DISUGUAGLIANZE

Titolo nota 05/09/2013

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

Proof:

1) $a_i a_j b_i b_j$ $\qquad a_i^2 b_j^2$

$$\sum_{i,j} (a_i b_j - a_j b_i)^2 \geq 0$$

$$\sum_{i,j} \underbrace{a_i^2 b_j^2 + a_j^2 b_i^2} - 2 a_i b_j a_j b_i \geq 0$$

2) Disomogeneizzazione

$f(a, b, c, \ldots)$ omogenea

$$f(\lambda a, \lambda b, \lambda c, \ldots) = \lambda^g f(a, b, c)$$

$$\sum_c a \sqrt{\frac{b^2 + ac}{2}} \qquad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \qquad \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

$g = 2 \qquad g = -1 \qquad g = 0$

Th $f(a, b, c, \ldots) \geq 0$ omogenea di grado d_f

è vera se e solo se è vera

per tutte le n-uple che sodisfano un qualche vincolo omogeneo $g(a,b,c)=1$ (di grado $d_g \neq 0$)

Dim:

$$\overset{f}{a^3+b^3+c^3-3abc \geq 0} \qquad \overset{g}{a+b+c=1}$$

$\Rightarrow$) ovvia

$\Leftarrow$) $a, b, c, \dots \qquad g(a,b,c)=K$

$$g(\underset{\alpha}{\lambda a}, \underset{\beta}{\lambda b}, \underset{\gamma}{\lambda c}) = \lambda^{d_g} \cdot K = 1$$

scelgo λ in modo che

$$a+b+c=1?$$

$$\frac{a}{a+b+c} = \alpha, \quad \frac{b}{a+b+c} = \beta, \quad \frac{c}{a+b+c} = \gamma$$

$$0 \leq f(\alpha,\beta,\gamma) = f(\lambda a, \lambda b, \lambda c) = \lambda^{d_f} f(a,b,c)$$

se $abc=1$

$$(abc)^{1/3}(a^2+b^2+c^2)+3 \geq 3(ab+bc+ca)(a+b+c)$$

abc

poi vedremo tecniche per trattare disug. omogenee

Perché non ci sono disug. fatte così?

$$a^3+b^3+c^3 \geq a^2+b^2+c^2 \qquad a=b=c=\frac{1}{N}$$

$$a^3+b^3+c^3 \leq a^2+b^2+c^2 \qquad a=b=c=N$$

$$a^2+1 \geq 2a$$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

Disomogeneizzo: $\sum a_i^2 = 1$, $\sum b_i^2 = 1$ $(*)$

Devo dim. che $\sum a_i b_i \leq 1$ se vale $(*)$

$$\sum_i a_i b_i \leq \sum_i \frac{a_i^2+b_i^2}{2} = \frac{\sum a_i^2 + \sum b_i^2}{2} \overset{*}{=} 1$$

$$\sum_i a_i b_i c_i \leq \underbrace{\left(\sum a_i^3\right)^{1/3}}_{=1} \underbrace{\left(\sum b_i^3\right)^{1/3}}_{=1} \underbrace{\left(\sum c_i^3\right)^{1/3}}_{=1} \quad \text{disomogeneizzo}$$

$$\sum a_i b_i c_i \leq \sum \frac{a_i^3+b_i^3+c_i^3}{3}$$

TST 09

$$(a_1 a_2 \cdots a_n + b_1 b_2 \cdots b_n)^n \leq (a_1+b_1)^n (a_2+b_2)^n \cdots (a_n+b_n)^n$$

$(a_1, b_1) \quad (a_2, b_2) \quad \cdots \quad (a_n, b_n)$

Hölder: se $\frac{1}{p}+\frac{1}{q}=1$

$$\sum_i a_i b_i \leq \Big(\underbrace{\sum_i a_i^p}_{=1}\Big)^{1/p} \Big(\underbrace{\sum_i b_i^q}_{=1}\Big)^{1/q}$$

$$\sum_i a_i b_i \leq$$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \qquad \frac{1}{p}+\frac{1}{q}=1$$

$w_1 = \frac{1}{p}$ $w_2 = \frac{1}{q}$ $\qquad x = a^{1/p}$, $y = b^{1/q}$

$$\sqrt[w_1+w_2]{x^{w_1} y^{w_2}} \leq \frac{w_1 x + w_2 y}{w_1+w_2}$$

$$\underbrace{x, x, x, x}_{w_1} \qquad \underbrace{y, y, y, y, y}_{w_2}$$

$$\sum a_i b_i c_i \leq \Big(\sum a_i^p\Big)^{1/p} \Big(\sum b_i^q\Big)^{1/q} \Big(\sum c_i^r\Big)^{1/r}$$

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$$

$$\underbrace{\begin{matrix} a_1 & a_1 \\ b_1 & b_1 \end{matrix}}_{w_1} \quad \underbrace{\begin{matrix} a_2 \\ b_2 \end{matrix}}_{w_2} \quad \underbrace{\begin{matrix} a_3 & a_3 & a_3 \\ b_3 & b_3 & b_3 \end{matrix}}_{w_3}$$

$$\Big(\sum w_i \cdot a_i b_i\Big)^2 \leq \Big(\sum w_i a_i^2\Big)^{1/2} \Big(\sum_i w_i b_i^2\Big)^{1/2}$$

Nesbitt a,b,c

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{3}{2}$$

$$\frac{a}{b+c} = a_i^2$$

$$a_i = \frac{\sqrt{a}}{\sqrt{b+c}} \qquad b_i = \sqrt{b+c}\sqrt{a}$$

$$\left(\sum_{cyc} a\right)^2 \leq \left(\sum_{cyc} \frac{a}{b+c}\right)\left(\sum_{cyc} (b+c)a\right)$$

$$\left(\sum_{cyc} a\right)^2 \geq \frac{3}{2}\left(\sum_{cyc} (b+c)a\right)$$

$$a^2+b^2+c^2+2ab+2bc+2ca \geq 3\,(ab+bc+ca)$$

$$a^2+b^2+c^2 \geq ab+bc+ca$$

$$\sum a^2 \geq \sum ab$$

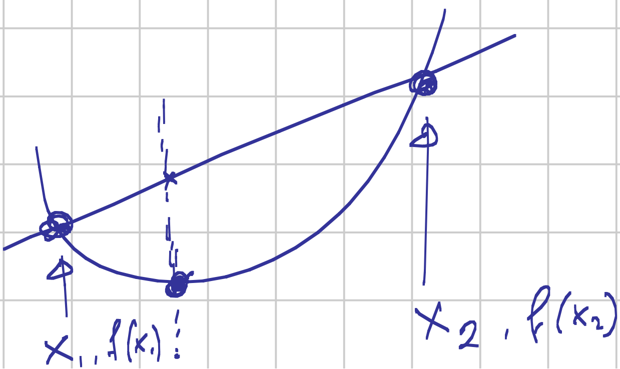
C.S.

$$\sum \frac{a}{b+c} \geq \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} (b+c)a} = \frac{\sum_{cyc} a^2 + 2\sum ab}{2\sum_{cyc} ab} \geq \frac{3\cancel{\sum ab}}{2\cancel{\sum ab}}$$

$$\geq \qquad \geq \qquad \leq$$

Convessità

Def: f convessa se "tiene l'acqua" "sorride"



$\lambda x_1 + (1-\lambda) x_2 \qquad \lambda \in [0,1]$

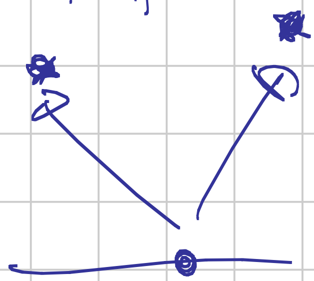
$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

$n=14$

$2 \to 4 \to 8 \to 16 \to 15 \to 14$

Jensen $\quad w_1 + w_2 + \dots + w_n = 1$, f convessa
$x_1, x_2, \dots x_n$ nel dominio di f

$$f\left(\sum w_i x_i\right) \le \sum w_i f(x_i)$$

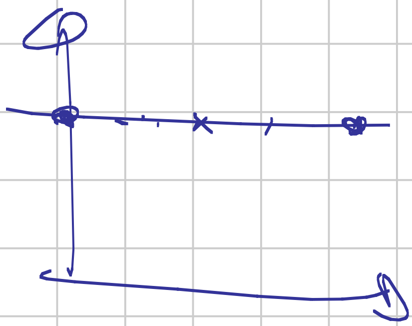


$$M \le \sum w_i f(x_i) \le \sum w_i M = M$$

Rmk: f convessa assume massimo sui bordi

f convessa su $[-1, 1]$

$$\max f = \begin{cases} f(1) \\ f(-1) \end{cases}$$



(Cos'hanno di speciale i punti $-1, 1$? Sono gli unici che non si scrivono come $w_1 x_1 + w_2 x_2$ in modo non banale

$$f\left(w_1 x_1 + \dots + w_n x_n + \quad w_{n+1} x_{n+1}\right) =$$

$\alpha = w_1 + w_2 + \dots + w_n \qquad \beta = w_{n+1} \qquad \alpha + \beta = 1$

$$A = \frac{w_1 x_1 + \dots + w_n x_n}{\alpha}$$

$$= f(\alpha A + \beta B) \overset{J2}{\leq} \alpha f(A) + \beta f(B) \overset{Jn}{\leq} \quad \text{pesi} = \frac{w_n}{\alpha}$$

$$\leq \alpha \cdot \sum_{i \leq n} f(x_i) \cdot \frac{w_i}{\alpha} + \beta \cdot f(x_{n+1})$$

Es: f convessa, a, b, c nel dominio

$$\frac{4}{3}\left(f\left(\frac{a+b}{2}\right) + f\left(\frac{b+c}{2}\right) + f\left(\frac{c+a}{2}\right)\right) \leq$$

$$\leq f(a) + f(b) + f(c) + f\left(\frac{a+b+c}{3}\right)$$

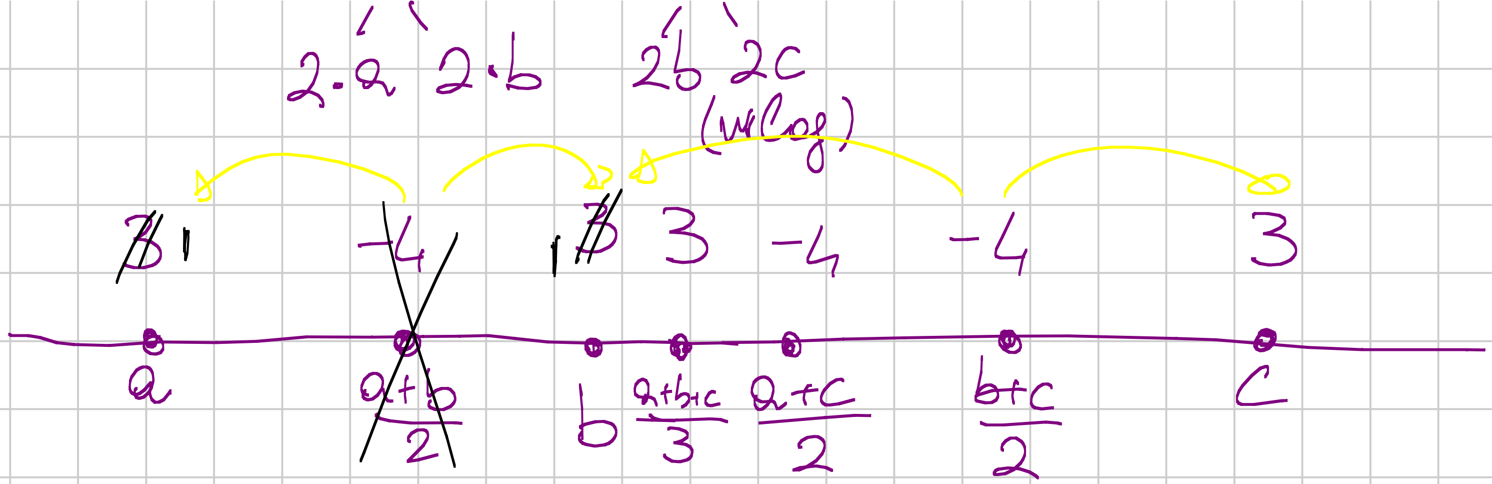
$f(x) = x$

$$4\left(\underbrace{\frac{a+b}{2}}_{\wedge} + \underbrace{\frac{b+c}{2}}_{} + \underbrace{\frac{c+a}{2}}_{\wedge}\right) = 3\left(a+b+c+\frac{a+b+c}{3}\right)$$

$$4 f\left(\frac{a+b}{2}\right) \leq 4\tfrac{1}{2} f(a) + 4\tfrac{1}{2} f(b)$$

$$f\left(\frac{b+c}{2}\right) \leq \tfrac{1}{2} f(b) + \tfrac{1}{2} f(c)$$

-4	-4	-4	$+3$	$+3$	$+3$	$+3$
$\frac{a+b}{2}$	$\frac{b+c}{2}$	$\frac{c+a}{2}$	a	b	c	$\frac{a+b+c}{3}$



$$4\,\frac{a+c}{2} = 2a+2c = a + \frac{a+b+c}{3}\cdot\lambda + (3-\lambda)\,c$$

$$2a+2c-a-3c = \left(\frac{a+b+c}{3} - c\right)\lambda$$

$$a-c = \left(\frac{a+b-2c}{3}\right)\lambda$$

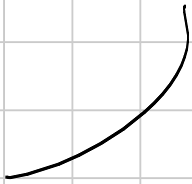
$$\lambda = 3\,\frac{c-a}{2c-b-a}$$

$$3-\lambda = 3\left(1-\frac{c-a}{2c-b-a}\right) = 3\cdot\frac{c-b}{2c-b-a}$$

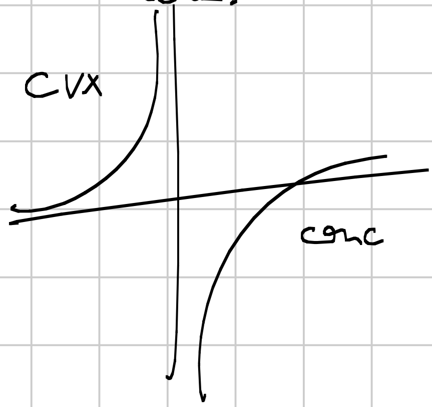
$$4f\left(\frac{a+c}{2}\right) \leq f(a) + \frac{3c-a}{2c-b-a}\,f\left(\frac{a+b+c}{3}\right) + \underbrace{3\,\frac{c-b}{2c-b-a}}_{\geq 3}\,f(c)$$

se ho uso troppi c

Come si riconosce una f. convessa?



$y=x^2$ convessa



$$\frac{a+bx}{c+dx}$$

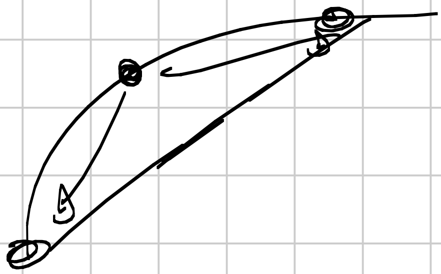
$f(x)+g(x)$ convessa

$f(x)\cdot g(x)$ non sempre!

rette = concave convesse

$f(x)$ concava $\quad -f(x)$ convessa

ES. per $\sqrt{x}$ vale "Jensen al contrario"



Oppure: derivata seconda positiva $\cup$ $\cap$

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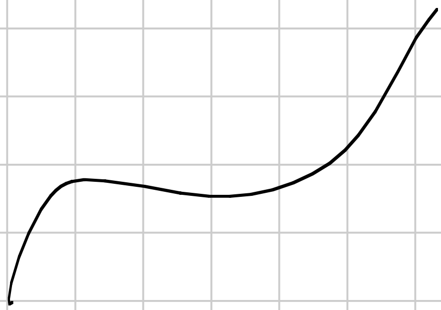
f. continua

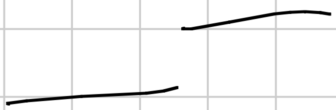
"si disegna senza staccare la penna"

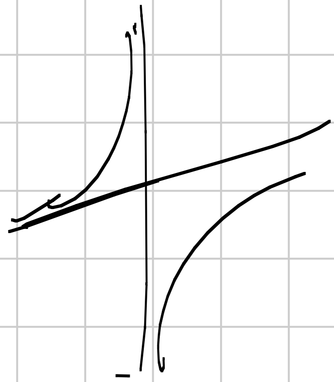


→ polinomi → funzioni notevoli

→ somme → prodotti → valori assoluti

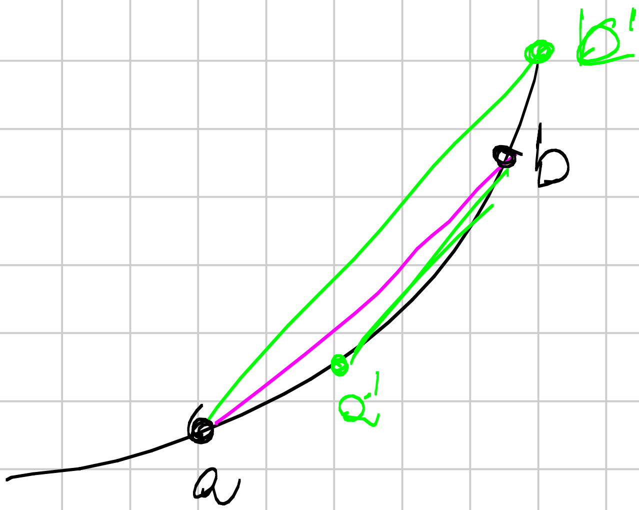
→ tutto quello che non annulla denominatori





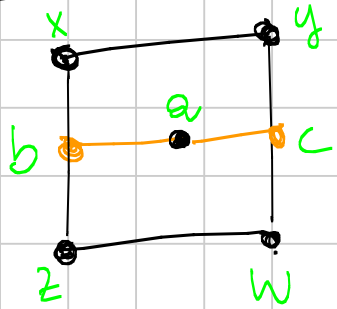
f continua è convessa $\Leftrightarrow$ "midpoint convex"

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$



$$M(a,b) \quad \frac{f(b)-f(a)}{b-a}$$

f convessa $\Leftrightarrow$ $M(a,b)$ crescente in a, b



$$f(a) \le \square \cdot f(b) + \square \cdot f(c) \le$$

$$\le \square \cdot f(x) + \square \cdot f(z) + \square\, f(y) + \square\, f(w)$$

Bunching:

$$\sum_{sym} a^2b = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

$$\sum_{sym} a^3 = 2a^3 + 2b^3 + 2c^3$$

Def p, q vettori di numeri ordinati in modo decrescente

p maggiorizza q $(p \succ q)$ se

$p_1 \geq q_1$, $p_1 + p_2 \geq q_1 + q_2$ .... fino all'ultimo dove vale uguaglianza

$$p_1 + p_2 + \dots + p_n = q_1 + q_2 + \dots + q_n$$

Thm (Bunching) se $p \succ q$, allora

$$\sum_{sym} a^{p_1} b^{p_2} c^{p_3} \dots \geq \sum_{sym} a^{q_1} b^{q_2} c^{q_3} \dots$$

per ogni n-upla $(a, b, c, \dots) \geq 0$

$$\sum_{sym} a^3 \geq \sum_{sym} a^2 b$$

Come altro si dimostra?

Riarrangiamento

$$\begin{bmatrix} a^2 & b^2 & c^2 \\ b & c & a \end{bmatrix}$$

AM-GM PESATA

$$a^2 b \le \frac{2}{3}a^3 + \frac{1}{3}b^3$$

$$\sum_{sym} a^5 b \ge \sum_{sym} a^3 b^2 c$$

$$[5 \quad 1 \quad 0] \qquad [3 \quad 2 \quad 1]$$

$$5 \ge 3$$

$$5+1 \ge 3+2$$

$$5+1+0 = 3+2+1$$

a cosa serve? Smonta coi conti disuguaglianze omogenee, simmetriche, polinomiali

$$\sum_{cyc} \frac{a}{b+c} \overset{?}{\ge} \frac{3}{2}$$

$$2\sum_{cyc} a(a+b)(a+c) \overset{?}{\ge} 3(a+b)(b+c)(c+a)$$

$$2\sum_{cyc}\left(a^3+a^2b+a^2c+abc\right) \ge 3\left[2abc+\sum a^2 b\right]$$

$$\sum_{sym} a^3 + \cancel{2\sum_{sym} a^2 b} + \cancel{\sum_{sym} abc} \ge \cancel{\sum_{sym} abc} + \cancel{3}\sum_{sym} a^2 b$$

$$7\cdot[3,1,0] + 2\cdot[3,2,1] \overset{?}{\ge} 3\cdot[4,2,0] + 6\cdot[2,2,2]$$

$$3\cdot[5,1,0] \ge 3[4,2,0]$$

$4[3,1,0] \geq 4\cdot[2,2,2]$

$2\cdot[3,2,1] \geq 2[2,2,2]$

Qualche volta non funziona

$$[3,0,0]+[1,1,1] \geq 2[2,1,0]$$

"esponenti concentrati battono esponenti sparpagliati"

Disuguaglianza di Schur:

$$\sum a(a-b)(a-c) \geq 0$$

per ogni $a,b,c \geq 0$

vale l'= se sono tutti uguali o ci sono due uguali e il terzo zero

Dim $a \geq b \geq c$

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-b)(c-a) \overset{?}{\geq} 0$$

$+ \quad + \quad + \quad - \quad - \quad -$

$+$ I $\qquad - $ II $\qquad +$ III

I + II ≥ 0 $\qquad$ III ≥ 0

$$(a-b)\left[a^2-ac+bc-b^2\right]=(a-b)(a-b)(a+b-c) \geq 0$$

$$\sum_{cyc} a^r(a-b)(a-c) \geq 0$$

se x, y, z sono ordinati nello stesso modo di a, b, c

allora $\sum_{cyc} x(a-b)(a-c) \geq 0$ [Schur-Vornicu]

Schur:

$$0 \leq \sum_{cyc} a(a-b)(a-c) = \sum_{\cancel{cyc}\ sym} a^3 - a^2b - a^2c + abc$$

$$\sum_{sym} a^3 + abc \geq 2\sum_{sym} a^2 b$$

$$[3,0,0] + [1,1,1] \geq 2[2,1,0]$$

$$[r+2,0,0] + [r,1,1] \geq 2[r+1,1,0]$$

(E) Bunch/Schur: BMO 2012-2

$$\sum_{cyc} \underbrace{(x+y)}_{c} \sqrt{\underbrace{(z+y)}_{a}\underbrace{(z+x)}_{b}} \geq 4(xy+yz+zx)$$

[Ravi substitution
Se ho disuguaglianze su a, b, c lati di un triangolo
$a \to x+y$ e cicliche
e mi semplifica le condizioni $x \geq 0$, $y \geq 0$, $z \geq 0$]

$$\sum_{cyc} c\sqrt{ab} \overset{?}{\geq} 4\sum_{cyc} \frac{(b+c-a)}{2}\frac{(c+a-b)}{2} \qquad (a,b,c\geq 0)$$

$a \to p^2$

$b \to q^2$

$c \to r^2$

$$\sum_{\cancel{cyc}\ sym} r^2pq \overset{?}{\geq} \sum_{\cancel{cyc}\ sym} (p^2+q^2-r^2)(r^2+p^2-q^2)$$

$$\cancel{2}\sum_{sym} r^4 + \sum_{sym} r^2pq + \text{roba} \geq \sum_{sym} \cancel{p^4} + 2\sum p^2q^2$$

$$[4,0,0] + [2,1,1] \geq 2\cdot[2,2,0]$$

$$\vee\!/$$

$$2\cdot[3,1,0] \geq$$

$$\max_{\sigma} \sum a_i a_{\sigma(i)}$$

$$a_i a_{\sigma(i)} + a_j a_{\sigma(j)} < a_i a_{\sigma(j)} + a_j a_{\sigma(i)}$$

se $a_i > a_j$

$a_{\sigma(i)} < a_{\sigma(j)}$

$(a, b, c, \dots)$ A somma fissata $a+b+c+\dots$, quando massimizzo il prodotto?

Supponiamo che ci sia un max e che non sono tutti uguali

wlog $a<b$

li rimpiazzo con $\frac{a+b}{2}, \frac{a+b}{2}$

$$\frac{(a+b)^2}{4} \geq ab$$

Thm: il più grande numero naturale è 1

Supponiamo che sia $a>1$

$a^2 > a$ assurdo!

"Smoothing": se devo dimostrare che una cosa ha max per $a=b=c=\dots$,
provo ad avvicinarli e vedo cosa succede

"unsmoothing"

Come si aggiusta la proof. di AM-GM?

$1,1,5 \to 1,3,3 \to 2,2,3 \to 2,\frac{5}{2},\frac{5}{2} \dots$

$\dots \left(\frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)$

A media aritmetica

$$\left(\overset{A}{\hat{a}}, \underset{A}{\underset{\wedge}{b}}\right) \not\to \left(\frac{a+b}{2}, \frac{a+b}{2}\right)$$

$$\to \left(A, a+b-A\right)$$

$$abcd \cdots \le Ab'cd \cdots \le AAc'd \cdots \le \cdots \le (A \cdot A \cdot A \cdot A)^n$$

Altro modo di aggiustarlo: dimostrare che $\exists$ massimo!

Weierstrass   Weierstraß

Funzione continua su un compatto assume un valore massimo M e un minimo m (e assume anche tutti i valori in mezzo almeno una volta, se è connesso)

compatto insieme chiuso + limitato

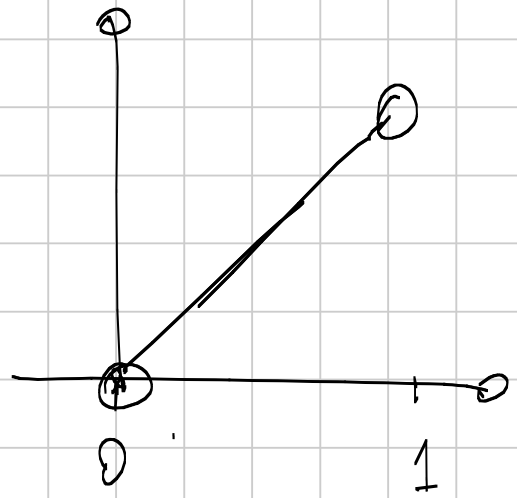
Limitato: le variabili non crescono troppo

$$S: \begin{cases} a+b+c+d=1 \\ a \geq 0 \quad d \geq 0 \\ b \geq 0 \\ c \geq 0 \end{cases} \qquad f(a,b,c,d) = a \cdot b \cdot c \cdot d$$

$$abc = 1$$

$$M^2 \cdot \frac{1}{M} \cdot \frac{1}{M}$$

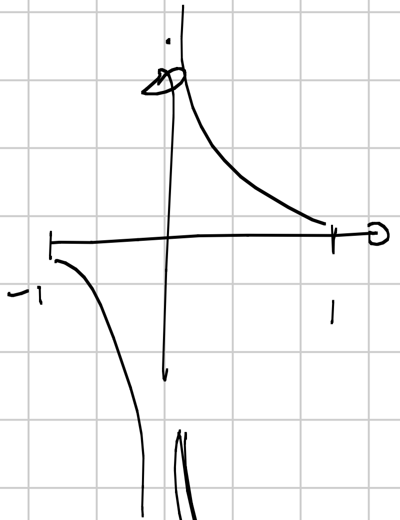
Chiuso: "definito solo con $=$ e $\geq$ e $\leq$"



$S = (0,1)$

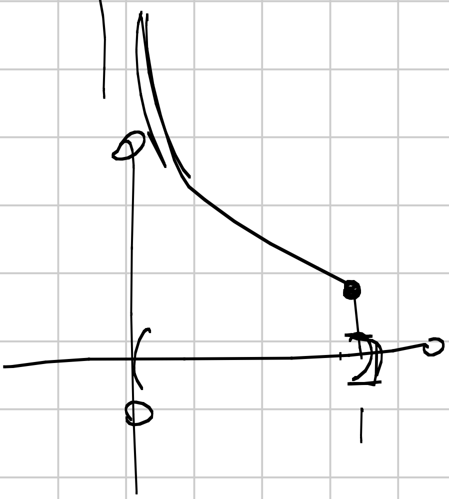
$f(x) = x$

$0 < x < 1$



$f(x) = 1/x$

$x = [-1, 1]$

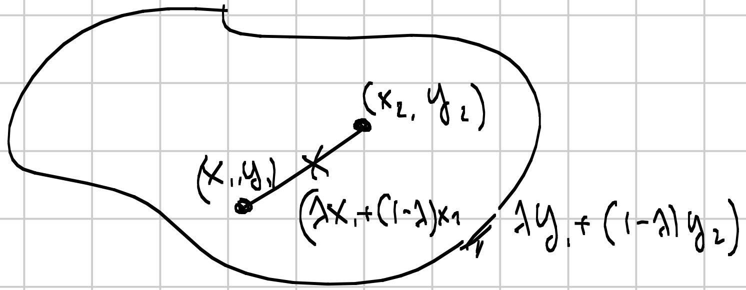


$(0,1]$ $\quad 0 < x \leq 1$

$$C_1 \leq \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \frac{x_4}{x_5} + \frac{x_5}{x_1} \leq C_2$$

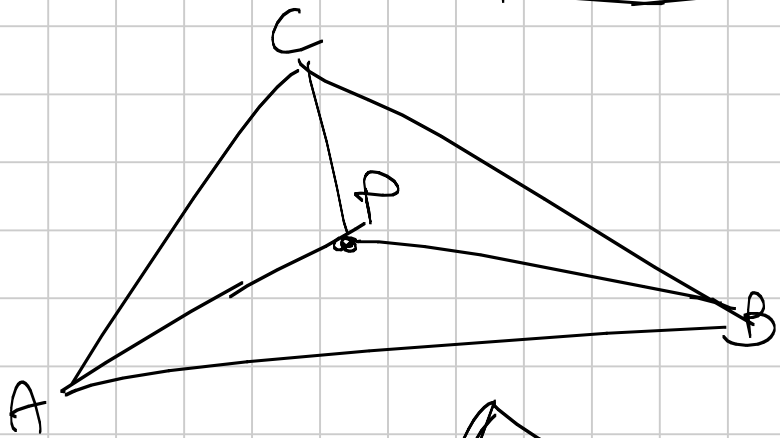
$x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}^+$

$$C_1 \le \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \le C_2$$

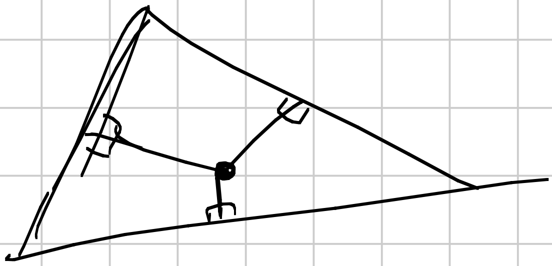


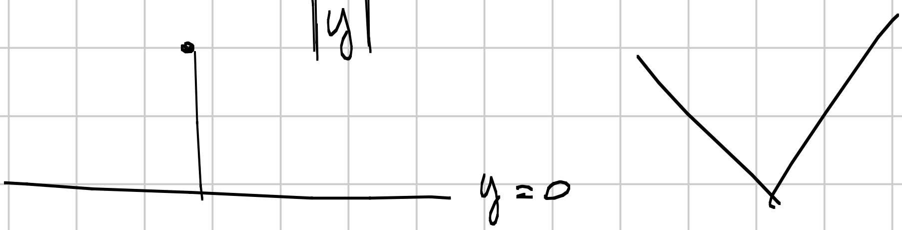
$$\lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2) \ge f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$$

distanza da un punto è convessa



$PA + PB + PC$





ABC, PQR, PQS

$$\sum_{cyc} a(a-b)(a-c) \geq 0$$

$$\sum_{sym} a^3 - 2a^2b + abc \geq 0$$

$$\begin{cases} S = a+b+c \\ Q = ab+bc+ca \\ P = abc \end{cases}$$

$$\sum a^2 = S^2 - 2Q$$

$$S^3 - 2QS = \sum a^2(a+b+c) = \sum_{cyc} a^3 + \sum_{cyc} a^2b + \sum_{cyc} a^2c =$$

$$= \frac{1}{2}\sum_{sym} a^3 + \sum_{sym} a^2b$$

$$S \cdot Q = \sum_{cyc} ab(a+b+c) = \sum_{cyc} abc + ab^2 + a^2b =$$

$$= \sum_{sym} a^2b + \frac{1}{2}\sum_{sym} abc$$

$$\boxed{\sum_{sym} a^2b = SQ - 3P}$$

$$S^3 - 2QS = a^3 + b^3 + c^3 + SQ - 3P$$

Schur: $\sum_{cyc} a^3 - a^2b - a^2c + abc \geq 0$

$$= S^3 - 2QS - 2\sum_{cyc}(a^2b + a^2c) + 3P \geq 0$$

$$= S^3 - 2QS - 2(QS - 3P) + 3P =$$

$$\boxed{= S^3 - 4SQ + 9P \geq 0}$$

Newton Maclaurin

$$\sqrt[3]{abc} \leq \sqrt{\frac{ab+bc+ca}{3}} \leq \frac{a+b+c}{3}$$

$$\sqrt[4]{abcd} \leq \sqrt[3]{\frac{abc+abd+acd+bcd}{4}} \leq \sqrt{\frac{ab+bc+cd+\dots}{6}} \leq$$

$$\leq \frac{a+b+c+d}{4}$$

$$\underbrace{abc}\underbrace{\left(\frac{a+b+c}{3}\right)} \leq \underbrace{\left(\frac{ab+bc+ca}{3}\right)^2}$$

$$d_k = \frac{\sum(\prod \text{a } k \text{ a } k \text{ delle } n \text{ variabili})}{\binom{n}{k}}$$

(controlla segno)

$$\sqrt[k]{d_k} \leq \sqrt[k-1]{d_{k-1}} \qquad d_{k+1}\, d_{k-1} \leq d_k^2$$

$$S^3 - 4SQ + 9P \geq 0$$

Metodo ABC

<u>Th</u>: Fissati S, Q, P di una certa terna a, b, c di reali positivi, esiste una terna di reali positivi (a', b', c') che ha gli stessi S, Q, P maggiore (o minore, a nostra scelta) e vale una di queste condizioni: $\begin{cases} a' = b' \\ c' = 0 \end{cases}$ $(*)$

Perché serve?

Per esempio, per dimostrare Schur!

$$S^3 - 4SQ + 9P \geq 0$$

Supponiamo di aver dimostrato Schur per terne del tipo $(*)$; allora,

(a, b, c)

prendo a', b', c' tali che
$$S' = S$$
$$Q' = Q$$
$$P' \leq P$$

<u>allora</u>

$$S^3 - 4SQ + 9P \geq \underbrace{S^3 - 4SQ + 9P'}_{\text{Questo è Schur per } (a', b', c')} \overset{!}{\geq} 0$$

<u>Se</u> so dimostrare Schur per terne $(*)$ allora ho vinto!

Dimostriamolo per terne (*):

1) $a' = b'$

$$\cancel{a'(a'-b')(a'-c')} + \cancel{b'(b'-a')(b'-c')} + \cancel{c'(c'-a')(c'-b')} \geq 0$$

$$c'(c'-a')^2$$

2) $c' = 0$

$$a^2(a-b) + b^2(b-a) \geq 0$$
$$= (a^2-b^2)(a-b)$$
$$= (a-b)^2(a+b) = 0$$

Questo trucco funziona se la disug. è monotona in P

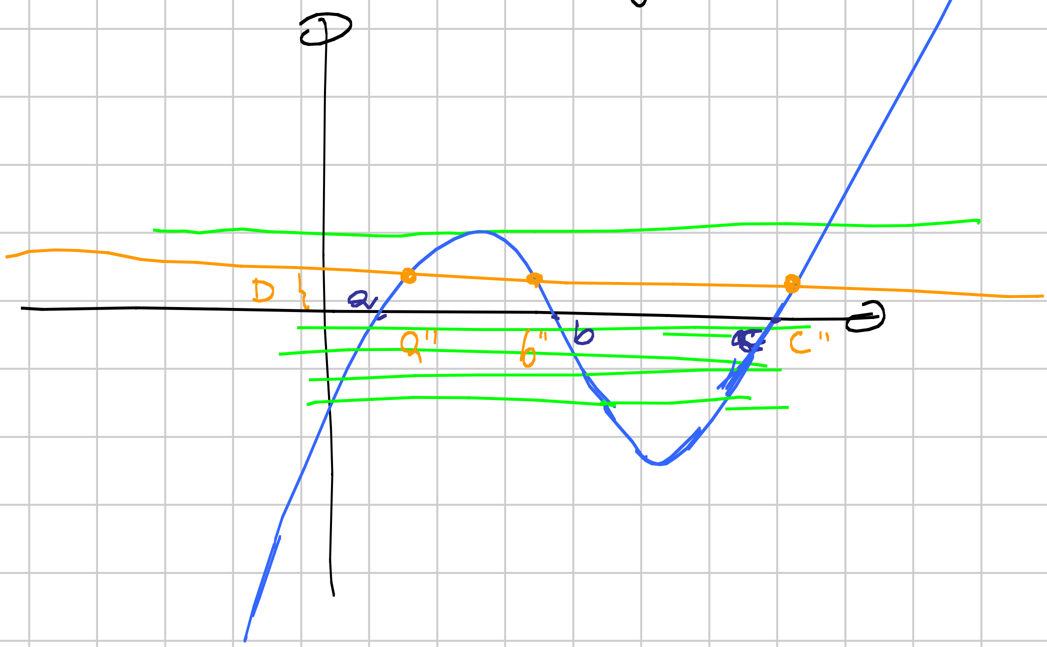
$$f(S,Q,P) \underset{\text{monotonia}}{\geq} f(S,Q,P') \underset{\text{caso facile}}{\geq} 0$$

Dim. teorema PQS

a, b, c sono radici di

$$p(x) \quad (x-a)(x-b)(x-c) = x^3 - Sx^2 + Qx - P$$

Com'è fatto il grafico di $p(x)$?



a'', b'', c'' sono radici $p(x) - D =$

$$= x^3 - Sx^2 + Qx - P - D$$

Faccio "salire o scendere la retta"
finché non trovo una ferma non più valida

Cosa succede nei casi estremi?

- $c = 0$ (una delle due sol. smette di essere positiva)

- $a = b$ (pto di tangenza, se alzo/abbasso ancora non ho più sol. reali)

$f(S, Q, P)$ monotona in P

Se il poly. simmetrico in a, b, c da cui partivo ha grado al più 5, allora ha grado in P al più 1 (perché P^2 ha già grado 6)

$\Rightarrow$ Questo trucco funziona sempre se il poly. simmetrico ha grado ≤ 5

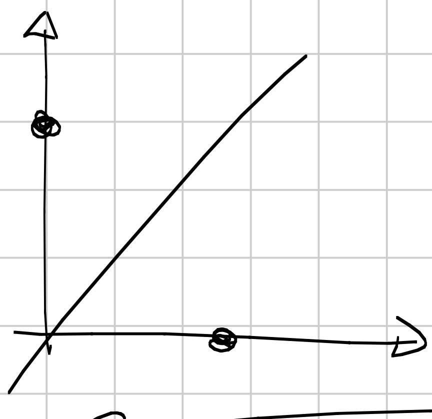
Data disuguaglianza simmetrica, polinomiale di grado ≤ 5 in 3 variabili, allora mi basta dimostrarla nei due casi particolari: $\begin{cases} \text{due uguali} \\ \text{uno zero} \end{cases}$

$$\sum P^2 - \cancel{QSP} + S$$

$$\sum \frac{a}{b+c} \geq \frac{3}{2}$$

Se io ho una disuguaglianza di grado $d \geq 2$ in n variabili, allora per vedere se vale per tutte le n-uple $(a_1, a_2 \dots a_n) \geq 0$, mi basta verificare che vale per tutte le n-uple che hanno al più $\left\lfloor \frac{d}{2} \right\rfloor$ valori $\boxed{\text{positivi}}$ distinti

"half-degree principle"



IMO 01-2

$$\sum \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

CS su $\frac{\sqrt{a}}{\sqrt[4]{a^2+8bc}}$, $\sqrt{a}\sqrt[4]{a^2+8bc}$

$$LHS \geq \frac{\left(\sum a\right)^2}{\sum a\sqrt{a^2+8bc}}$$

Mi manca

$$\sum a\sqrt{a^2+8bc} \leq \left(\sum a\right)^2$$

Provo...

$$a\sqrt{a^2+8bc} \leq \frac{a^2+a^2+8bc}{2} \qquad (*)$$

$$LHS \geq \text{robe} \geq \text{robe} \geq \text{robe} \geq 1$$

se uguaglianza, tutti =

Ma $(*)$ non soddisfa le stesse...

AM-GM su

$$3a \cdot \sqrt{a^2+8bc} \leq \frac{9a^2+a^2+8bc}{2}$$

"POI" Point of incidence

$$a\sqrt{a^2+8bc} \le \frac{1}{3}\left(5a^2+4bc\right)$$

$$\sum \underbrace{a}_{x_i} \underbrace{\sqrt{a^2+8bc}}_{y_i} \le (a+b+c)^2$$

$$\sum_i a\sqrt{a^2+8bc} \le \left(\sum_i a^2\right)^{1/2}\left(\sum_i a^2+8bc\right)^{1/2} \overset{\text{HOPE}}{\le} (a+b+c)^2$$

$$\left(\sum_{\text{cyc}} a^2\right)\left(\sum_{\text{cyc}} a^2+8bc\right) \overset{\text{HOPE}}{\le} (a+b+c)^4$$

$$1\cdot\left(a^4+b^4+c^4\right) \quad \checkmark \quad 1\cdot\left(a^4+b^4+c^4\right)$$

$$8\sum_s \boxed{a^3b} \qquad 4\,a^3b$$

"

CS

$$\sum \underbrace{\sqrt{a}}_{x_i}\underbrace{\sqrt{a^3+8abc}}_{y_i}$$

Questa volta funziona! =)