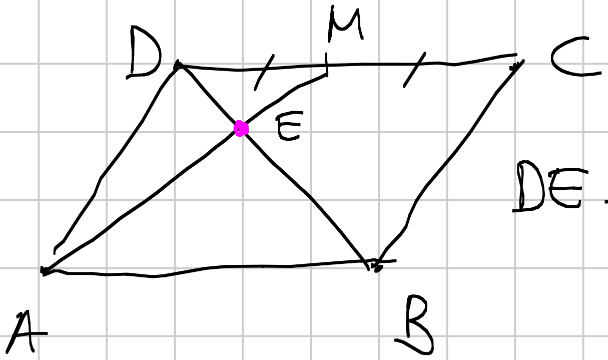
GEOMETRIA 1 (medium)

Titolo nota 02/09/2013

VETTORI



parallelogramma :)
pto medio :)

$DE = \frac{1}{3} BD$:)

$$\vec{AE} = a\vec{AM} = \vec{AB} + b\vec{BD}$$

$\vec{AD} - \vec{AB}$

$\vec{AD} + \frac{1}{2}\vec{AB}$

$\vec{DM}$

$$\Rightarrow \vec{AD}(a-b) = \vec{AB}\left(1-b-\frac{1}{2}a\right)$$

vorrei dire $= 0$; è vero?

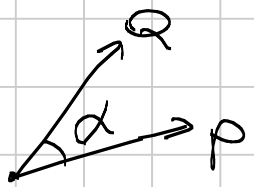
Sì, perché A, B, D NON sono allineati.

ottengo $a=b$, $1-\frac{3}{2}b=0 \rightarrow b=a=\frac{2}{3}$

$\rightarrow$ abbiamo vinto!

Prodotto scalare

$$\vec{P}\cdot\vec{Q} = |\vec{P}||\vec{Q}|\cos\alpha \leftarrow$$

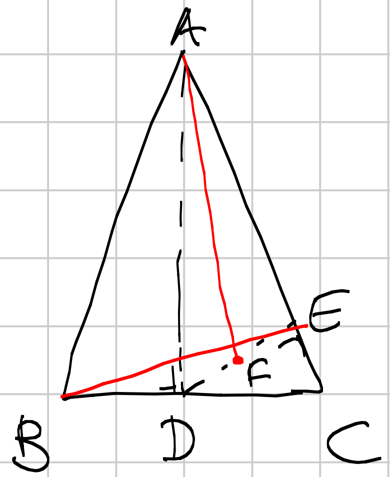


$$(x_1, y_1)\cdot(x_2, y_2) = x_1x_2 + y_1y_2$$

$\vec{P}\cdot\vec{Q} = 0$ quando sono ortogonali

$\vec{P}\cdot\vec{P} = |\vec{P}|^2$

ES.



$AB = AC$:) proiezioni :)

$DF = FE$:) ortogonalità

tesi: $AF \perp BE$

$$\vec{AF}\cdot\vec{BE} = (\vec{AE}+\vec{EF})\cdot(\vec{BD}+\vec{DE}) =$$

$$= \vec{AE}\cdot\vec{BD} - \frac{1}{2}\vec{DE}\cdot\vec{BD} + \cancel{\vec{AE}\cdot\vec{DE}} - \frac{1}{2}\vec{DE}\cdot\vec{DE}$$

ortogonali

$$\boxed{(\vec{AD}+\vec{DE})\cdot\vec{BD}}$$

$$\overset{\shortparallel}{\vec{DE}\cdot\vec{BD}}$$

$$= \frac{\vec{DE}}{2}\left(\vec{BD} - \vec{DE}\right) = \frac{\vec{DE}}{2}\cdot(\underbrace{\vec{DC}-\vec{DE}}_{\vec{EC}})$$

$= \vec{DC}$

isoscele!

$= 0$ per hp.

Conti di lunghezze!

incentro

$$OI^2 = R^2 - 2rR$$

circocentro

metto il centro in O

come si scrive I ?

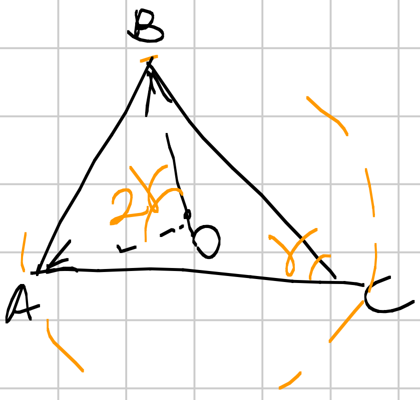
$$\frac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c}$$

$$\left|\frac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c}\right|^2 = \frac{1}{(2p)^2}\langle a\vec{A}+b\vec{B}+c\vec{C},\ a\vec{A}+b\vec{B}+c\vec{C}\rangle =$$

$$= \frac{1}{(2p)^2}\Big(a^2 \underbrace{\vec{A}\cdot\vec{A}}^{R^2} + b^2 \overbrace{\vec{B}\cdot\vec{B}}^{R^2} + c^2 \overbrace{\vec{C}\cdot\vec{C}}^{R^2} + 2\sum_{cyc} ab\, \underbrace{\vec{A}\cdot\vec{B}}_{R^2\cos 2\gamma}\Big) =$$

$\cos^2\gamma - \text{sen}^2\gamma$

$1 - 2\,\text{sen}^2\gamma$



$$= \frac{1}{(2p)^2}\Big(R^2(a^2+b^2+c^2+2ab+2bc+2ca) - 4R^2(ab\,\text{sen}^2\gamma + bc\,\text{sen}^2\alpha + ac\,\text{sen}^2\beta)\Big) =$$

$$= R^2 - \frac{\cancel{4}R^2}{\cancel{4}p^2} 2S(\text{sen}\,\alpha + \text{sen}\,\beta + \text{sen}\,\gamma)$$

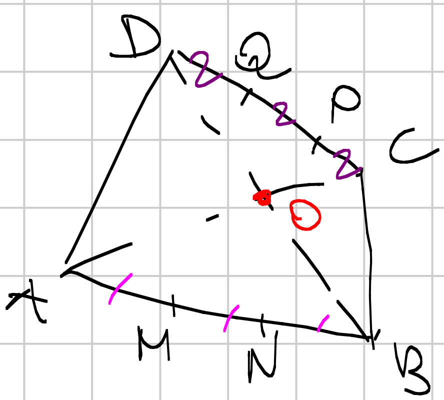
$r = \frac{S}{p}$ $\quad 2R\,\text{sen}\,\alpha = a$

$\rightarrow 2R(\text{sen}\,\alpha + \text{sen}\,\beta + \text{sen}\,\gamma) = 2p$

$$OI^2 = R^2 - Rr\,\frac{2\cancel{p}}{\cancel{p}} = R(R-2r)$$

$\rightarrow$ COROLLARIO $\quad R - 2r \geq 0$

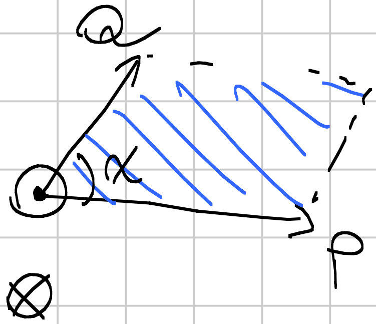
$R \geq 2r$



tesi: Area(MOP) = = Area(NOQ)

posso farlo coi vettori?

Aree → :) PRODOTTI VETTORI



$\vec{P}, \vec{Q}$

$\vec{P}\times\vec{Q}$, modulo $S = |P||Q| \operatorname{sen}\alpha$

ortog al piano di $\vec{P}, \vec{Q}$ con verso dato dalla regola mano dx.

proprietà: $a \times b = -b \times a$

è 0 se a, b allineati

è prod dei moduli se ortogonali (±)

$(a+b)\times c = a\times c + b\times c$

occhio: l'associatività è FALSA!

Risolviamo problema:

$$\underbrace{\vec{OM}}_{\frac{2}{3}\vec{A}+\frac{1}{3}\vec{B}} \times \underbrace{\vec{OP}}_{\frac{1}{3}\vec{D}+\frac{2}{3}\vec{C}} \overset{?}{=} -\left(\underbrace{\vec{OQ}}_{\frac{2}{3}\vec{D}+\frac{1}{3}\vec{C}} \times \vec{ON}\right) \leftarrow \frac{1}{3}\vec{A}+\frac{2}{3}\vec{B}$$

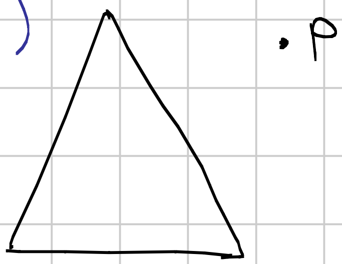
$$\frac{2}{3}\vec{A}\times\frac{1}{3}\vec{D} + \cancel{\frac{2}{3}\vec{A}\times\frac{2}{3}\vec{C}} + \cancel{\frac{1}{3}\vec{B}\times\frac{1}{3}\vec{D}} + \frac{1}{3}\vec{B}\times\frac{2}{3}\vec{C}$$

A, O, C allineati   //

$$-\left(\frac{2}{9}\vec{D}\times\vec{A} + \frac{2}{9}\vec{C}\times\vec{B}\right)$$

Altro esempio (x es.)

BONUS: è vero per qualunque poligono regolare

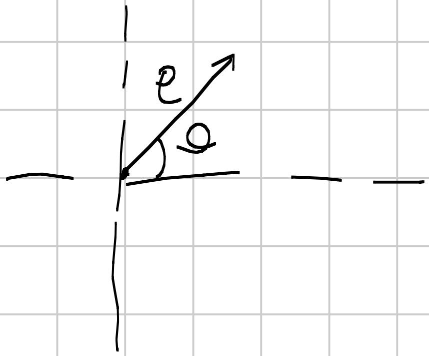


equilatero

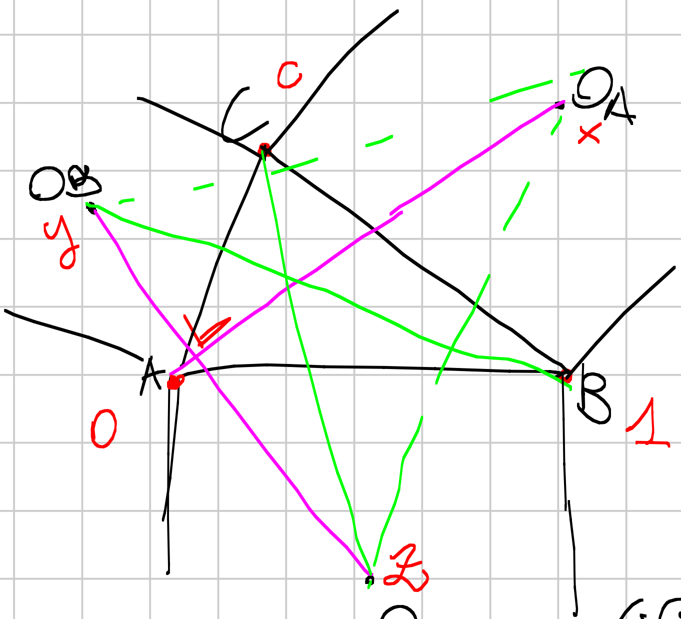
la somma "con segno" delle distanze di P dai lati NON dipende da P!

COMPLESSI

$x + iy$ $\qquad$ $\rho(\cos\vartheta + i\,\mathrm{sen}\,\vartheta) = \rho e^{i\vartheta}$



Esercizio 1



tesi

AO_A, BO_B, CO_C concorrono;

$AO_A = O_BO_C$

$c\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$y = c\frac{\sqrt{2}}{2}e^{i\pi/4}$ $= c\left(\frac{1}{2} + \frac{i}{2}\right)$

$$z = 1\,\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \frac{(1-i)}{2}$$

$$x = (c-1)\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) + 1 =$$

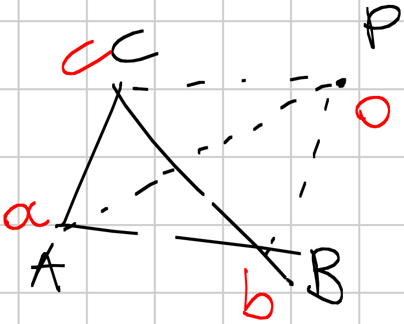
$$= (c-1)\frac{(1-i)}{2} + 1 =$$

$$= \frac{c+1+i(1-c)}{2}$$

$$y - z = \frac{c-1+i(c+1)}{2} = ix$$

$\rightarrow y - z \perp x \rightarrow$ concorrenza "gratis"

Esercizio 2



tesi

$$\underbrace{\left(\frac{PA}{BC}\right)^2}_{x} + \underbrace{\left(\frac{PB}{AC}\right)^2}_{y} + \underbrace{\left(\frac{PC}{AB}\right)^2}_{z} \geq 1$$

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

$$\sum_{cyc} \frac{PA \cdot PB}{AC \cdot BC} \geq 1$$

metto il centro in P!

cosa diventa?

$$\sum_{cyc} \frac{|a||b|}{|a-c||c-b|} \; *$$

$$\sum_{cyc} \frac{ab}{(a-c)(c-b)} = \sum_{cyc} \frac{ab(b-a)}{(a-c)(c-b)(b-a)}$$

$$= \frac{1}{(a-c)(c-b)(b-a)} \sum_{cyc} ab(b-a)$$

$$ab^2 - a^2b$$

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2a)$$

$\cancel{abc} - \cancel{abc} \quad + a^2b + b^2c + c^2a$
$- (ab^2 + bc^2 + ca^2)$

l'espressione iniziale fa -1 !

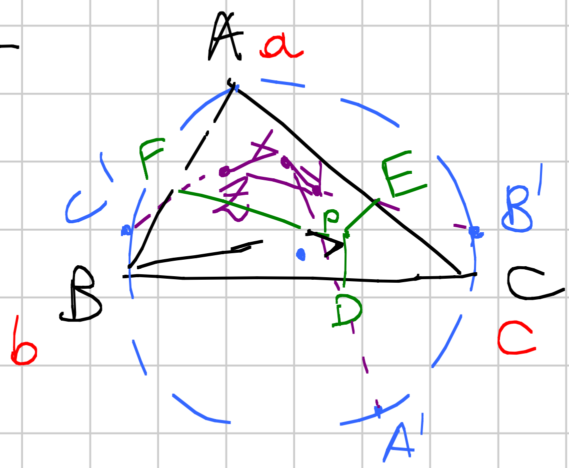
$* \ \sum_{cyc} | \qquad | \geq \left| \sum_{cyc} \dots \right| = 1 \rightarrow$ tesi!

Esercizio 3 (CHINA TST '11)

ABC O circocentro
A', B', C' simmetrici rispetto a O
pto P, proiezioni D, E, F sui lati di ABC, X, Y, Z simmetrici di A', B', C' risp. a D, E, F.

$\widehat{XYZ} \sim \widehat{ABC}$

$\text{Re}\, a = \frac{a + \bar{a}}{2}$

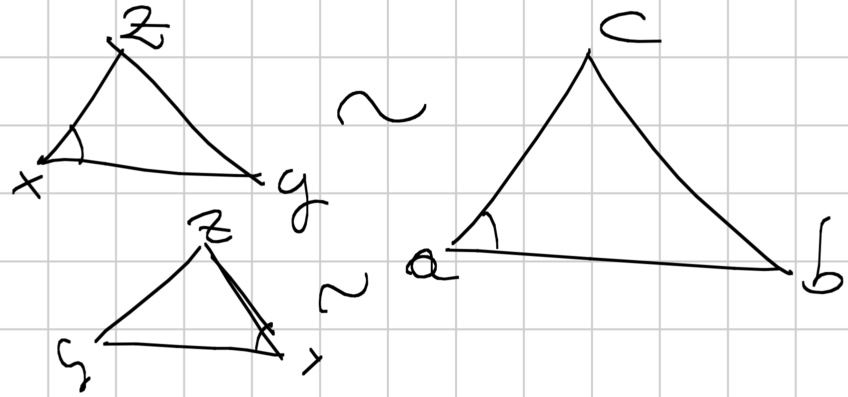


O è in O, R = 1

$a\bar{a} = b\bar{b} = c\bar{c} = 1$

PIANO: calcolarmi x in f. di a, b, c, p

condizione di similitudine



$\frac{c-a}{b-a} \stackrel{?}{=} \frac{z-x}{y-x}$

(oppure $\frac{c-a}{b-a} = \frac{\bar{z} - \bar{x}}{\bar{y} - \bar{x}}$)

calcolo d.

$$d = \left[\frac{\left(\frac{p-b}{c-b}\right)+\overline{\left(\frac{p-b}{c-b}\right)}}{2}\right](c-b)+b =$$

$$2d = p-b+\frac{\bar p-\bar b}{\bar c-\bar b}(c-b)+2b =$$

$$= p+b+\frac{bc}{\cancel{b-c}}\,\cancel{(c-b)}\,(\bar p-\bar b) =$$

$$\uparrow -1$$

$$= p+b-bc\bar p+c$$

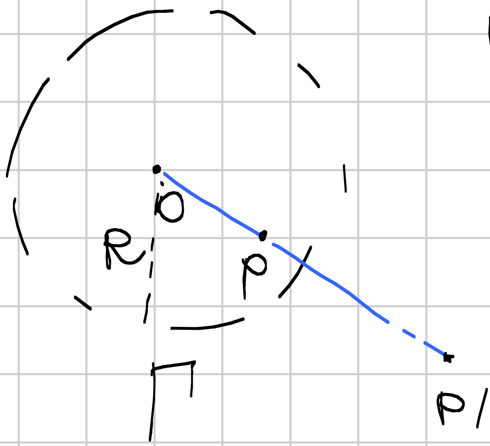
$$x+a' = 2d = p+b+c-bc\bar p$$

$$x = \boxed{p+a+b+c}-bc\bar p \qquad z = p+a+b+c-ab\bar p$$

$$\frac{\bar z-\bar x}{\bar y-\bar x} = \frac{\bar b\bar c\cancel{p}-\bar a\bar b\cancel{p}}{\cancel{p}(\bar b\bar c-\bar a\bar c)} = \frac{abc(\bar b\bar c-\bar a\bar b)}{abc(\bar b\bar c-\bar a\bar c)}$$

$$\frac{a-c}{a-b} = \frac{c-a}{b-a}$$

INVERSIONE circolare



$$\text{piano}\setminus\{O\} \to \text{piano}\setminus\{O\}$$

P va in P' sulla semiretta OP

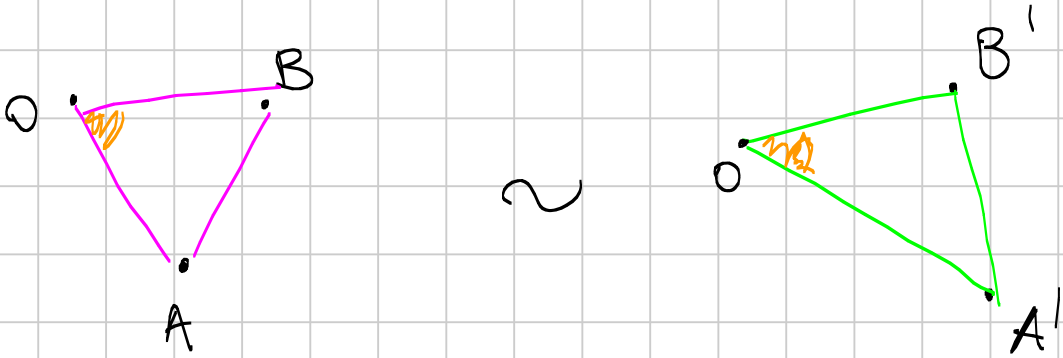
$$OP \cdot OP' = R^2$$

rette per O $\rightarrow$ se stesse

circonferenze per O $\leftrightarrow$ rette non per O

circonferenze non per O $\circlearrowright$

angoli "si mantengono" (fra rette/circonferenze e loro immagini)

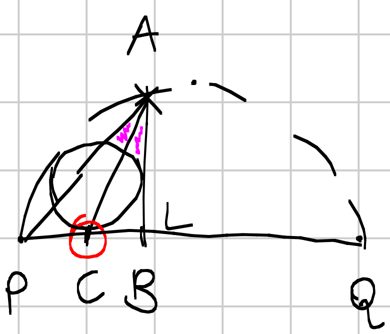


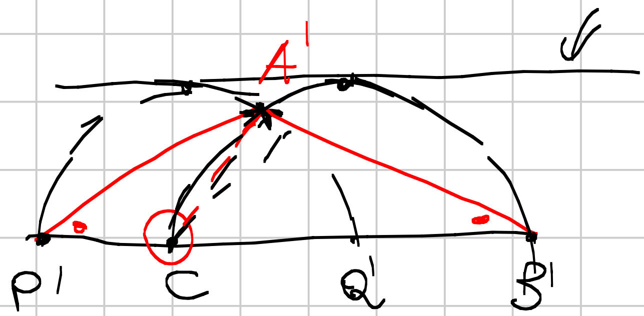
$$\frac{OA'}{OB'} = \frac{R^2}{OA}\,\frac{OB}{R^2} = \frac{OB}{OA}$$

$$A'B' = AB\cdot\frac{OA'}{OB} = \frac{AB\cdot R^2}{OA\cdot OB}$$

Es. 1

$P\hat{A}C \cong C\hat{A}B$





$C\hat{A}P \cong A'\hat{P'}C$

$C\hat{A}B \cong A'\hat{B'}C$

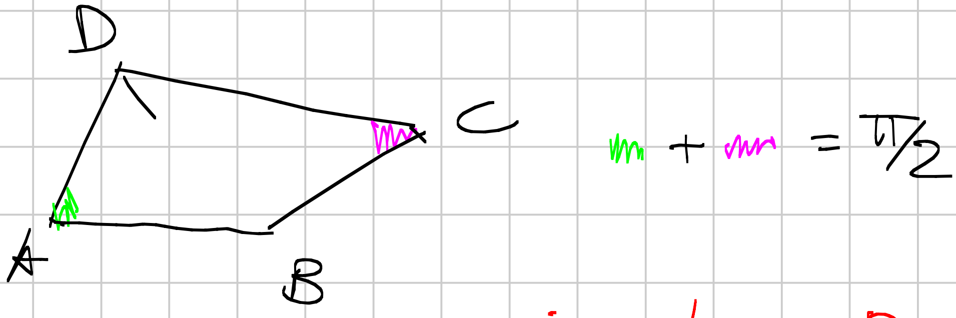
$\overset{\triangle}{CAP} \sim \overset{\triangle}{CP'A'}$

$\overset{\triangle}{A'B'C} \sim \overset{\triangle}{BAC}$

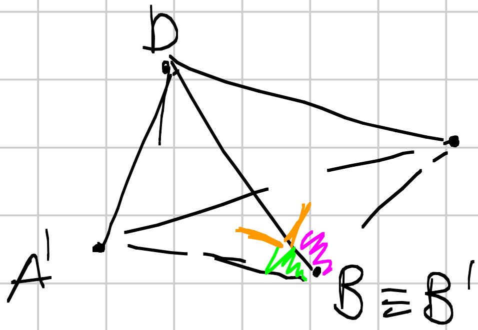
Es. 2

ABCD quad. convesso $\hat{A} + \hat{C} = \pi/2$

$$(AB \cdot DC)^2 + (AD \cdot BC)^2 = (AC \cdot DB)^2$$



inverto in D
e lascio fermo B



$\widehat{DA'B} \sim \widehat{DBA}$

$\widehat{DC'B} \sim \widehat{DBC}$

pitagora su $\widehat{A'BC'}$

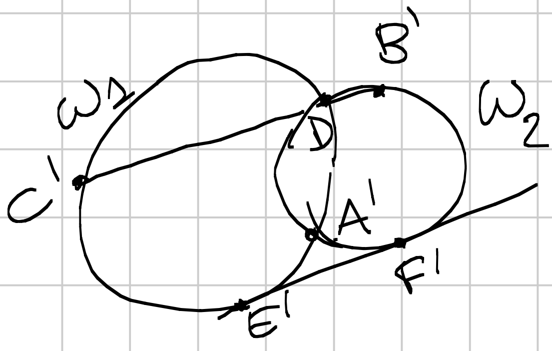
$$A'C' = \frac{AC \cdot BD^2}{AD \cdot DC} \qquad A'B = \frac{AB \cdot BD^2 \cdot DC}{AD \cdot BD \cdot DC}$$

$$C'B = \frac{BC \cdot BD^2 \, AD}{BD \cdot DC \, AD}$$

$$A'B^2 + C'B^2$$

$$\cancel{\frac{BD^4}{(BD \cdot DC \cdot AD)^2}}\left((AB \cdot DC)^2 + (BC \cdot AD)^2\right) = \frac{(AC \cdot BD)^2 \cdot \cancel{BD^2}}{\cancel{(AD \cdot DC \cdot BD)^2}}$$

Esercizio



tesi
circ. circoscritte
a B'DE' e
C'DF' si incontrano su A'D
(di nuovo)