A3 − MEDIUM SCAMBRET

Titolo nota 05/09/2014

$$a_{n+2} = \alpha a_{n+1} + \beta a_n + \text{Mostro}(n) \qquad (*)$$

a_n e $\overline{a_n}$ che risolvono $(*)$

$$a_n - \overline{a_n}$$

$$a_{n+2} = \alpha a_{n+1} + \beta a_n + M(n)$$
$$\overline{a_{n+2}} = \alpha \overline{a_{n+1}} + \beta \overline{a_n} + M(n)$$

$$a_n = \underbrace{(a_n - \overline{a_n})}_{\substack{\text{SOLUZIONE} \\ \text{GENERALE DI} \\ (*) \text{ SENZA MOSTRO}}} + \underbrace{\overline{a_n}}_{\substack{\text{SOLUZIONE} \\ \text{SPECIALE DI} \\ (*) \text{ CON MOSTRO}}}$$

$$a_n - \overline{a_n} = a_n^*$$

Esempio 1: $a_{n+1} = c a_n + d \qquad (c \neq 1)$

1) $a_1 = c a_0 + d$
$a_2 = c^2 a_0 + cd + d$
$a_3 = c^3 a_0 + c^2 d + cd + d$

$$a_n = c^n a_0 + d(c^{n-1} + c^{n-2} + \dots + 1)$$
$$= c^n a_0 + d\,\frac{c^n - 1}{c - 1}$$

2) $a_{n+1} = c a_n + d$

$a^*_{n+1} = c a^*_n \longrightarrow a^*_n = c^n \cdot \alpha$

$\overline{a_{n+1}} = c\,\overline{a_n} + d$

Euristica: • se mostro (n) è un polinomio di grado g, allora $\overline{a_n}$ un polinomio di grado g

• se mostro (n) è b^n, $\overline{a_n} = \phi \cdot b^n$

$\overline{a_n} = \ell$

$\ell = c\ell + d \longrightarrow \ell = -\frac{d}{c-1}$

$a_n = c^n \alpha - \frac{d}{c-1}$

<u>Esempio 2</u> $a_{n+1} = 5a_n - 4n$ $\quad a_0 = b$

$a^*_n = 5^n \cdot \alpha$

$\overline{a_n} = cn + d$

$c(n+1) + d = 5(cn+d) - 4n$

$cn + c + d = 5cn + 5d - 4n$

$c = 1 \quad d = \frac{1}{4}$

$a_n = 5^n \cdot \alpha + n + \frac{1}{4}$

Esempio 3 $\underbrace{a_{n+2} = 6a_{n+1} - 8a_n}_{} + 4^n$

$$a_n^* = \lambda_1 \cdot 2^n + \lambda_2 \cdot 4^n$$

$$\overline{a_n} = \phi \cdot 4^n$$

$$\phi \cdot \cancel{4}^{n+2} = 6\phi \cdot \cancel{4}^{n+1} - 8\phi \cancel{4}^n + \cancel{4}^n$$

$$16\phi = 24\phi - 8\phi + 1$$

$$\rightsquigarrow 0 \cdot \phi = 1$$

Euristica: se mostro $(n) = b^n$, ma b è soluzione dell'equazione associata della ricorrenza senza mostro (con molteplicità m)

$$\overline{a_n} = \phi \cdot n^m \cdot b^n$$

Esempio (n+1) $a_{n+2} = 4a_{n+1} - 4a_n + 2^n$

$$\overline{a_n} = \phi \cdot n^2 \cdot b^n$$

Euristica: se mostro $(n) = f(n) + g(n)$

provate $\overline{a_n} = \overline{f_n} + \overline{g_n}$

Esempio (n+2): $\underbrace{a_{n+2} = 5a_{n+1} - 6a_n}_{} + n^2 \cdot 3^n$

BMO 2004/1 $\quad a_n \in \mathbb{N}$

$$\begin{cases} a_{m+n} + a_{m-n} - m + n - 1 = \frac{1}{2}(a_{2m} + a_{2n}) & m \geq n \geq 0 \\ a_1 = 3 \end{cases}$$

$a_{2004} = ?$

$n=0$ $\quad 2a_m - m - 1 = \frac{1}{2}(a_{2m} + a_0)$

$m=0$ $\quad a_0 = 1$

$$4a_m - 2m - 3 = a_{2m}$$

$n=1$ $\quad a_{m+1} + a_{m-1} - m = \frac{1}{2}(4a_m - 2m - 3 + 7)$

$(**)$ $\boxed{a_{m+1} = 2a_m - a_{m-1} + 2}$ $\quad \forall m \geq 1$

$a_0 = 1, \; a_1 = 3$

$\underline{a_m}^* = \lambda_1 \cdot 1^n + \lambda_2 \, n \cdot 1^n = \underline{\lambda_1} + \lambda_2 n$

$\overline{a_m} = am^2 + \underline{bm + c}$

$a = 1$

$a_m = m^2 + \lambda_2 m + \lambda_1$

$a_m = m^2 + m + 1$

BMO '03/3 $\quad f: \mathbb{Q} \to \mathbb{R}$

i) $f(1) > -1$

ii) $f(x+y) - xf(y) - yf(x) + x + y - xy - f(x)f(y) = 0$

iii) $f(x) - 2f(x+1) - x - 2 = 0$

i) $x = y = 0 \quad \to f(0) \in \{0, 1\}$

iii) $x = 0 \quad \to f(1) \in \{-1, -\frac{1}{2}\}$

$$f(1) = -\frac{1}{2} \quad f(0) = 1$$

iii) $x = n \quad f(n) = 2f(n+1) + n + 2$

$$f(n) = a_n$$

$$a_{n+1} = \frac{a_n}{2} - \frac{n+2}{2}$$

$$a_n^* = \alpha \cdot \frac{1}{2^n}$$

$$\overline{a_n} = cn + d$$

$$cn + c + d = \frac{cn + d - n - 2}{2}$$

$$c = -1 \quad d = 0$$

$$a_n = f(n) = \frac{\alpha}{2^n} - n$$

$$1 = a_0 = f(0) = \frac{\alpha}{1} - 0 \quad \to \alpha = 1$$

$$f(n) = \frac{1}{2^n} - n \quad \forall n \in \mathbb{N}$$

$$g(x) = f(x) + x$$

ii) $g(x+y) = g(x)\, g(y) \leftarrow$

iii) $g(x) = 2g(x+1)$

IMO SL 1992 A2

$a, b \in \mathbb{R}^+$ $\quad f: \mathbb{R}^+ \to \mathbb{R}^+$

$$f(f(x)) + af(x) = b(a+b)x \quad \forall x \in \mathbb{R}^+$$

Euristica: $f(f(x)), f(x), x \longrightarrow$ successioni

$x_0 \in \mathbb{R}^+ \qquad x_{n+1} = f(x_n)$

$x = x_n \qquad x_{n+2} + ax_{n+1} = b(a+b)x_n \quad \forall n$

$$x^2 + ax - b(a+b) = 0$$

$R_1 = b \qquad R_2 = -(a+b)$

$$x_n = \lambda_1 \cdot b^n + \lambda_2 (-a-b)^n$$

$a + b > b$

Se $\lambda_2 \neq 0 \quad |\lambda_2 (-a-b)^n| > |\lambda_1 b^n|$

$$\to \exists M: \; x_M < 0$$

$\lambda_2 = 0 \quad \to x_n = \lambda_1 \cdot b^n$

$$x_0 = \lambda_1 \cdot b^0 = \lambda_1$$

$$\to x_n = x_0 \cdot b^n$$

$$x_1 = f(x_0) = x_0 \cdot b$$

$$\to f(x) = bx \quad \forall x > 0$$

BMO 2002/4

$f: \mathbb{N}^+ \to \mathbb{N}^+ \quad 2n + 2001 \le f(f(n)) + f(n) \le 2n + 2002$

BMO 2009/4 $\quad f: \mathbb{N}^+ \to \mathbb{N}^+$

$$f\left(f^2(m) + 2f^2(n)\right) = m^2 + 2n^2$$

$n = 2015 \quad f\left(f^2(m) + 2f^2(2015)\right) = m^2 + 2 \cdot 2015^2$

$$m^2 = m_1^2 \quad \to m = \pm m_1$$

Se $\quad a^2 + 2b^2 = c^2 + 2d^2 \ (\bullet) \Longrightarrow$

$\Rightarrow \cancel{f}\left(f^2(a) + 2f^2(b)\right) = \cancel{f}\left(f^2(c) + 2f^2(d)\right)$

$a = x + p$
$b = x + q$
$c = x + r$
$d = x + s$

$\begin{cases} p^2 + 2q^2 = r^2 + 2s^2 \\ p + 2q = r + 2s \end{cases}$

$p = 0$

$\begin{cases} 2q^2 = r^2 + 2s^2 \\ 2q = r + 2s \end{cases}$

$r = 2\alpha \qquad q = \alpha + s$

$q^2 = 2\alpha^2 + s^2$

$\alpha^2 + 2\alpha s + s^2 = 2\alpha^2 + s^2$

$(p, q, r, s) = (0, 3, 4, 1)$

$$f^2(x) + 2f^2(x+3) = f^2(x+4) + 2f^2(x+1)$$

$f^2(x) = u_x$

$$\sum_{k=0}^{n} k x^n \qquad 1^4 + 2^4 + \ldots + m^4$$

C1 – 2009 M
(TEPPIC)

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$f: \mathbb{R} \to \mathbb{R}$ $\boxed{f(x+y) = f(x) + f(y)}$ ←

$$f(x) = \lambda x \quad \forall x \in \mathbb{Q}$$

- f è continua
- f è monotona
- f è limitata (almeno in un intervallo)
- $\exists$ un pallino nel piano in cui non passa il grafico di f.

BST 2012 / 9

$f: \mathbb{Q} \to \mathbb{Q}$

$$f(x + f(y + f(z))) = y + f(x+z) \qquad \forall x,y,z \in \mathbb{Q}$$

E : 1° passaggio → mai azzerare le variabili senza f

$x = z = 0$ $\quad f(f(y + f(0))) = y + f(0)$

$$f(f(w)) = w \quad \forall w \in \mathbb{Q}$$

f è biunivoca

E : f è iniettiva $f(M) = f(N) \Rightarrow M = N$

f è surgettiva

(1) $\exists \alpha : f(\alpha) =$ quello che volete voi

(2) $f(y) = z$

$$f(f(x)) = 2f(x)$$
$$f(\alpha) = 2z$$

E : se RHS è simmetrico nelle variabili x e z $\rightarrow$ LHS è simmetrico.

$$f(x+f(y+f(z))) = y+f(x+z) = f(z+f(y+f(x)))$$

$$x+f(y+f(z)) = z+f(y+f(x)) \quad (*)$$

$x = f(a)$, $z = f(c)$ $\quad y = b$

$$f(a)+f(b+f(f(c))) = f(c)+f(b+f(f(a)))$$

$$f(a)+f(b+c) = f(c)+f(b+a)$$

$c=0$ $\quad f(a)+f(b) = f(a+b) + \cancel{f(0)}$

$\quad -f(0) \quad -f(0) \quad\quad -f(0) \quad \cancel{-f(0)}$

$$\left(f(a)-f(0)\right)+\left(f(b)-f(0)\right) = \left(f(a+b)-f(0)\right)$$

$$g(x) = f(x)-f(0)$$

$$g(x)+g(y) = g(x+y) \quad \forall x,y \in \mathbb{Q}$$

2° ORA BMO 2007/2

$\because$

$\forall x, y \in \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(f(x)+y) = f(f(x)-y) + 4f(x)y$

E.: azzerare f, cioè $f(\square)$ pongo $\square = 0$

oppure $f(\ q_1\ ) = f(\ q_A\ ) + \_\_\ \_$

$\Longrightarrow q_1 = q_A$

$\cancel{f(x)} + y = \cancel{f(x)} - y \longrightarrow y = 0$

$y = f(x)$ $\quad f(2f(x)) = 4f(x)^2 + f(0)$

$= [2f(x)]^2 + f(0)$

$\to f(z) = z^2 + f(0) \qquad z \in 2\,\mathrm{Im}(f)$

$y = f(x) - m$

$f(2f(x)-m) = f(m) + 4f(x)^2 - 4f(x)\,m$

$m = 2f(z)$ $\quad f(2f(x) - 2f(z)) = 4f(z)^2 + f(0) + 4f(x)^2$

$- 8f(x)f(z)$

$f(2f(x) - 2f(z)) = [2f(x) - 2f(z)]^2 + f(0)$

$f(\omega) = \omega^2 + f(0) \qquad \omega \in 2(\mathrm{Im}\, f_1 - \mathrm{Im}\, f_1)$

$2f(f(x)+y) - 2f(f(x)-y) = 8f(x)y$

- se $\forall x \ f(x) \equiv 0$
- se $\exists x_0 : f(x_0) \neq 0$

$$\text{LHS} = 2\left(f(\quad) - f(\quad)\right) = 8f(x_0)y$$

$$y = \frac{r}{8f(x_0)} \qquad f(x) = x^2 + f(0)$$

IMO 1999/6 (x case)

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TST 2008/6

$f: \mathbb{R}^+ \to \mathbb{R}^+$ $\quad f(x+y) \geq yf(f(x)) + f(x) \longrightarrow \nexists f$

$$f(x+y) \geq yf(f(x))$$

$$\frac{f(x+y)}{y} \geq f(f(x))$$

$y=1$ $\quad f(x+1) \geq f(f(x))$

$x=1$ $\quad f(y+1) \geq yf(f(1))$

$$\lim_{x\to+\infty} f(x) = +\infty$$

se sapessi che $f(f(x_0)) \geq n$

$$f(x_0+y) \geq yn$$

$$f(y) \geq ny - nx_0 > (n-1)y \qquad \forall y > nx_0$$

$$f(x+y)$$

$x+y = f(z)$

$x=1$

$y = f(z)-1$

$$f(f(z)) \geq \left(f(z)-1\right) f(f(1))$$

$$\longrightarrow \lim_{x\to+\infty} f(f(x)) = +\infty$$

$$f(x+y) \geq y f(f(x))$$

$$x+y=f(x)$$

$$\begin{cases} x = z \\ y = f(z) - z \end{cases}$$

per ogni z abbastanza grandi $\left(z > 2^{2^{2^{x_0}}}\right)$

$$\cancel{f(f(z))} \geq (f(z)-z)\cancel{f(f(z))}$$

$$1 \geq f(z) - z$$

$$f(z) \leq z+1$$

$$f(z) > nz$$

IMO 2013/5 $\quad f : \mathbb{Q}_{>0} \to \mathbb{R}$

i) $f(x)f(y) \geq f(xy) \quad \forall x, y \in \mathbb{Q}_{>0}$

ii) $f(x+y) \geq f(x)+f(y) \quad \forall x, y \in \mathbb{Q}_{>0}$

iii) $\exists\, a \in \mathbb{Q}_{>1} : f(a) = a$

$$\Longrightarrow f(x) = x \quad \forall x \in \mathbb{Q}_{>0}$$

$$f(x+y+z) \geq f(x) + f(y+z) \geq f(x)+f(y)+f(z)$$

① $f(nx) \geq n f(x) \quad \forall x \in \mathbb{Q}_{>0} \quad \forall n \in \mathbb{N}$

$$f(x)f(y)f(z) \geq f(x)f(yz) \geq f(xyz)$$

② $\left(f(x)\right)^n \geq f(x^n) \quad \forall x \in \mathbb{Q}_{>0} \quad \forall n \in \mathbb{N}$

③ $x = a \qquad f(a)f(y) \geq f(ay)$

$y = 1 \qquad f(1) \geq 1$

$$f(n) \geq n \quad \forall n \in \mathbb{N}$$

④ $x = \frac{p}{q} \quad y = q$

$$f\left(\frac{p}{q}\right) \geq \frac{f(p)}{f(q)} > 0 \quad \forall \frac{p}{q}$$

⑤ $\to f(x) > 0 \quad \forall x \in \mathbb{Q}_{>0}$

$$f(x+y) \geq f(x) + f(y) > f(x)$$

ii) $\to f$ è crescente

⑥ $a^n \geq f(a^n)$

$b_n = \lfloor a^n \rfloor$

$$b_n \leq f(b_n) = f(\lfloor a^n \rfloor)$$
$$\leq f(a^n) \leq a^n < b_n + 1$$

$$b_n \leq f(b_n) < b_n + 1$$

(7) Se esistesse un c : $f(c) \geq c+1$

$$f(c+1) \geq f(c) + f(1) \geq c+2$$

$$f(y) \geq y+1 \qquad \forall y \geq c$$

per un qualche b_n : $b_n \geq c$

$$\to f(b_n) \geq b_n + 1$$

$$\to f(n) < n+1$$

(8) Suppongo che $f(c) \geq c + \frac{1}{m}$ con $c \in \mathbb{N}$, $m \in \mathbb{N}$

(1) $f(nx) \geq nf(x)$

$n = m$, $x = c$ $\quad f(mc) \geq mf(c)$

$$\geq mc + m \cdot \frac{1}{m}$$
$$= mc + 1$$

$$\to f(n) = n \quad \forall n \in \mathbb{N}$$

$$f\left(\frac{p}{q}\right) f(q) \geq f(p) \qquad p, q \in \mathbb{N}$$

$$f\left(\frac{p}{q}\right) \geq \frac{p}{q} \qquad (*)$$

1 $\quad f(nx) \geq n f(x)$

$n = q \ , \ x = \frac{p}{q} \qquad f(p) \geq q f\left(\frac{p}{q}\right)$

$$f\left(\frac{p}{q}\right) \leq \frac{p}{q} \quad (*)$$

TST USA SL 2004 $\quad f: \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1}$

i) $f(x) \leq 2(x+1)$

ii) $x f(x+1) = f(x)^2 - 1$

① $$f(x) = \sqrt{x f(x+1) + 1}$$
$$\leq \sqrt{x \cdot 2 \cdot (x+2) + 1}$$
$$= \sqrt{2x^2 + 4x + 1}$$
$$< \sqrt{2}(x+1)$$

② $$f(x)^2 = x f(x+1) + 1$$
$$< x\sqrt{2}(x+2) + 1$$
$$< \sqrt{2}(x+1)^2$$
$$f(x) < 2^{1/4}(x+1)$$

③ $\to$ $$f(x) \leq 2^{1/2^k}(x+1) \quad \forall k \in \mathbb{N}$$
$$\nexists x \in \mathbb{R}_{\geq 1} : f(x) > x+1$$
$$2^{1/2^k}(x_0+1) \geq f(x_0) = x_0 + 1 + \varepsilon$$
$$\to f(x) \leq x+1 \quad \forall x \in \mathbb{R}_{\geq 1}$$

$$f(x) \geq 1 \quad \forall x \in \mathbb{R}_{\geq 1}$$
④ $$f(x)^2 = x f(x+1) + 1 \geq x+1$$
$$\to f(x) \geq \sqrt{x+1} > \sqrt{x}$$

(5) $f(x)^2 = xf(x+1)+1 \geq x\sqrt{x+1}+1$
$> x\sqrt{x}$

$f(x) \geq x^{3/4}$

$f(x) \geq x^{1-\frac{1}{2^k}}$

$\rightarrow \nexists x : f(x) < x \quad (\ )$
$\rightarrow f(x) \geq x$

(7) $f(x)^2 = xf(x+1)+1 \geq x(x+1)+1$
$= x^2+x+1$
$= (x+\frac{1}{2})^2 + \frac{3}{4}$
$> (x+\frac{1}{2})^2$

$\rightarrow f(x) > x + \frac{1}{2}$

(8) $f(x)^2 = xf(x+1)+1 > x(x+\frac{3}{2})+1$
$= (x+\frac{3}{4})^2 +$ scarto
$> (x+\frac{3}{4})^2$

$\rightarrow f(x) > x + \frac{3}{4}$

$f(x) \geq x+1-\frac{1}{2^k}$

$\rightarrow \nexists x : f(x) < x+1$
$\rightarrow f(x) \geq x+1 \quad \forall x \geq 1$

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3° ORA

TST VIETNAM 2003

Sia A l'insieme delle $f: \mathbb{R}^+ \to \mathbb{R}^+$ t.c.

$$f(3x) \geq f(f(2x)) + x \quad \forall x \in \mathbb{R}^+$$

Trovare max α : $\forall f \in A \quad f(x) \geq \alpha x$

$$f(x) = cx$$

$$3cx \geq 2c^2x + x$$

$$3c \geq 2c^2 + 1$$

$$\to \frac{1}{2} \leq c \leq 1 \qquad \alpha \leq \frac{1}{2}$$

$$f(x) \geq a_n x$$

$$f(3x) \geq 2a_n^2 x + x = \frac{2a_n^2+1}{3} \cdot 3x$$

$$a_{n+1} = \frac{2a_n^2+1}{3}$$

TST 2006/3 $\quad f:\mathbb{Z}\to\mathbb{Z}$

$$f(m-n+f(n)) = f(m)+f(n) \quad \forall m,n\in\mathbb{Z}$$

E: se $a,b\in \operatorname{Im}(f)$
avete m_0 e n_0 : $f(m_0)=a$
$f(n_0)=b$

$$f(\qquad\qquad) = a+b$$

$\to a+b \in \operatorname{Im}(f)$

- $0\in \operatorname{Im}(f) \quad \to |\operatorname{Im}(f)|=1$
- $a\neq 0\in \operatorname{Im}(f) \quad na\in \operatorname{Im}(f)$

$\to |\operatorname{Im}(f)|=\infty$

$m=n \qquad f(f(n)) = 2f(n) \qquad \forall n\in\mathbb{Z}$

$f(n)=2n \qquad \forall n\in \operatorname{Im}(f)$

$$f(m-n+f(n)) = f(n-m+f(m))$$

Se f è iniettiva $\quad m-n+f(n) = n-m+f(m)$

$$\underbrace{f(n)-2n} = \underbrace{f(m)-2m}$$

Se {espressione x} = {espressione in y}

$\to$ {espressione x} $=k$

$$f(n) = 2n+k$$

f non è iniettiva $\Longrightarrow \exists\, a<b : f(a)=f(b)$

$n=a \qquad f(m-a+f(a)) = f(m)+f(a)$

$n=b \qquad f(m-b+f(b)) = f(m)+f(b)$

$$f(m-a+f(a)) = f(m-b+f(b))$$

$$m = k + a - f(a)$$

$$f(k) = f(k+a-\cancel{f(a)}-b+\cancel{f(b)})$$

$$f(k) = f(k+(a-b))$$

$\rightarrow f$ è periodica

$\rightarrow |Im(f)| = 1$

$\rightarrow f(n)=0 \quad \forall n \in \mathbb{N}$

IMO SL A2 2005 $\quad f:\mathbb{R}^+ \to \mathbb{R}^+$

$$f(x)f(y) = 2f(x+yf(x)) \quad \forall x,y \in \mathbb{R}^+$$

① $f(x)=c \quad \to c^2 = 2c \to c=2$

② $\cancel{2}f(x+yf(x)) = \frac{f(x)f(y)}{2} = \cancel{2}f(y+xf(y))$

Supponiamo f iniettiva

$$x+yf(x) = y+xf(y)$$

$x=1 \qquad 1+yf(1) = y+f(y)$

$$f(y) = cy+1 \qquad \text{NO}$$

$x = x+yf(x) \to yf(x) = 0 \qquad$ no

$y = x+yf(x) \to y_f = \frac{x}{1-f(x)}$

Se $f(x) < 1 \to y_f > 0$

$\to f(x)\cancel{f(y)} = 2\cancel{f(\quad)}$

$$f(x) = 2$$

$\to f(x) \geq 1 \quad \forall x>0$

Se $f(x)f(y) = 2f(\quad) \geq 2f(x)$

$$f(y) \geq 2$$

Claim: $f(x) \geq 2$

$$\underbrace{f(x)}_{2}\underbrace{f(y)}_{2} = \underbrace{2f(\qquad)}_{\leq}$$

$a, b \in Im(f) \to \frac{ab}{2}$

$$\left(\frac{f(x)}{2}\right)\left(\frac{f(y)}{2}\right) = \frac{f(\quad)}{2}$$

$$g(x)g(y) = g(\qquad)$$

se $a, b \in Im(g) \to ab \in Im(g)$

$\to a^k \in Im(g)$

$g(x) \geq \frac{1}{2}$ Supponiamo che esiste α : $g(\alpha) < 1$

$\beta < 1$ e $\beta \in Im(g)$

$\to \beta^k \in Im(g)$

ma $\beta^k \geq \frac{1}{2}$ $\forall k$ $\lightning$

$\to g(x) \geq 1 \to f(x) \geq 2$

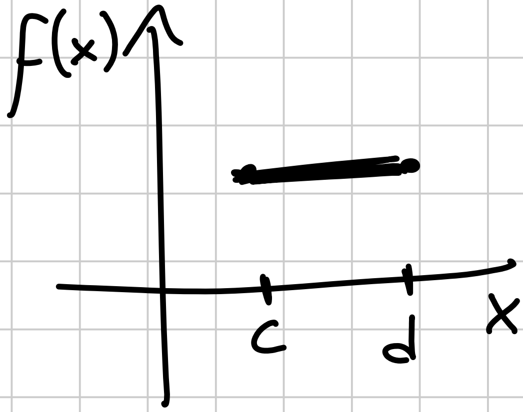
$2f(x) \leq f(x)f(y) = 2f(x+q_s)$

$\to f$ è monotona

$\exists$ c, d : $f(c) = f(d)$ e $c < d$

$x = c$

$f(c)f(y) = 2f(c + yf(c))$



Se $c + yf(c) \leq d$

$\cancel{f(c)}f(y) = 2\cancel{f(\quad)}$

$0 < y \leq \frac{d-c}{f(c)}$

Se $f(\alpha) = 2$

$x = y = \alpha$

$4 = 2f(\alpha + 2\alpha) \to f(3\alpha) = 2$

$f(3^k \alpha) = 2$

$x \in (0, \alpha) : f(x) = 2$

$\to x \in (0, 3^k \alpha) : f(x) = 2$ $\square$

$f : \mathbb{R}^+ \to \mathbb{R}^+$

$f(x + f(y)) = f(x+y) + f(y)$

MEMO 2012/1 $\qquad f: \mathbb{R}^+ \to \mathbb{R}^+$

$$f(x+f(y)) = y f(xy+1)$$

$$x f(x+f(y)) = y f(y+f(x)) \xleftarrow{\text{SIM}} \text{(NO)}$$

$x = \frac{1}{y}$ (NO)

$x+f(y) = xy+1$ (FORSE)

$x = \frac{f(y)-1}{y-1}$ $(y \neq 1)$

$y=1$

$\frac{f(y)-1}{y-1} \leq 0$ $\begin{cases} y>1 \to f(y) \leq 1 \\ y<1 \to f(y) \geq 1 \end{cases}$

E: se avete $f(\text{Mostro}) = f(x)$

$xy+1 = x \to y = \frac{x-1}{x} = 1-\frac{1}{x}$ $(x>1)$

$f(x+f(y)) = y f(x)$

$f(x+f(1-\frac{1}{x})) = (1-\frac{1}{x}) f(x)$ NO

$xy+1 = y \longrightarrow x = 1-\frac{1}{y}$ $(y>1)$

$f(1-\frac{1}{y}+f(y)) = y f(y)$

Se $\exists y_0 : f(y_0) > \frac{1}{y_0}$

$$f(y_0) - \frac{1}{y_0} + 1 > 1 \rightarrow f(\quad) \le 1$$

$$1 = y_0 \cdot \frac{1}{y_0} < y_0 f(y_0) \le 1 \quad ⚡$$

Se $\exists y_0 : f(y_0) < \frac{1}{y_0}$ ⚡

$$\rightarrow f(y) = \frac{1}{y} \quad \forall y > 1$$

$$f(x + f(y)) = \frac{y}{xy+1} \qquad \forall x, y \in \mathbb{R}^+$$

se $y > 1$

$$f\left(x + \frac{1}{y}\right) = \frac{y}{xy+1} \qquad \forall x \in \mathbb{R}^+ \; \forall y > 1$$

$\frac{1}{y} = a$ (con $a < 1$)

$$f(x+a) = \frac{\frac{1}{a}}{\frac{x}{a}+1} = \frac{1}{x+a}$$

Claim : $f(z) = \frac{1}{z}$

$x + a = z$ con $z < 1$

$x = z - a$

$$\rightarrow f(x) = \frac{1}{x} \quad \forall x \in \mathbb{R}^+$$

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$f : \mathbb{R} \to \mathbb{R} \qquad f(x + xy + f(y)) = \left(f(x) + \frac{1}{2}\right)\left(f(y) + \frac{1}{2}\right)$

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