# Senior 2013. Algebra 3

!!Sthe W!!

Titolo nota 06/09/2013

## Eq. funzionali

**Es.** (Cauchy) Trovare tutte le funzioni

$$f : \mathbb{Z} \to \mathbb{Z}$$

tali che

(1) $f(x+y) = f(x) + f(y) \quad \forall x,y \in \mathbb{Z}$

**Sol.**

$P(a,b)$: $\quad f(a+b) = f(a) + f(b)$

$P(x,0)$: $\quad f(x+0) = f(x) + f(0) \quad (\forall x \in \mathbb{Z})$

$$\cancel{f(x)} = \cancel{f(x)} + f(0)$$

$$f(0) = 0.$$

$P(x,-x)$: $\quad 0 = f(0) = f(x) + f(-x)$

$$f(-x) = -f(x) \quad (f \text{ è dispari})$$

$P(1,1)$: $\quad f(2) = f(1) + f(1) = 2f(1)$

$P(x,x)$: $\quad f(2x) = 2f(x)$

$P(x,1)$ : $f(x+1) = f(x) + f(1)$ $(x \in \mathbb{Z})$

Possiamo provare per induzione che $f(x) = x f(1)$

Dim. Passo base $x=0$ ok.

Passo induttivo : $f(n) = n f(1)$ Hp. ind.

$\rightsquigarrow f(n+1) \overset{?}{=} (n+1) f(1)$

- con $x=n$ $\rightsquigarrow$ $f(n+1) = f(n) + f(1) = (n+1) f(1)$.

Abbiamo dimostrato che $f(n) = n f(1)$ $\forall n \in \mathbb{N}$

$f(-n) = -f(n) \longrightarrow f(z) = z f(1)$ $\forall z \in \mathbb{Z}$.

Abbiamo trovato che $f(x) = x f(1)$. Chi può essere $f(1)$? Sappiamo che $f(1) \in \mathbb{Z}$ $(f: \mathbb{Z} \to \mathbb{Z})$; va bene qualunque intero.

$$f(x) = ax \qquad (a \in \mathbb{Z})$$

(verifico che funziona) $\leftarrow$ SEMPRE.

$f: \mathbb{Q} \longrightarrow \mathbb{Q}$ $\qquad f(nx) = n f(x)$ (ind. su $n$.)

$x = \frac{p}{q}$ $\quad p f(1) = f\left(q \cdot \frac{p}{q}\right) = q f\left(\frac{p}{q}\right)$

$$f\left(\frac{p}{q}\right) = \frac{p}{q} f(1) \qquad \left(f(x) = x f(1) \ \forall x \in \mathbb{Q}\right).$$

$$f(x+y) = f(x) + f(y) \quad \forall\, x,y \in \begin{smallmatrix}\mathbb{Z}\\ \mathbb{Q}\end{smallmatrix}$$

$$\Longrightarrow \quad f(x) = ax \quad \text{per qualche } a. \quad \left(x \in \begin{smallmatrix}\mathbb{Z}\\ \mathbb{Q}\end{smallmatrix}\right)$$

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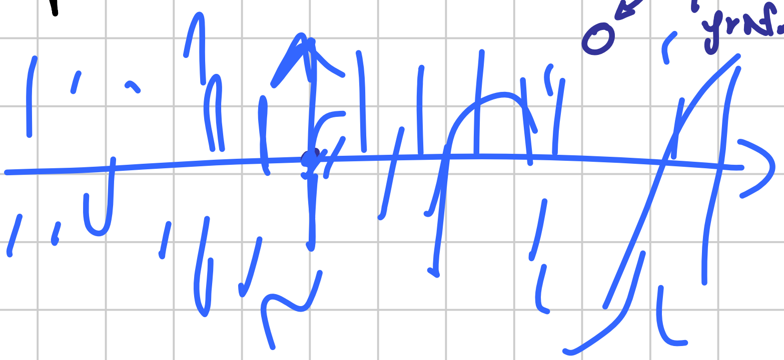
$f: \mathbb{R} \to \mathbb{R}$ $\quad \exists a:$ $\quad f(x+y) = f(x) + f(y).$

posso dire che $f(x) = ax \quad \forall\, x \in \mathbb{R}$ ?

**NO!!**, Esistono funzioni brutte a piacere che sono "lineari" ma non sono delle rette. ("basi di Hamel").

**Remark.** Appena ho su $f$ una ipotesi di "regolarità", riottengo $f(x) = ax \quad \forall\, x \in \mathbb{R}$.

Es.
- $f$ crescente (o decrescente)
- $f$ limitata in un intervallo
- il grafico di $f$ non contiene una pallina in $\mathbb{R}^2$:



qui non c'è il grafico di $f$

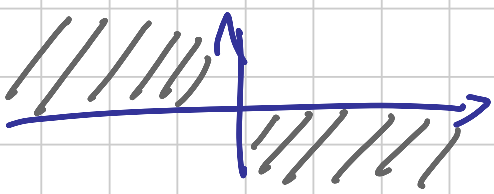
$\Downarrow$

$f(x) = ax \quad \forall x \in \mathbb{R}$

(Es. se $f$ è crescente $\Rightarrow$ $f(x) \leq f(0) \quad \forall x \leq 0$

$f(x) \geq f(0) \quad \forall x \geq 0$

il grafico di $f$ non c'è


)

$f: \mathbb{R} \to \mathbb{R}$.

**Ex.** $\cdot\ f(x^2+y) = f(x)^2 + f(y) \quad \forall\ x,y \in \mathbb{R}$

$P(0,0) \quad \cancel{f(0)} = f(0)^2 \cancel{+ f(0)} \rightsquigarrow f(0) = 0$

$P(x,0) \quad f(x^2) = f(x)^2$

$\overset{1}{=}\ f(x^2+y) = f(x^2) + f(y)$

$x^2 = a$

$\downarrow$

$\overset{2}{=}\ f(a+y) = f(a) + f(y) \quad \left(\begin{array}{l} a \geq 0 \\ y \in \mathbb{R} \end{array}\right)$

f è cresente : perchè?

f è positiva sui positivi $\left(f(x^2) = (f(x))^2 \geq 0\right)$.

$f(a-a) = f(a) + f(-a) \rightsquigarrow f(a) = -f(-a)$

(dispari).

$x, y \in \mathbb{R}$

$f(x+y) = f(x) + f(y)$ (se almeno uno tra x e y è $\geq 0$)

(2)

$x, y \leq 0$

f dispari

$\downarrow$

$f(x+y) = -f((-x)+(-y)) \overset{(2)}{=} -\left(f(-x) + f(-y)\right)$

$\overset{f \text{ dispari}}{=} f(x) + f(y)$.

Cauchy con $f(a) \geq 0$ se $a \geq 0$. fine.

# Esistono funzioni brutte!!

$$f : [0,1] \longrightarrow [0,1]$$

$$f(f(x)) = x$$

$$\left\{ \begin{array}{l} f(f(\underbrace{f(f(x))}_{x})) = f(f(x)) = x \\ f^{(2n)}(x) = x \end{array} \right\}$$

$f(x) = x$ funziona

$f(x) = 1 - x$ (funziona)

$f(f(a)) = a$

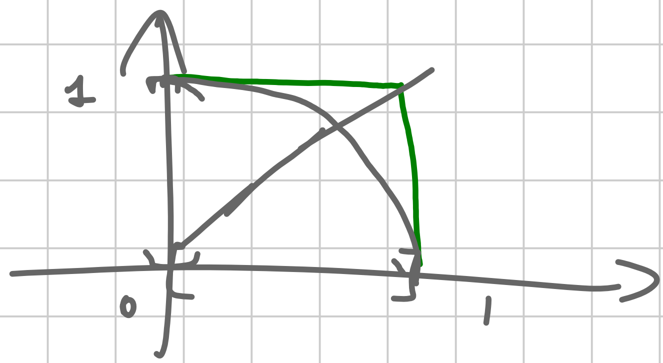
$\quad \| $

$f(b)$

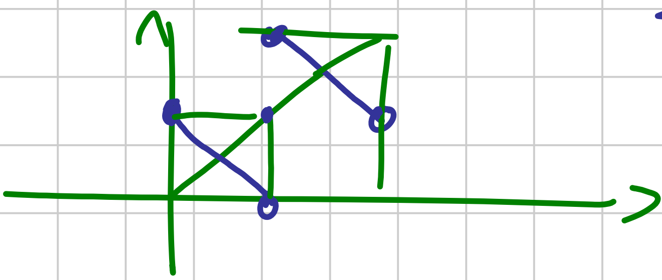
$f(a) = b \iff f(b) = a$

$(a,b) \in \Gamma_f \iff (b,a) \in \Gamma_f$

$f$ risolve $\iff$ $\Gamma_f$ e' simmetrico rispetto a $y = x$




$f(x) = \sqrt{1-x^2}$



$$f(x) = \begin{cases} \frac{1}{2} - x & x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{3}{2} - x & x > \frac{1}{2} \end{cases}$$

$\left( f(2x) = f(x) + 2 \right)$.

I 2012 (B1). $f : \mathbb{Z} \to \mathbb{Z}$

$$f(a)^2 + f(b)^2 + f(c)^2 = 2\left(f(a)f(b) + f(b)f(c) + f(c)f(a)\right)$$

$\forall\ a,b,c$ t.c. $a+b+c=0$.

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$P(0,0,0)$ : $3f(0)^2 = 6f(0)^2 \Rightarrow f(0)=0$

$P(a,-a,0)$ . $f(a)^2 + f(-a)^2 = 2f(a)f(-a)$

$$\left(f(a) - f(-a)\right)^2 = 0 \qquad f(a)=f(-a)$$

(f è pari).

$c = a+b$

• $f(a)^2 + f(b)^2 + f(a+b)^2 = 2\left(f(a)f(b) + \left(f(a)+f(b)\right)f(a+b)\right)$

( $f(a+b) \in \mathbb{Z}$ $\rightsquigarrow$ $\Delta$ eq. quadratica DEVE essere un quadrato )

$$f(a+b) = f(a)+f(b) \pm \sqrt{(f(a)+f(b))^2 - (f(a)-f(b))^2}$$

• $f(a+b)^2 - 2\left(f(a)+f(b)\right) f(a+b) + \left(f(a)^2 - 2f(a)f(b) + f(b)^2\right) = 0$

$f(a+b) = f(a) + f(b) \pm 2\sqrt{f(a)f(b)}$ $\forall\ a,b \in \mathbb{Z}$

$\cancel{m}\ k(a+b)^2 = \cancel{m}\ k^2(a) + \cancel{m}\ k^2(b) \pm 2\cancel{m}\ k(a)k(b)$

$f(a)f(b) = \square$ per ogni $a,b$

$f(a)f(1) = \square$ $\qquad f(a) = \square \cdot k$

$f(1) = n^2 \cdot \underbrace{m}_{\text{squarefree}}$ $\qquad m = p_1 p_2 \cdots p_j$ (tutti distinti!)

$f(a)\, f(1) = \square$ $\qquad f(a) \cdot \overset{p_1 \cdots p_j}{m} = \square \quad p_1 \cdots p_j$

$p_1 \cdots p_j$

$m \mid f(a)$ $\qquad f(a) = m \cdot g(a)$

$m^2\, g(a) = \square \rightsquigarrow g(a) = k(a)^2$

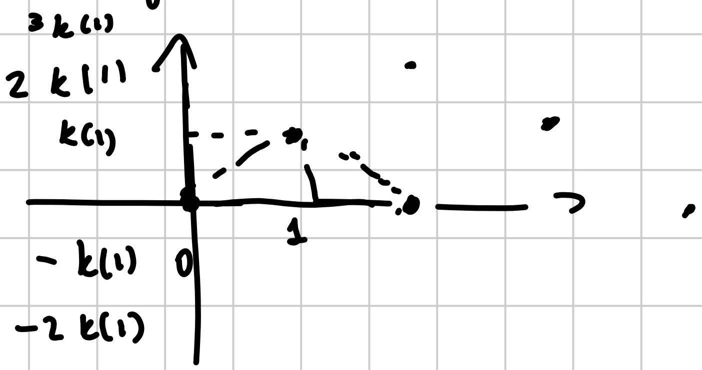
$f(a) = m \cdot k(a)^2$

$k(a+b)^2 = k(a)^2 + k(b)^2 \pm 2k(a)k(b)$

$k(a+b)^2 = (k(a) \pm k(b))^2$ (mi restringo a $k \geq 0$)

fattizia $\rightsquigarrow$ $\boxed{k(a+b) = \pm k(a) \pm k(b)}$ $\qquad k(n+1) = \pm k(n) \pm k(1)$

ho già dim. che $f(a) = f(-a)$ $\qquad k(1) = 0 \Rightarrow f \equiv 0$



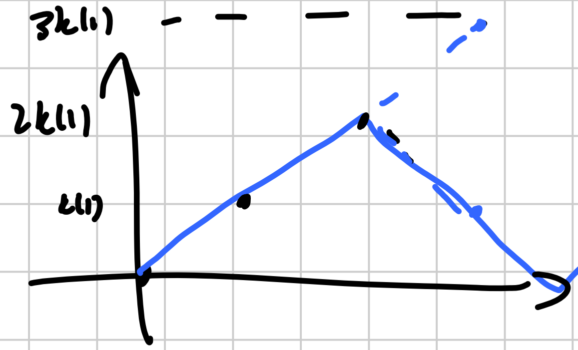

$k(2) \leftarrow 0,\ 2k(1)$ $\qquad \pm k(1)$

$k(3) =$

$k(4) =$

$$k(n) = \begin{cases} 0 & \text{se } n \text{ pari} \\ k(1) & \text{se } n \text{ dispari} \end{cases}$$

$k(2) = 2k(1)$




$k(3) = \pm 2k(1) \pm k(1)$

$k(3) = 3k(1) \to k(n) = nk(1)$

$k(3) = k(1) \to \begin{cases} k(4) = k(1) \\ k(2) = 2k(1) \\ k(3) = k(1) \end{cases}$ (mod 4)

3 tipi di soluzioni

- $f(n) = an^2 \qquad a \in \mathbb{Z}$
- $f(n) = \begin{cases} 0 & n \text{ pari} \\ \alpha \in \mathbb{Z} & n \text{ dispari} \end{cases}$
- $f(n) = \begin{cases} 0 & 4 \mid n \\ a & n \text{ dispari} \\ 4a & \text{negli altri casi.} \end{cases}$

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## Succ. per ricorrenza lineare

$$\begin{cases} x_{n+2} = 3x_{n+1} + x_n \\ x_0 = 0 \\ x_1 = 1 \end{cases} \qquad (*)$$

$$\left( x_{2009} > 2^{2009} \, ? \right)$$

1ª cosa: Se $y_n$ risolve $(*)$ e $z_n$ ris. $(*)$ allora $w_n = \alpha y_n + \beta z_n$ risolve $(*)$.

$$w_{n+2} = \alpha y_{n+2} + \beta z_{n+2} = \alpha(3y_{n+1} + y_n) + \beta(3z_{n+1} + z_n) =$$

$$= 3w_{n+1} + w_n \qquad \text{(linearità)}$$

II provo a trovare sol. della forma $x_n = \rho^n$
cosa deve essere verificato?

$$\rho^{n+2} = 3\rho^{n+1} + \rho^n$$

divido per $\rho^n$

$$\rho^2 = 3\rho + 1$$

$$\rho^2 - 3\rho - 1 = 0 \quad \leftarrow \text{(eq. caratteristica della succ. per ricorrenza)}$$

$$\rho_{1,2} = \begin{cases} \frac{3+\sqrt{13}}{2} \\ \frac{3-\sqrt{13}}{2} \end{cases}$$

$y_n = \rho_1^n \qquad z_n = \rho_2^n$ risolvono $(*)$ ; scegliendo
opportunamente $\alpha$ e $\beta$ ho che

$$\alpha\rho_1^n + \beta\rho_2^n = w_n$$

sono TUTTE le possibili soluzioni di $(*)$.

$$\alpha + \beta = 0 \qquad \alpha = -\beta$$

$$\alpha\rho_1 + \beta\rho_2 = 1$$

$$\alpha(\rho_1-\rho_2)=1 \qquad \alpha=\frac{1}{\sqrt{13}}$$

$$\beta=-\frac{1}{\sqrt{13}}$$

$$x_n=\frac{1}{\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^n-\frac{1}{\sqrt{13}}\underbrace{\left(\frac{3-\sqrt{13}}{2}\right)}_{|\cdot|<1}^{n}$$

$$\frac{3+\sqrt{13}}{2}>3 \qquad x_n \gtrsim \frac{1}{\sqrt{13}}3^n \geq 2^n \quad (n\geq 10).$$

chi è $\lfloor(2+\sqrt{3})^{2013}\rfloor$ modulo 5 ?

$$\underbrace{(2+\sqrt{3})}_{\rho_1}{}^{2013}+\underbrace{(2-\sqrt{3})}_{\rho_2}{}^{2013}=a_{2013}$$

$a_{2013}$ è intero.

eq. per $\rho_1,\rho_2 \longrightarrow x^2-4x+1$

$$a_{n+2}=4a_{n+1}-a_n$$

$$\begin{cases} a_{n+2}=-a_{n+1}-a_n \\ a_0=2 \\ a_1=-1 \end{cases}$$

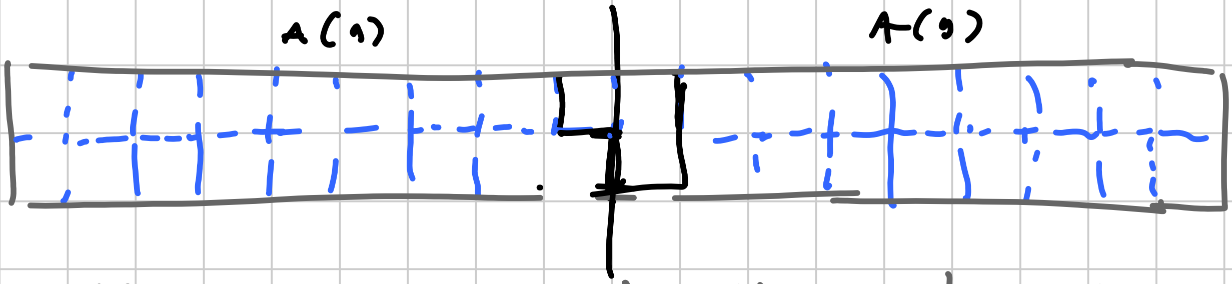
2
-1
-1
2
-1
-1
2
⋮

$$\lfloor(2+\sqrt{3})^{2013}\rfloor \equiv 1 \quad (5).$$

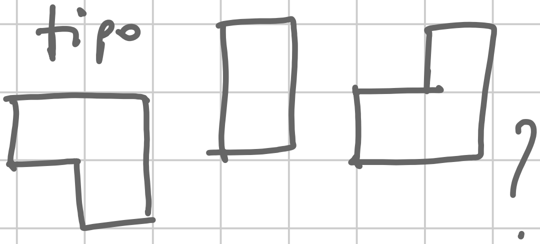
$$\frac{B(n)^2}{2}+A(n)^2+A(n-1)^2+2B(n)A(n-1)=A(2n)$$

$$A(9)^2+\frac{B(9)^2}{2}+A(8)^2+2A(8)\cdot B(9)=A(18)$$

**Es.**




in quanti modi posso tassellare il rettangolo $2 \times 18$ con mattonelle del tipo



?

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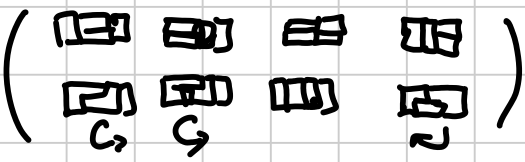
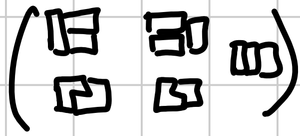
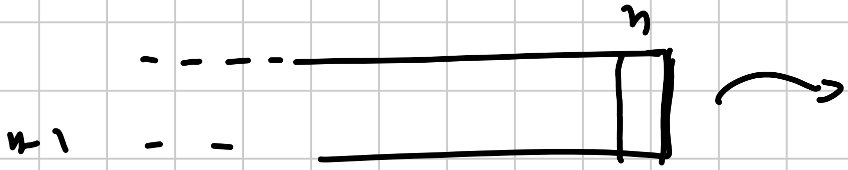
$A(n)$ = n° di modi di tassellare il $2 \times n$.

$A(1) = 1$ $A(2) = 2$ $A(3) = 5$

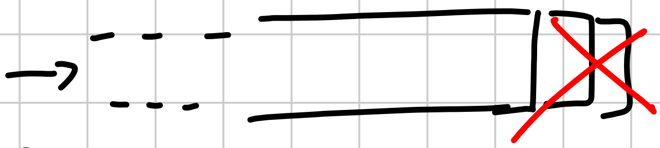
$A(4) = 11$



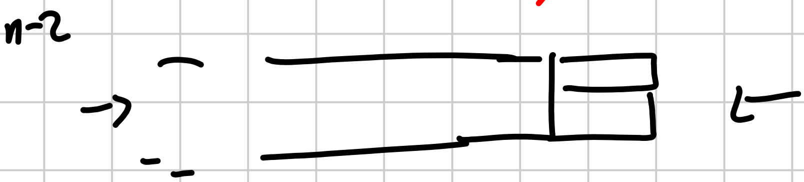





mi viene da una delle $A(n-1)$ conf. complete del $2 \times (n-1)$

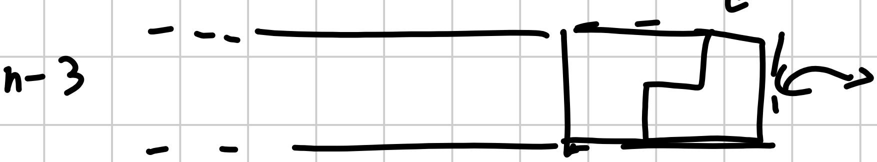


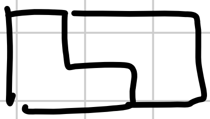
l'ho già contata



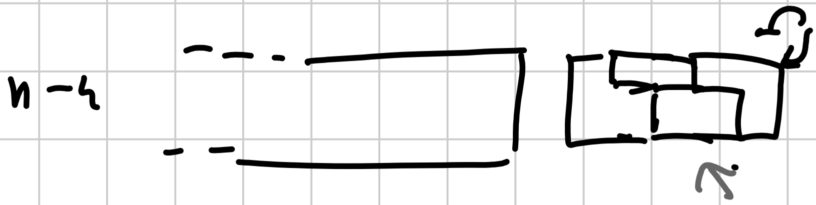

$A(n-2)$

senza altre possibili divisioni possibili all'interno



$2\,A(n-3)$




$2\,A(n-4)$

$$A(n) = A(n-1) + \cancel{A(n-2)} + 2\left(A(n-3) + \dots + A(1)+1\right)$$

$$A(n-1) = \cancel{A(n-2)} + A(n-3) + 2\left(A(n-4) + \dots + A(1)+1\right)$$

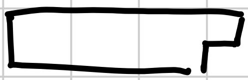
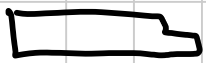
$$A(n) - A(n-1) = A(n-1) + A(n-3)$$

$$x_n = 2x_{n-1} + x_{n-3} \qquad \rho^3 - 2\rho^2 - 1 = 0$$

$$\left( x_n \sim c\cdot \rho^n + \varepsilon_n \right)$$

o

$A(n)$ n° modi  $2\times n$

$B(n)$ n° modi  opp. 

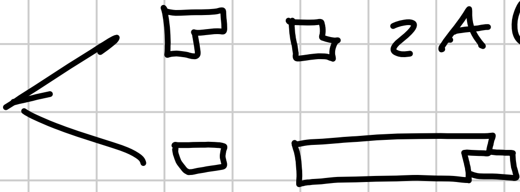
$2\times n - 1$

$A(n) =$ 

$A(n-1)$

$A(n-2)$

$B(n-1)$

$B(n) =$ 

$2A(n-2)$

$B(n-1)$

$$\begin{cases} A(n) = A(n-1) + A(n-2) + B(n-1) \leftarrow \\ B(n) = 2A(n-2) + B(n-1) \end{cases}$$

siamo int. ad $A(n)$

$$B(n-1) = A(n) - A(n-1) - A(n-2)$$

$$A(n+1) - A(n) - \cancel{A(n-1)} = 2A(n-2) + A(n) - \cancel{A(n-1)} - A(n-3)$$

$$A(n+1) = 2A(n) + A(n-2)$$

$$A(n) = \alpha \rho_1^n + \beta \rho_2^n + \gamma \rho_3^n .$$

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Trovare tutte le $g: \mathbb{N} \to \mathbb{N}$ t.c.

$$(g(m)+n)(g(n)+m)$$

è un quadrato $\forall\ m, n \in \mathbb{N}$.

$P(n,n)$ $\quad (g(n)+n)^2$ è un quadrato vero.

$P(0,n)$ $\quad (g(0)+n)\, g(n)$ è un quadrato.

$m = A^2 - g(n)$ $\quad A > g(n)$

$g(A^2 - g(n)) + n$ è un quadrato

$\left[\begin{array}{l} g(n) = n \\ \phantom{x} \end{array}\right.$ Funzione $g(n) = n + a$

$\left.(n+a+m)(n+m+a) \text{ è un quadrato}\right]$

$g(n) = g(n+1)$ ?

$$\left( (g(n) + n+1)(g(n+1)+n) \text{ è quadrato.} \right)$$

$$(g(n)+n)^2 < (g(n)+n+1)(g(n)+n) < (g(n)+n+1)^2$$

NON può essere

$g(n) = g(n+2)$?

$$(g(n)+n+2)(g(n)+n) = (g(n)+n+1)^2 - 1 \neq \square$$

$$|g(n) - g(n+1)| > 1 \qquad p \mid g(n) - g(n+1)$$

$$(g(n)+m)(g(m)+n) = \square$$

$d = g(n+1) - g(n)$.

$$(g(n)+m+d)(g(m)+n+1) = \square$$

prendo $m$ t.c. $p \mid g(n)+m$ $\qquad p^{2k+1} \parallel g(n)+m$

$$v_p(g(n)+m) = 2k+1 > v_p(d)$$

$\Downarrow$

$p \mid g(m)+n$

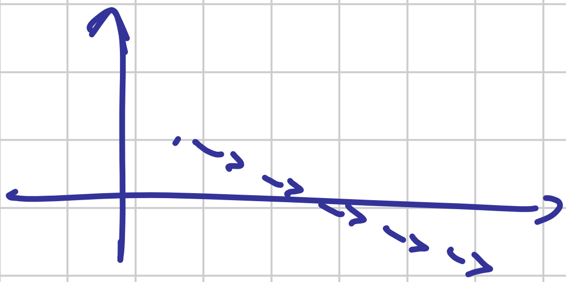
$$v_p(g(n)+m+d) = v_p(d)$$

se $v_p(d)$ è dispari ok. $\qquad$ se $v_p(d)$ è pari allora $v_p(d) \geq 2$ $\qquad k=0$

$$v_p(g(n)+m+d) = 1 \qquad \text{ok.}$$

$| g(n) - g(n+1) | \leq 1$

$g(n+1) = g(n) \pm 1$



$g(n) = n + a.$

$g(n+2) = g(n)$

$g : \mathbb{N} \to \mathbb{N}$

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