Algebra 2

Titolo nota 06/09/2014

<≥< DISUGUAGLIANZE ≥<>≤><≥<>≤><≥

$\forall a, b, c \in \mathbb{R} \quad a^2+b^2+c^2 \geq ab+bc+ca$

$(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$

RIARRANGIAMENTO

$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \qquad a_i, b_i \in \mathbb{R}$

$b_1 \geq b_2 \geq \dots \geq b_n$

$$\sum_{cyc} a_1 b_1 \geq \sum_{i=1}^{n} a_i b_{\sigma(i)} \geq \sum_{i=1}^{n} a_i b_{n+1-i}$$

$\| \quad a_1b_1+a_2b_2+\dots+a_nb_n$

$\| \quad \sum_{i=1}^{n} a_i b_i$

come si dimostra?

Supp. disug. siano strette.

Suppongo σ non sia "ottimale" cioè non sia l'id.

$\exists i, j \quad b_{\sigma(i)} < b_{\sigma(j)}$

$i < j$

loro compaiono in $\boxed{a_i b_{\sigma(i)} + a_j b_{\sigma(j)}}$

Se li scambio al posto di $\boxed{}$

$a_i b_{\sigma(j)} + a_j b_{\sigma(i)} \overset{?}{>}$

$a_i b_{\sigma(i)} + a_j b_{\sigma(j)}$

$(a_i - a_j)(b_{\sigma(j)} - b_{\sigma(i)}) \overset{?}{>} 0$ SI

$a_i > a_j \qquad b_{\sigma(i)} < b_{\sigma(j)}$

.... → con le uguaglianze.

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

NOTA: simmetria → posso supporre $a \geq b \geq c$

→ riarrangiamento su $\begin{matrix}(a, b, c)\\(a, b, c)\end{matrix}$

Altro esempio: $x, y, z > 0$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$$

riarrangiamento su $\begin{matrix}(x \geq y \geq z)\\(1/z \geq 1/y \geq 1/x)\end{matrix}$

→ generalizzata $x_i > 0$

$$\sum_{cyc} x_2/x_1 \geq n$$

AM - GM $\quad a_1, \dots, a_n \geq 0$

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq (a_1 \dots a_n)^{1/n}$$

media aritmetica

media geometrica $= g$

(se $g = 0$ ovvia)

$x_1 = a_1/g \quad x_2 = a_1 a_2/g^2$

$x_3 = a_1 a_2 a_3 / g^3 \quad \dots \quad x_n = 1$

$$\sum_{cyc} x_2/x_1 = a_2/g + a_3/g + \dots + a_n/g + a_1/g$$

$$= \frac{\sum_{i=1}^n a_i}{g} \geq n$$

$$\Rightarrow \sum_{i=1}^n a_i / n \geq g = (\Pi a_i)^{1/n}$$

CHEBYSHEV

$a_1 \geq \dots \geq a_n \quad a_i, b_i \in \mathbb{R}$

$b_1 \geq \dots \geq b_n$

$$\frac{\sum_{i=1}^{n} a_i b_i}{n} \geq \frac{\sum a_i}{n} \frac{\sum b_i}{n} \geq \frac{\sum_{i=1}^{n} a_i b_{n+1-i}}{n}$$

$$\frac{1}{n^2} \sum_{j=0}^{n-1} \boxed{\sum_{i=1}^{n} a_i b_{i+j}}$$

$\quad\quad\quad\quad\quad\quad\quad\quad i+j$ "ciclea"

$$\geq \quad\quad\quad\quad \geq$$

$$\frac{1}{n^2} \cancel{n} \sum_{i=1}^{n} a_i b_i \quad\quad \frac{1}{n^2} \cancel{n} \sum a_i b_{n+1-i}$$

CAUCHY-SCHWARZ

$a_1 \dots a_n$

$b_1 \dots b_n \in \mathbb{R}$

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

$$\sum_{i,j} (a_i b_j - a_j b_i)^2 \geq 0$$

$$\parallel$$

$$2 \sum_{i,j} a_i^2 b_j^2 - 2 \sum_{i,j} a_i a_j b_i b_j$$

$$\boxed{\left(\sum a_i^2\right)\left(\sum b_i^2\right)} \quad \boxed{\left(\sum a_i b_i\right)^2}$$

$$\geq$$

$=$ si ha se $a_i b_j = a_j b_i$

(se $b_i, b_j \neq 0 \rightarrow a_i/b_i = a_j/b_j$)

se (nel caso $\underline{b} = (b_1 \dots b_n) \neq 0$)

$\underline{a} = \lambda \underline{b}$ per qualche λ

(cioè $a_i = \lambda b_i$).

In $\mathbb{R}^2$ $\quad a = (x_a, y_a) \quad b = (x_b, y_b)$

$$\underbrace{(x_a^2 + y_a^2)^{1/2}}_{\text{norma di } a} \underbrace{(x_b^2 + y_b^2)^{1/2}}_{\text{norma di } b} \geq x_a x_b + y_a y_b$$

$$\|a\| \|b\| \cos\vartheta = \langle a, b \rangle$$

≤ 1

quando $=$? Quando a, b sono paralleli

Torniamo a $a^2 + b^2 + c^2 \geq ab + bc + ca$

$$(a^2 + b^2 + c^2)(b^2 + c^2 + a^2) \geq (ab + bc + ca)^2$$

questo è sempre positivo

$$a^2 + b^2 + c^2 \geq |ab + bc + ca|$$

"Lemma di Titu" $\quad x_1 \dots x_n$, $a_1 \dots a_n \in \mathbb{R}$

$$\frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} x_i} \leq \sum_{i=1}^{n} a_i^2 / x_i$$

CS su

$\left(a_1/\sqrt{x_1} \dots a_n/\sqrt{x_n}\right)$

$\left(\sqrt{x_1} \dots \sqrt{x_n}\right)$

$\sum_{cyc} a_1^2 / x_1$

ESEMPIO

$$* \sum_{cyc} \frac{1}{c^3(a+b)} \geq 3/2 \qquad abc = 1$$

$a_1^2 = 1/c^2 \quad x_1 = c(a+b)$

$a_1 = 1/c$

$$* \quad \geq \frac{\left(\sum_{cyc} 1/c\right)^2}{\sum_{cyc} c(a+b)} = \frac{\left(\sum_{cyc} ab\right)^{\cancel{2}}}{2\cancel{\sum_{cyc} ab}}$$

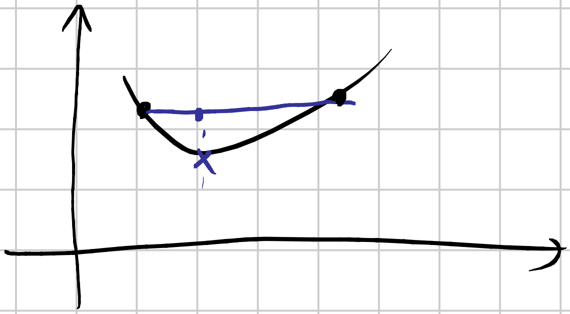
$$\sum_{cyc} ab \overset{?}{\geq} 3$$

AM - GM $\quad \dfrac{\sum_{cyc} ab}{3} \geq \left[(\underbrace{abc}_{1})^2\right]^{1/3}$

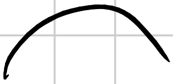
JENSEN

f funzione convessa

= felice



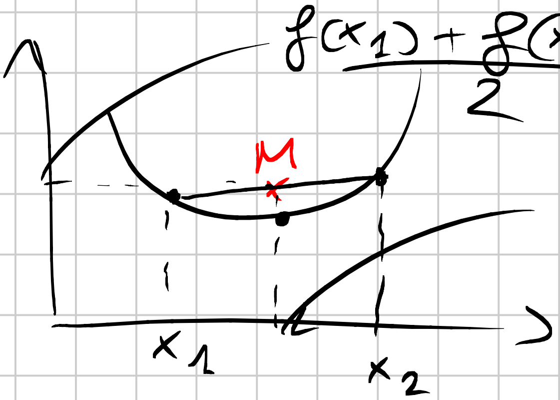
= 2 pti sul grafico → il segmento che li unisce sta sopra



triste = concava

f è convessa $x_1, \dots, x_n$

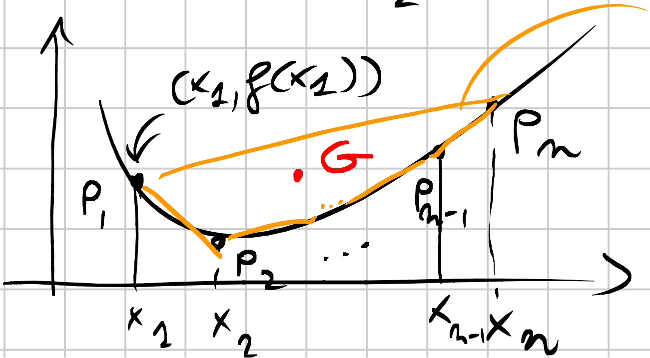
$$\frac{\sum_{i=1}^{n} f(x_i)}{n} \geq f\left(\frac{\sum x_i}{n}\right)$$



$\dfrac{x_1+x_2}{2}$

M sta sopra il grafico

$$\frac{f(x_1)+f(x_2)}{2} \geq f\left(\frac{x_1+x_2}{2}\right)$$



poligono convesso

G, baricentro di $P_1 \dots P_n$, sta sopra al grafico.

$$G = \left(\frac{\sum x_i}{n}, \frac{\sum f(x_i)}{n} \right)$$

$$\frac{\sum f(x_i)}{n} \geq f\left(\frac{\sum x_i}{n} \right)$$

$\lambda_i \geq 0$

$\lambda_1 \dots \lambda_n \quad \lambda_1 + \dots + \lambda_n = 1$

$$f\left(\sum \lambda_i x_i \right) \leq \sum \lambda_i f(x_i)$$

Ma quando una funzione è convessa? $f''(x) \geq 0$

$|x|$ è convessa e anche concava;

CONVESSE:

x^2 per $x \geq 0$ $x^3 \dots x^n$

x^α $\alpha \geq 1$, $x \geq 0$, e^x

sen$x$

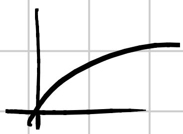
$1/x$, $x > 0$

$1/x$

CONCAVE

$\sqrt{x}$

$\log x$



x^α, $x > 0$, $\alpha < 0$

convesso

$f + g$ convessa

crescente $f(g(x))$ convesse

$f(ax+b)$ convessa

$f(x)\,g(x)$, convesse crescenti positive

MEDIE p-ESIME $\quad a_1 \dots a_n > 0$

$$\left(\frac{\sum a_i^p}{n}\right)^{1/p} \geq \left(\frac{\sum a_i^q}{n}\right)^{1/q}$$

se $p \geq q$

$$\underset{3}{CM} \geq \underset{2}{QM} \geq \underset{1}{AM} \left(\geq \underset{\text{"0"}}{GM}\right) \geq \underset{-1}{HM}$$

$f(x) = x^{p/q}$ ◡ convessa $(x \geq 0)$

$$\left(\sum x_i / n\right)^{p/q} \leq \frac{1}{n} \sum x_i^{p/q}$$

$x_i := a_i^q$

$$\left(\frac{1}{n}\sum a_i^p\right)^{1/p} \geq \left(\frac{\sum a_i^q}{n}\right)^{1/q}$$

commento: ho dim anche le "medie pesate" ...

NESBITT

$$\sum_{cyc} \frac{a \,/\, (a+b+c)}{(b+c) \,/\, (a+b+c)} \geq 3/2 \qquad a, b, c > 0$$

$x \qquad y \qquad z$

- RIARRANGIAMENTO

$$\boxed{\begin{gathered} a \geq b \geq c \\ a+b \geq a+c \geq b+c \\ \frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b} \end{gathered}}$$

$$\sum_{cyc} \frac{a}{b+c} \geq \frac{1}{2}\left[\sum_{cyc} \frac{a}{a+b} + \sum_{cyc} \frac{b}{a+b}\right] = \frac{1}{2} \sum_{cyc} \frac{a+b}{a+b} = 3/2$$

- é omogenea
  → posso supporre $a+b+c = 1$

$$\sum_{cyc} \frac{a}{1-a} = f(a)+f(b)+f(c)$$

$\mathcal{C} = b+c$

$$\geq 3 f\left(\frac{a+b+c}{3}\right) = 3f(1/3)$$

↑ $1/2$

$\frac{x}{1-x}$ "◡" é convessa su $[0,1)$

$$= -1 + \frac{1}{1-x}$$

→ fra 0 e 1 ha la stessa "convessità" di $1/x$

- CS ("Titu")

$$\sum_{cyc} \frac{a^2}{a(b+c)} \geq \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a(b+c)} \overset{?}{\geq} 3/2$$

"$\sum \frac{a_i^2}{x_i}$"

$\sum_{cyc} a(b+c) = 2\sum_{cyc} ab$

$$\left(\sum_{cyc} a\right)^2 \overset{?}{\geq} 3\sum_{cyc} ab$$

$$\sum_{cyc} a^2 \overset{?}{\geq} \sum_{cyc} ab$$

VERO! :)

PROBLEMI DEL LIBRETTO 3, 4, 5, 9, 10

$$\sum_{cyc} \frac{a^2+b^2}{a+b} \geq a+b+c \qquad a,b,c \in \mathbb{R}^+$$

$$\sum_{cyc} x^2y \geq 2(x+y+z)-3 \leftarrow 3 = \sum_{cyc} 1/x$$

se $\sum_{cyc} xy = 3xyz \qquad x,y,z \in \mathbb{R}^+$

3 riarrangiamento

5. CS

4. max x^5yz

$x+y+z=1$

$x, y, z > 0$

$$\frac{\overbrace{\frac{1}{5}x+\frac{1}{5}x \qquad}^{5 \text{ volte}}+y+z}{7} = 1/7$$

$GM(\ldots) \leq 1/7$

$$\left(\frac{1}{5^5}x^5yz\right)^{1/7} \leq 1/7$$

$$\rightarrow x^5yz \leq \frac{5^5}{7^7}$$

cost. del 10 sono 1, 2