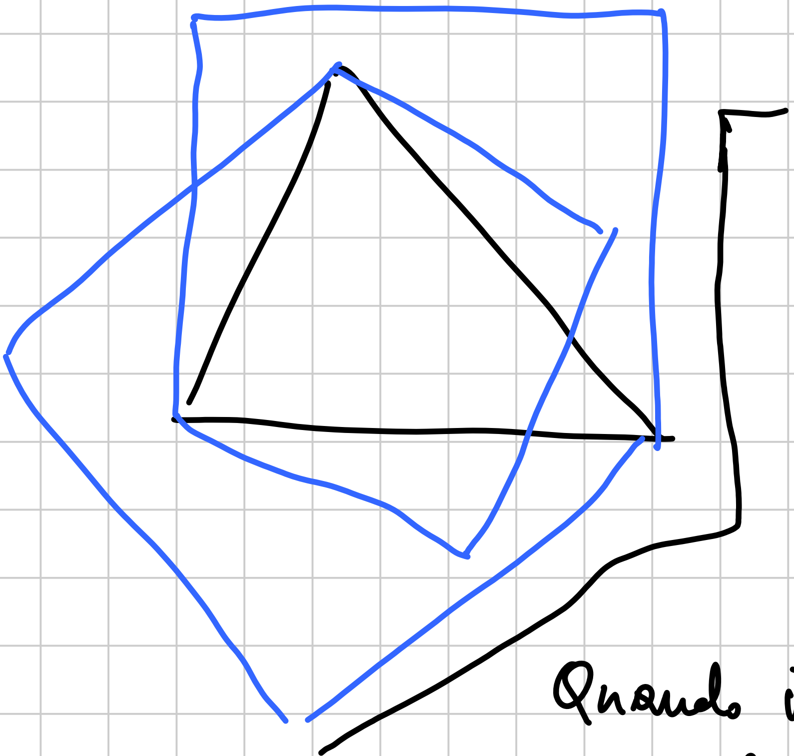
$$z - y = b\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\frac{\sqrt{2}}{2} - c\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\frac{\sqrt{2}}{2} =$$

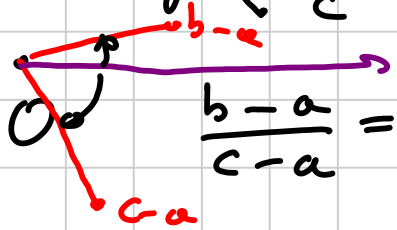
$$= \frac{b}{2} - \frac{bi}{2} - \frac{c}{2} - \frac{ci}{2}$$

$$x = (b-c)(1+i)\frac{1}{2} + c = \frac{b}{2} + \frac{ib}{2} - \frac{c}{2} - \frac{ic}{2} + c =$$

$$= \frac{b}{2} + \frac{ib}{2} + \frac{c}{2} - \frac{ic}{2} =$$

$$= i\left(\frac{b}{2} - \frac{ib}{2} - \frac{c}{2} - \frac{ic}{2}\right) = i(z-y)$$





$$\frac{b-a}{c-a} = \frac{x}{y} e^{i\alpha}$$

Quando il Triangolo a, b, c è equilatero?

Quando $\frac{b-a}{c-a} = e^{i\frac{\pi}{3}}$

$$b - a = e^{i\frac{\pi}{3}}(c-a)$$

$$b + a\left(e^{i\frac{\pi}{3}} - 1\right) - c e^{i\frac{\pi}{3}} = 0$$

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i - 1\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{2}{3}\pi}$$

$$-e^{i\frac{\pi}{3}} = e^{i\pi} \cdot e^{i\frac{\pi}{3}} = e^{i\frac{4}{3}\pi}$$

$$b + a e^{i\frac{2}{3}\pi} + c e^{i\frac{4}{3}\pi} = 0.$$

$$\omega = e^{i\frac{2}{3}\pi}$$

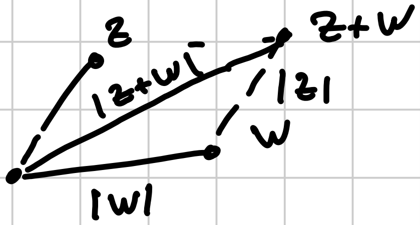
a, b, c formano un tri equilatero

se e solo se $b + \omega a + \omega^2 c = 0$

oppure

$a + \omega b + \omega^2 c = 0$

Oss: $|z+w| \le |z| + |w|$



Oss 2: z e $\bar{z}$ sono simmetrici rispetto alla retta reale.

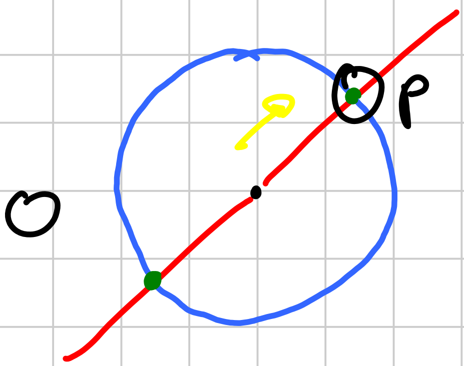
Oss 3: z, w, t sono allineati se $t = kz + (1-k)w$ con $k \in \mathbb{R}$

se e solo se $\frac{z-t}{w-t} \in \mathbb{R}$.

Vettori

Fisso un'origine O. Ad ogni punto P corrisponde un vettore $(\vec{OP}, \vec{P})$, che è una freccia.

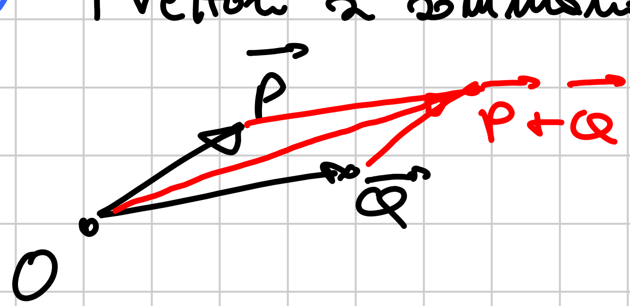
Vettore = direzione, intensità e verso



$\|\vec{P}\|$ = intensità = modulo =

$= \overline{OP}$

I vettori si sommano



I vettori si moltiplicano per un numero reale

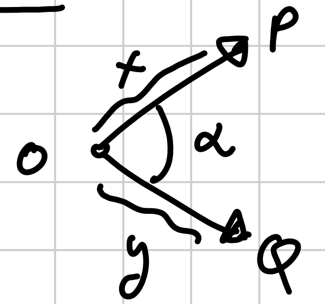
$k \in \mathbb{R}$, $\vec{P}$ vettore $\quad k\vec{P} =$ stessa direzione

modulo pari a $|k| \cdot \|\vec{P}\|$

verso uguale a $\vec{P}$ se $k>0$

verso opposto se $k<0$

Prodotto scalare: $\vec{P}\cdot\vec{Q} = \|\vec{P}\|\cdot\|\vec{Q}\|\cdot\cos(P\hat{O}Q)$



$xy\cos\alpha$

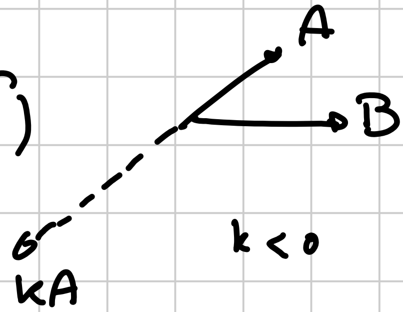
$OP \perp OQ \iff \vec{P}\cdot\vec{Q} = 0.$

Proprietà:

1) $(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C} + \vec{B}\cdot\vec{C}$

2) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

3) $(k\vec{A})\cdot\vec{B} = k(\vec{A}\cdot\vec{B})$



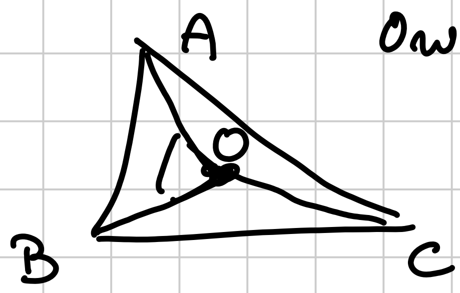
$k<0$

4) $\vec{A}\cdot\vec{A} = \|\vec{A}\|^2$

Es:

$$\|\vec{A}+\vec{B}\|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) =$$
$$= (\vec{A}+\vec{B})\cdot\vec{A} + (\vec{A}+\vec{B})\cdot\vec{B} =$$
$$= \vec{A}\cdot\vec{A} + \underbrace{\vec{B}\cdot\vec{A}}_{\parallel} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B} =$$
$$= \|\vec{A}\|^2 + 2\vec{A}\cdot\vec{B} + \|\vec{B}\|^2$$

Es:



Origine nel circocentro

$$\|\vec{A}\|^2 = R^2 = \|\vec{B}\|^2 = \|\vec{C}\|^2$$

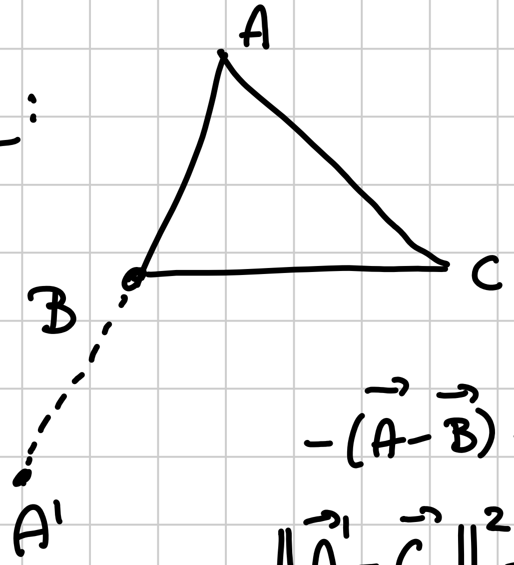
$$\vec{A}\cdot\vec{B} = R^2\cos A\hat{O}B = R^2\cos 2\gamma$$

$$\|\vec{A}-\vec{B}\|^2 = c^2$$

$$\begin{array}{ccccccc} \|\vec{A}-\vec{B}\|^2 & = & \|\vec{A}\|^2 & + & \|\vec{B}\|^2 & - & 2\vec{A}\cdot\vec{B} \\ \| & & \| & & \| & & \\ c^2 & = & R^2 & + & R^2 & - & 2\vec{A}\cdot\vec{B} \end{array}$$

$$\vec{A}\cdot\vec{B} = \frac{2R^2-c^2}{2} = R^2-\frac{c^2}{2}$$

Es:



$BA' = BA$

$A'C = ?$

Origine nel circocentro

$$-(\vec{A}-\vec{B})+\vec{B} = 2\vec{B}-\vec{A} = \vec{A}'$$

$$\|\vec{A}'-\vec{C}\|^2 = \|\vec{A}'\|^2+\|\vec{C}\|^2-2\vec{A}'\cdot\vec{C} =$$

$$= \|2\vec{B}-\vec{A}\|^2+R^2-2(2\vec{B}-\vec{A})\cdot\vec{C} =$$

$$= \|2\vec{B}\|^2+\|\vec{A}\|^2-2(2\vec{B})\cdot\vec{A}+R^2-$$

$$-4\vec{B}\cdot\vec{C}+2\vec{A}\cdot\vec{C} =$$

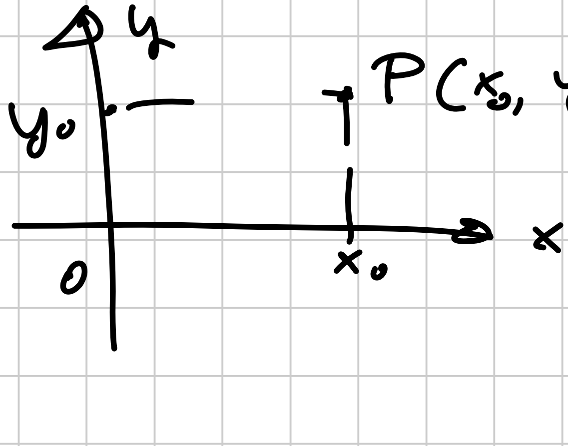
$$= 4R^2+R^2-4\left(R^2-\frac{c^2}{2}\right)+R^2-4\left(R^2-\frac{a^2}{2}\right)+$$

$$+2\left(R^2-\frac{b^2}{2}\right) =$$

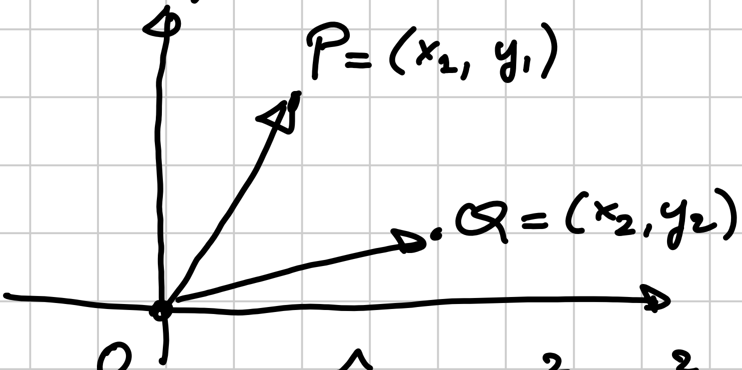
$$= 6R^2-4R^2-4R^2+2R^2+2c^2+2a^2-b^2 =$$

$$= 2c^2+2a^2-b^2.$$

Coordinate cartesiane



come si fa il prodotto scalare?



$$\vec{P} \cdot \vec{Q} = \|\vec{P}\| \cdot \|\vec{Q}\| \cos P\hat{O}Q$$

$$= \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} \cos P\hat{O}Q$$

$$\cos P\hat{O}Q = \frac{PO^2 + QO^2 - PQ^2}{2\, PO \cdot QO} \qquad \text{(Teo di Carnot)}$$

$$\vec{P} \cdot \vec{Q} = \frac{PO^2 + QO^2 - PQ^2}{2} = \frac{1}{2}\left[(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - \right.$$

$$\left. - (x_1 - x_2)^2 - (y_1 - y_2)^2 \right] =$$

$$= \frac{1}{2}\left[\cancel{x_1^2} + \cancel{y_1^2} + \cancel{x_2^2} + \cancel{y_2^2} - \cancel{x_1^2} - \cancel{x_2^2} + 2x_1x_2 - \cancel{y_1^2} - \cancel{y_2^2} + 2y_1y_2 \right] =$$

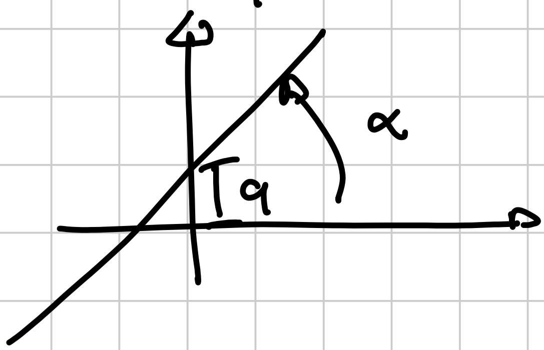
$$= \frac{1}{2}\left[2x_1x_2 + 2y_1y_2\right] = x_1x_2 + y_1y_2$$

- ep. della retta $\quad y = mx + q. \quad m = \mathrm{tg}\,\alpha$

$\quad ax + by + c = 0 \quad q = \text{intercetta}$

rette $\parallel \iff$ stessa m

rette $\perp \iff m \cdot m' = -1$



- circonferenza: $(x - x_0)^2 + (y - y_0)^2 = r^2$

$$x^2 + y^2 + 2\alpha x + 2\beta y + \gamma = 0$$

$$\alpha = -x_0 \qquad \beta = -y_0 \qquad \gamma = x_0^2 + y_0^2 - r^2$$

$$r^2 = x_0^2 + y_0^2 - \gamma = \alpha^2 + \beta^2 - \gamma > 0$$

$$\{ x^2 + y^2 + 2x + 2y + 3 = 0 \}$$

$1 + 1 - 3 = -1 < 0$ non è una circ.

$$pow_\Gamma(P) = OP^2 - R^2 = (x - x_0)^2 + (y - y_0)^2 - r^2$$

Γ: cerchio O e raggio R

per ottenere la pot. di P basta sostituire le sue coord nell'eq della circ.

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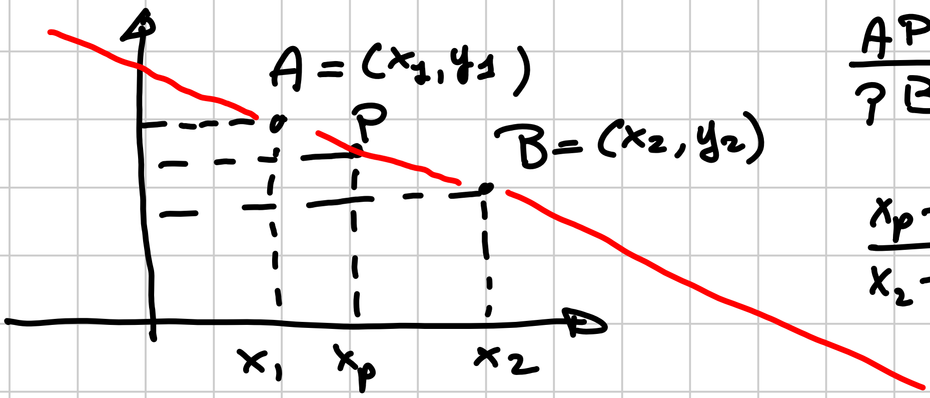


$$\frac{AP}{PB} = k$$

$$\frac{x_p - x_1}{x_2 - x_p} = k$$

$$x_p(1+k) = x_1 + k x_2$$

$$x_p = \frac{x_1 + k x_2}{1+k}$$

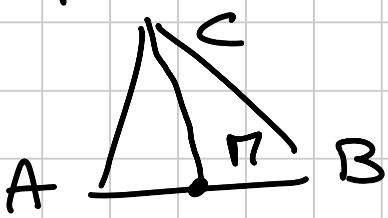
$$y_p = \frac{y_1 + k y_2}{1+k}$$

$$(x_p, y_p) = \frac{(x_1, y_1) + k(x_2, y_2)}{1+k} \Longleftarrow$$

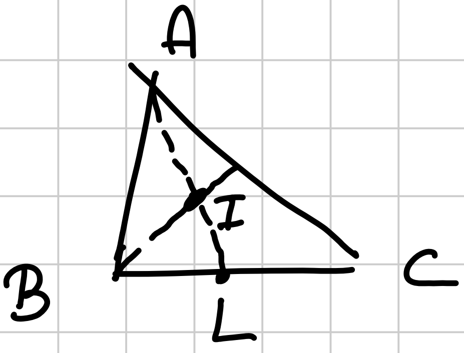
$$\vec{P} = \frac{\vec{A} + \boxed{k\vec{B}}}{1+k} \qquad \frac{AP}{PB} = k$$

$p = x_p + i y_p \qquad a = x_1 + i y_1 \qquad b = x_2 + i y_2 \qquad p = \frac{a + kb}{1+k}$

A, B, C triangolo pt. medio di AB è $\frac{\vec{A} + \vec{B}}{2} = \vec{M}$



$$\frac{CG}{GM} = 2 \implies \vec{G} = \frac{\vec{C} + 2\vec{M}}{3} = \frac{\vec{A}+\vec{B}+\vec{C}}{3}$$



$$\frac{BL}{LC} = \frac{BA}{AC} = \frac{c}{b}$$

$$L = \frac{\vec{B} + \frac{c}{b}\vec{C}}{1 + \frac{c}{b}} = \frac{b\vec{B} + c\vec{C}}{b+c}$$

$$I = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}$$

<u>Fatto vero</u>: Se l'origine è nel circocentro, allora $\vec{H} = \vec{A} + \vec{B} + \vec{C}$ è l'ortocentro.

———— o ————

Esercizi

$X = (1,1) \qquad Y = (0,2)$

$\{P : XP = 2PY\}$

$X = (x_1, y_1) \qquad Y = (x_2, y_2)$

$\{P : XP = \lambda YP\} \qquad \lambda \neq 1, 0$

$P = (x, y)$

$$(x-x_1)^2 + (y-y_1)^2 = \lambda^2\left[(x-x_2)^2 + (y-y_2)^2\right]$$

$$x^2 + x_1^2 + y^2 + y_1^2 - 2xx_1 - 2yy_1 = \lambda^2 x^2 + \lambda^2 x_2^2 + \lambda^2 y^2 + \lambda^2 y_2^2 - 2\lambda^2 y y_2 - 2\lambda^2 x x_2$$

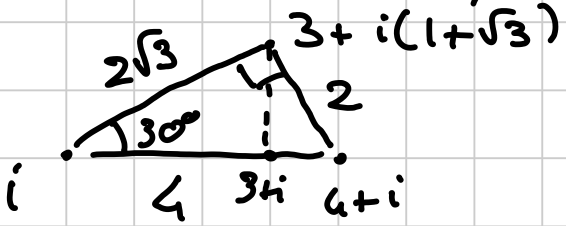
$$x^2(1-\lambda^2) + y^2(1-\lambda^2) - 2x(x_1 - \lambda^2 x_2) - 2y(y_1 - \lambda^2 y_2)$$
$$+ x_1^2 + y_1^2 - \lambda^2(x_2^2 + y_2^2) = 0$$

circonferenza di Apollonio

$$x_0 = \frac{x_1 - \lambda^2 x_2}{1-\lambda^2} \qquad y_0 = \frac{y_1 - \lambda^2 y_2}{1-\lambda^2}$$

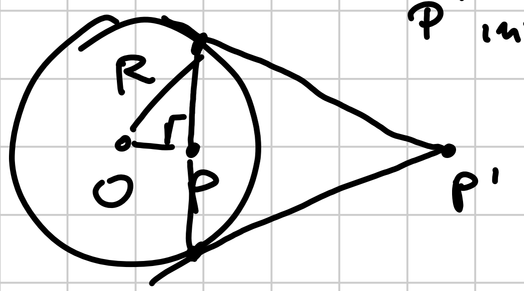
C t.c. X, Y, C sono all. e $\frac{XC}{CY} = -\lambda^2$

$\boxed{I}$ $\frac{a+b}{2}, \dots, \dots$



$$(2\sqrt{3})^2 + 2^2 = 12 + 4 = 16 = 4^2$$

$$\frac{-a\vec{A} + b\vec{B} + c\vec{C}}{-a+b+c}, \dots, \dots$$



P' inverso di P se O, P, P' sono allin.
e $OP \cdot OP' = R^2$
e P, P' stanno sulla stessa semiretta da O.

$\vec{N}$ t.c. $\vec{N} = k \cdot \vec{m_a}$ $k > 0$

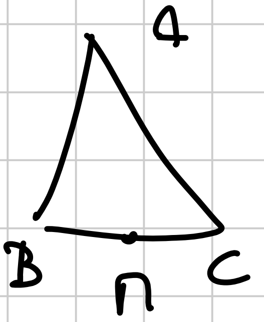
$$Om_a \cdot ON = R^2 \qquad ON = \|\vec{N}\| = \frac{R^2}{\|\vec{m_a}\|}$$

$$m_a = \frac{\vec{B} + \vec{C}}{2} \qquad \|m_a\|^2 = \frac{1}{4}(R^2 + R^2 + 2R^2 - a^2) = R^2 - \frac{a^2}{4}$$

$$\frac{ON}{Om_a} = \frac{R^2}{\|m_a\|^2} = \frac{R^2}{R^2 - \frac{a^2}{4}} = 1 + \frac{a^2}{4R^2 - a^2}$$

$$\vec{N} = \left(1 + \frac{a^2}{4R^2 - a^2}\right) \frac{\vec{B} + \vec{C}}{2}$$

6) $\sum \text{mediane}^2$



$$\vec{M} = \frac{\vec{B} + \vec{C}}{2}$$

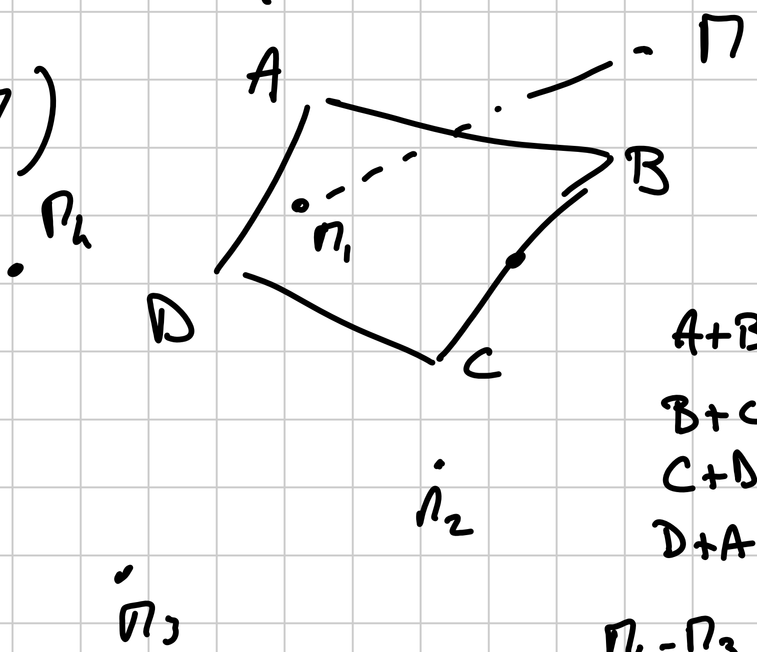
$$AM^2 = \left\| \vec{A} - \frac{\vec{B} + \vec{C}}{2} \right\|^2 =$$

$$a^2 = \|\vec{B} - \vec{C}\|^2 = \|\vec{B}\|^2 + \|\vec{C}\|^2 - 2\vec{B}\cdot\vec{C}$$

$$= \left\| \frac{\vec{A} - \vec{B}}{2} + \frac{\vec{A} - \vec{C}}{2} \right\|^2 = \frac{1}{4}(c^2 + b^2) + \frac{1}{2}(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{C})$$

$$\frac{3}{4}(a^2 + b^2 + c^2)$$

7)



$$\frac{A + B}{2}$$

$$A + B - M = M_1$$

$$B + C - M = M_2$$

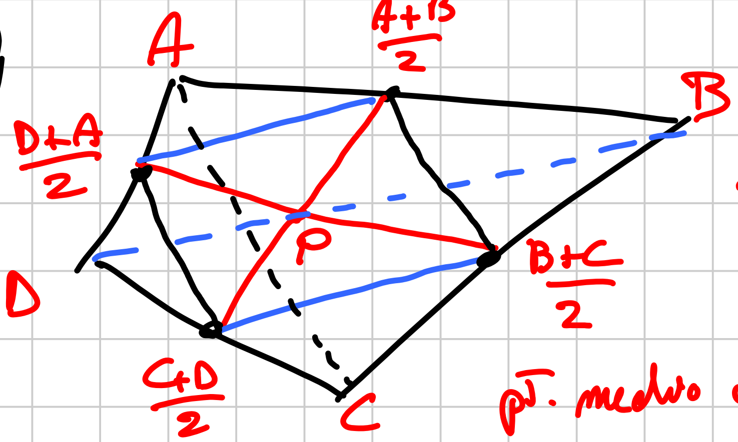
$$C + D - M = M_3$$

$$D + A - M = M_4$$

$$M_4 - M_3 = D + A - M - C - D + M = A - C$$

$$M_1 - M_2 = A + B - M - B - C + M = A - C$$

8)



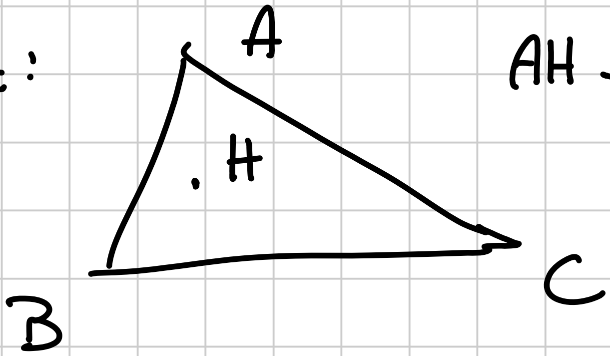
$$P = \frac{\frac{A+B}{2} + \frac{C+D}{2}}{2} = \frac{A+B+C+D}{4}$$

pt. medio di $BD = \frac{B+D}{2}$

pt medio di $AC = \frac{A+C}{2}$

$\Rightarrow$ pt medio tra i pt. medi $= \dfrac{A+B+C+D}{4}$

Ultima cosa:



$AH \perp BC \quad (*)$

Se l'origine è in O, $\vec{H} = \vec{A} + \vec{B} + \vec{C}$ soddisfa $(*)$

$$(\vec{H} - \vec{A}) \cdot (\vec{B} - \vec{C}) = (\vec{B} + \vec{C}) \cdot (\vec{B} - \vec{C}) =$$
$$= \|\vec{B}\|^2 - \|\vec{C}\|^2 = R^2 - R^2 = 0$$

$\Rightarrow$ Se l'origine è nel circocentro, $\vec{H} = \vec{A} + \vec{B} + \vec{C}$ sta sulle altezze $\Rightarrow$ è l'ortocentro.