G1 - Medium (Sam)

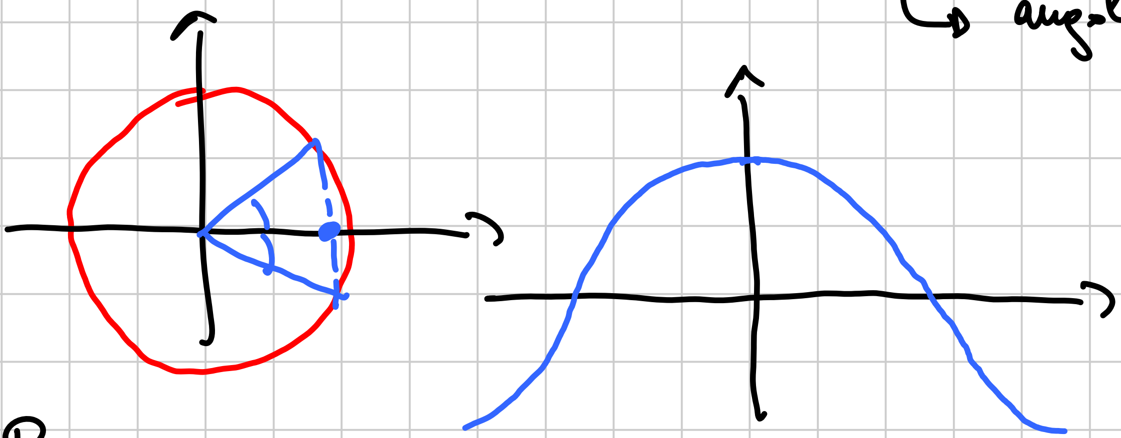
Titolo nota 02/09/2014

- Vettori
- Coordinate
- Complessi

Vettori

$$\vec{A} \cdot \vec{B} = \|\vec{A}\| \cdot \|\vec{B}\| \cdot \cos \widehat{AB}$$

↳ angolo tra i vettori



Piano

" in 2 coord $\vec{A} = (a_1, a_2)$ $\vec{B} = (b_1, b_2)$

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2$$

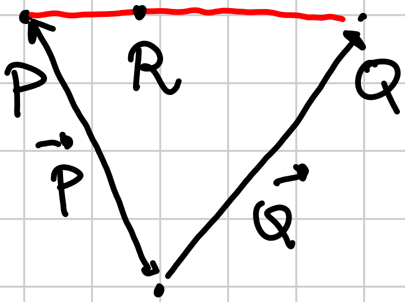
si può fare in $\underline{n}$ coordinate

Es: IF I = incentro F = centro della cfr. di Feuerbach

Fissate qualunque origine

$$IF^2 = \|\vec{I} - \vec{F}\|^2 = (\vec{I} - \vec{F}) \cdot (\vec{I} - \vec{F})$$

$$\vec{I} = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a+b+c}$$

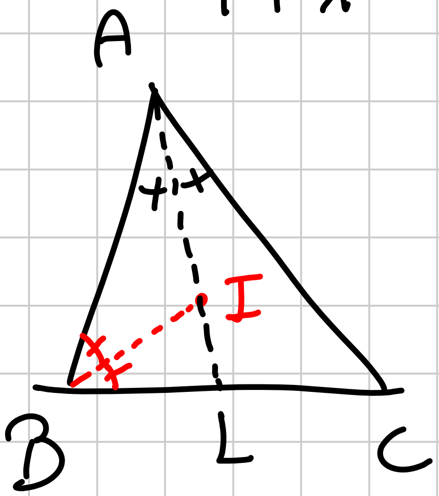


$$\frac{PR}{RQ} = \frac{\lambda}{1}$$

punti materiali $P_1, \ldots, P_N$
con masse $m_1, \ldots, m_N$

$$\frac{m_1 \vec{P}_1 + \ldots + m_N \vec{P}_N}{m_1 + \ldots + m_N}$$

$$\vec{R} = \frac{\vec{P} + \lambda \vec{Q}}{1 + \lambda}$$



$$\frac{BL}{LC} = \frac{c}{b} \qquad \vec{L} = \frac{c\vec{C} + b\vec{B}}{b + c}$$

$$\frac{LI}{IA} = \frac{BL}{c}$$

$$I = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a + b + c}$$

$\vec{F} = ?$ F pt. medio di OH (O circocentro H ortocentro)

se metto l'origine in O

$$\vec{H} = \vec{A} + \vec{B} + \vec{C}$$

$$\vec{F} = \frac{1}{2}(\vec{O} + \vec{H}) = \frac{\vec{A} + \vec{B} + \vec{C}}{2}$$

$$\|\vec{I}-\vec{F}\|^2 = \left\| \frac{aA+bB+cC}{a+b+c} - \frac{A}{2} - \frac{B}{2} - \frac{C}{2} \right\|^2 =$$

$$= \left\| \frac{A(a-b-c) + B(b-a-c) + C(c-a-b)}{2(a+b+c)} \right\|^2 =$$

$$= \frac{1}{4(a+b+c)^2} \left[\sum_{cyc} AA\,(a-b-c)^2 + 2\sum_{cyc} A\cdot B\,(a-b-c)(b-a-c) \right]$$

$$= \frac{1}{4p^2} \left[\sum_{cyc} R^2 (a-b-c)^2 + \sum_{cyc} (2R^2-c^2)(\underbrace{c^2-a^2-b^2}+2ab) \right],$$

$\vec{A}\cdot\vec{A} = R^2$ $\qquad 2\vec{A}\cdot\vec{B} = 2R^2 - c^2$

origine in O

$$= \frac{1}{4p^2} \left[\sum_{cyc} R^2 (a^2+b^2+c^2-2ab-2ac+2bc) + \sum_{cyc} \cdots \right] =$$

$$= \frac{1}{4p^2} \left[R^2 \left(3\sum_{cyc} a^2 - 2\sum_{cyc} a^2 + 2\sum_{cyc} ab \right) - \left(\sum_{cyc} a^4 - 2\sum_{cyc} a^2b^2 \right) - 2\sum_{cyc} a^2bc \right] =$$

$$= \frac{1}{4p^2} R^2 \underbrace{\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab \right)}_{= p^2} + \frac{1}{4p^2} 16S^2 - \frac{2abc \cdot p}{4p^2} =$$

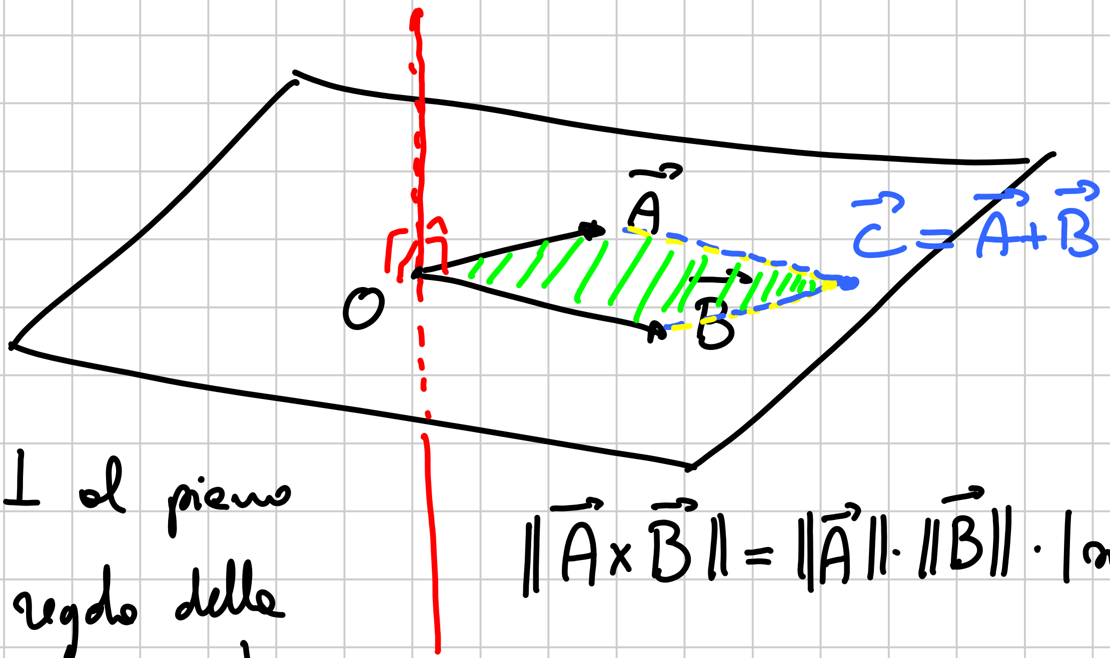
$$= \frac{1}{4}R^2 + \frac{4S^2}{p^2} - \frac{abc}{2p} = \frac{R^2}{4} + r^2 - 2\frac{R}{2}\cdot r =$$

$$= \left(\frac{R}{2} - r\right)^2 \qquad IF = \frac{R}{2} - r$$

$$\left[\begin{array}{l} OH = ? \quad GH = ? \quad GO = ? \quad GI = ? \quad IH = ? \\ \overset{\Delta}{GIH} \text{ è ottusangolo.} \end{array}\right]$$

<u>Oss</u>: $\vec{P}\cdot\vec{Q} = 0 \iff \vec{P} = \vec{0}$ o $\vec{Q} = \vec{0}$ o $PO \perp QO$ $\qquad$ O origine

<u>Prodotto vettore</u>: $\vec{A}, \vec{B} \longrightarrow \vec{A}\times\vec{B}$ in 3 dim.



dir = $\perp$ al piano

verso = regola della mano dx.

$$\|\vec{A}\times\vec{B}\| = \|\vec{A}\|\cdot\|\vec{B}\|\cdot|\sin A\hat{O}B|$$

$\searrow$ $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

A, O, B allineati $\iff \vec{A}\times\vec{B} = \vec{0}$

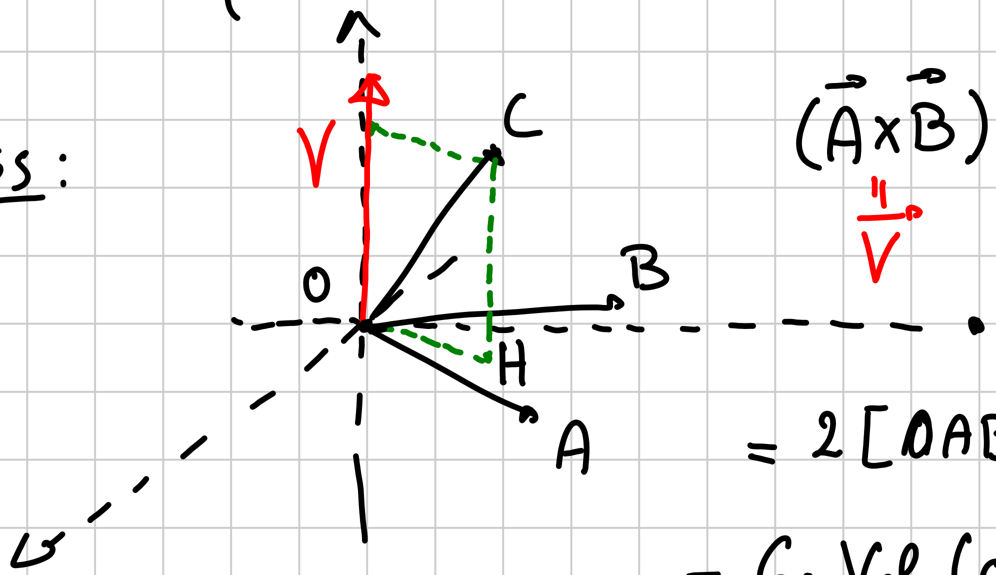
$\|A \times B\| = 2[AOB]$

Es per esse: $\vec{A} = (a_1, a_2, a_3)$

$\vec{B} = (b_1, b_2, b_3)$

$\vec{A} \times \vec{B} = (a_2 b_3 - b_2 a_3,\ b_1 a_3 - a_1 b_3,\ a_1 b_2 - b_1 a_2)$

Oss:



$(\vec{A} \times \vec{B}) \cdot \vec{C} =$

$\overset{\parallel}{\vec{V}}$

$= 2[OAB] \cdot CH =$

$= 6 \cdot Vol(OABC)$

$A = (a_1, a_2, a_3)$

$B = (b_1, b_2, b_3)$

$C = (c_1, c_2, c_3)$

$(A \times B) \cdot C =$

$= c_1 a_2 b_3 - c_1 b_2 a_3 + c_2 b_1 a_3 - c_2 a_1 b_3 + c_3 a_1 b_2 - c_3 b_1 a_2 =$

$$= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$A+B$

B

C

$C = D + E$

$((A+B) \times B) \cdot C =$

$$= (A \times B) \cdot C + \underbrace{(B \times B)}_{0} \cdot C$$

Coordinate baricentriche

ABC triangolo P punto del piano

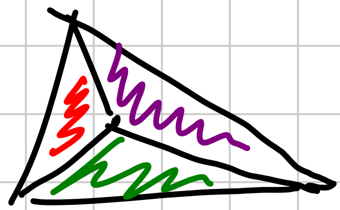
$\rightarrow \exists!\ \alpha, \beta, \gamma$ reali t.c.

(i) $\alpha + \beta + \gamma = 1$

(ii) $\alpha \vec{A} + \beta \vec{B} + \gamma \vec{C} = \vec{P}$

$[XYZ] > 0$ se X, Y, Z sono in senso antiorario

$$\gamma = \frac{[PAB]}{[ABC]}, \ldots,$$



$P = \alpha A + \beta B + \gamma C \qquad Q = \rho A + \sigma B + \tau C$

pt. medio $\frac{\alpha + \rho}{2} A + \frac{\beta + \sigma}{2} B + \frac{\gamma + \tau}{2} C$

Coord. baricentriche di $P = [p : q : r]$

$$\text{t.c. } \vec{P} = \frac{p\vec{A} + q\vec{B} + r \cdot \vec{C}}{p+q+r}$$

se $p + q + r = 1$ si dicono <u>normalizzate</u>

Es: $\{ lx + my + nz = 0 \}$ $\quad l, m, n$ reali

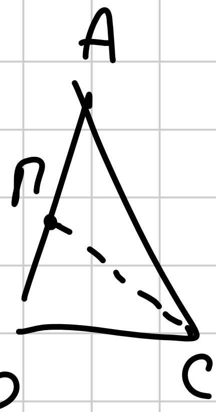
Lo è una retta

$x + y = 0 \qquad x = 0,\ y = 0,\ z = 0$ lati

Es: $A = [1:0:0] \quad B = [0:1:0] \quad C = [0:0:1]$

$M =$ pt medio di $AB = \left[\frac{1}{2} : \frac{1}{2} : 0\right] = [1:1:0]$

normalizzato $\qquad$ generico



$CM \to$ della forma $lx + my + nz = 0$

passare per C $\quad l \cdot 0 + m \cdot 0 + n \cdot 1 = 0$

" $\quad$ " M $\quad l \cdot 1 + m \cdot 1 + n \cdot 0 = 0$

$$\begin{cases} n = 0 \\ l + m = 0 \end{cases} \qquad x - y = 0$$

Un po' di punti: $\quad G = [1:1:1]$

$I = [a:b:c] \to$ excentri si cambia un segno

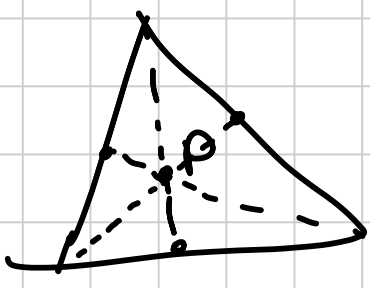
$M_a = [0:1:1] \quad M_b = [1:0:1]$

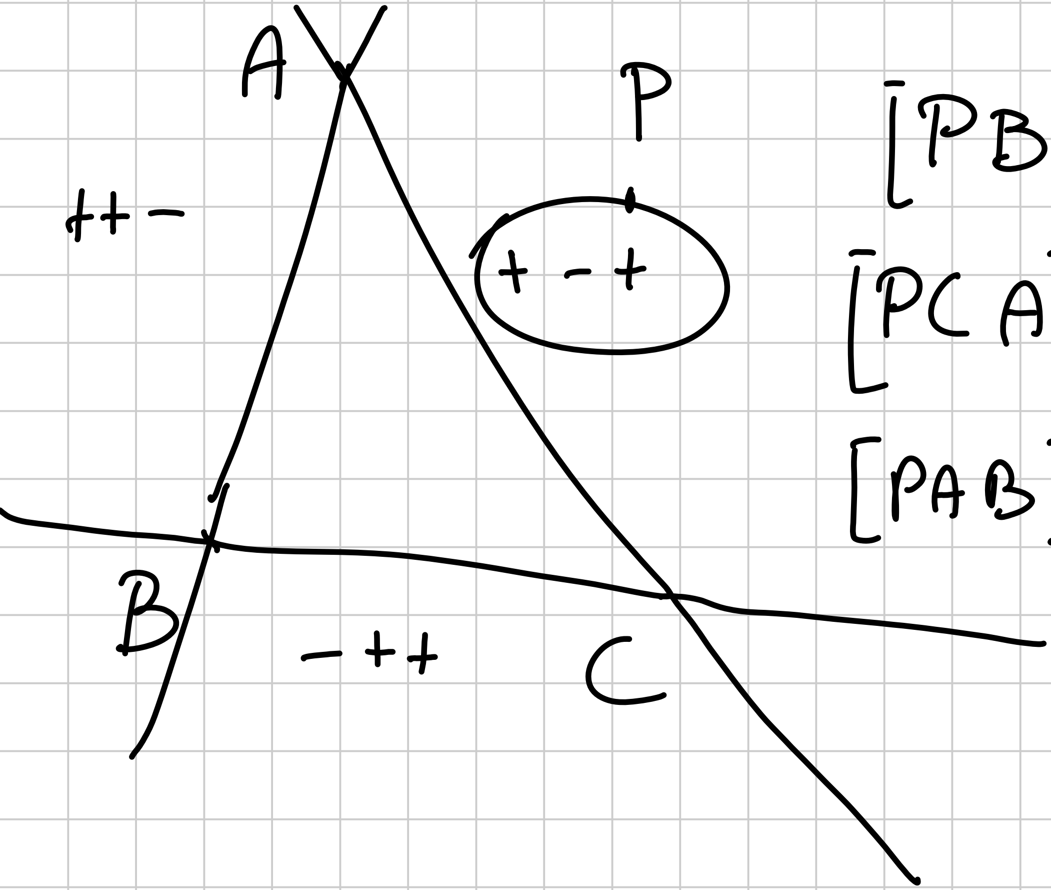
$M_c = [1:1:0]$

piede delle bisett: $\quad L_a = [0:b:c]$

$O = - - -$

$H = - - -$

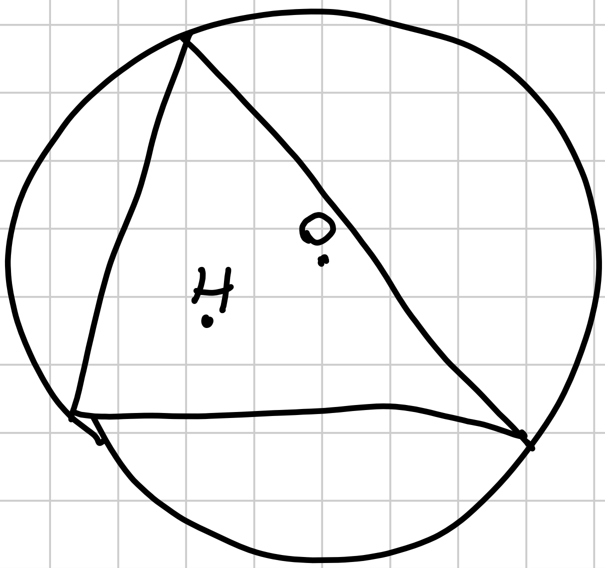


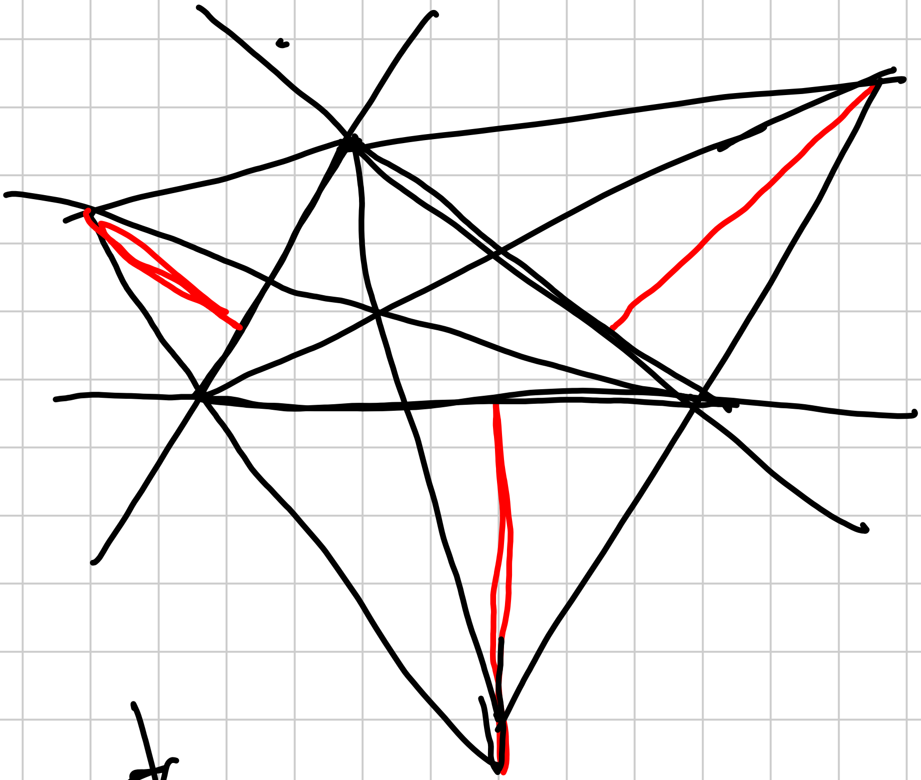


$[PBC] > 0$

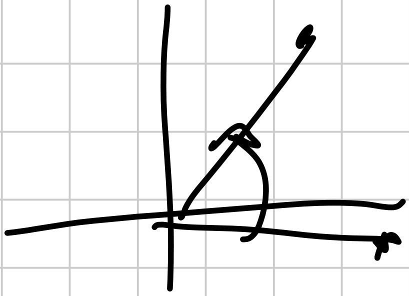
$[PCA] < 0$

$[PAB] > 0$

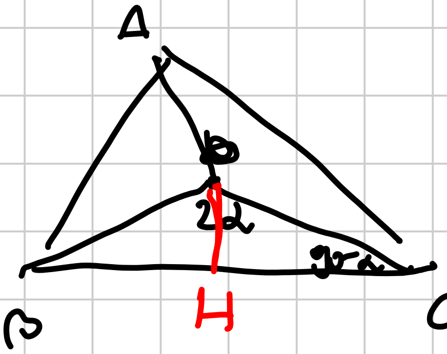




$\mathbb{R}^3$



- YIU: Introduction to Triangle Geometry
- I. to the geometry of Complex Numbers
-



$$\left[\Delta OBC : \Delta OCA : \Delta OAB\right]$$

$$\left[\frac{BC \cdot OH}{2} : \dots\right]$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$$\left[\frac{\cancel{2}R\sin\alpha R\cos\alpha}{\cancel{2}} : \dots\right]$$

$$\left[\sin\alpha\cos\alpha : \sin\beta\cos\beta : \sin\gamma\cos\gamma\right] \quad 1)$$

$$\left[\sin 2\alpha : \sin 2\beta : \sin 2\gamma\right]$$

1) $[2R\sin\alpha\cos\alpha \,.\, \dots]$

$[a\cos\alpha : b\cos\beta : c\cos\gamma]$

$\left[a\left(\frac{b^2+c^2-a^2}{2bc}\right) : \dots\right]$

$[a^2(b^2+c^2-a^2) : \dots]$

2) CONWAY

$S_A = 2S \cdot \cot\alpha = \frac{b^2+c^2-a^2}{2}$

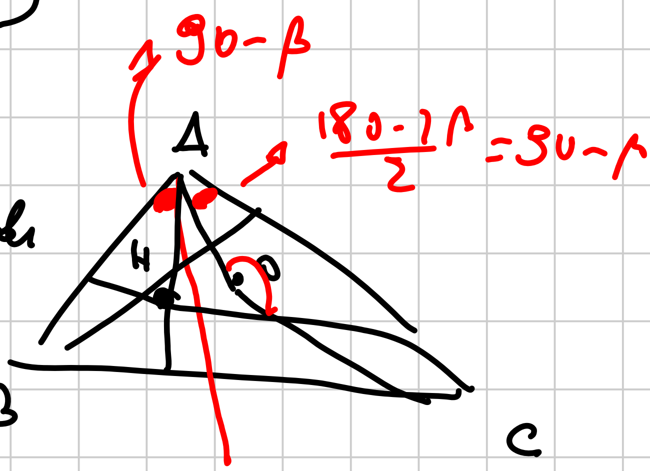
$2\frac{bc\cancel{\sin\alpha}}{2}\frac{\cos\alpha}{\cancel{\sin\alpha}}$ // Carnot

$S_B = \dots$

$S_C = \dots$

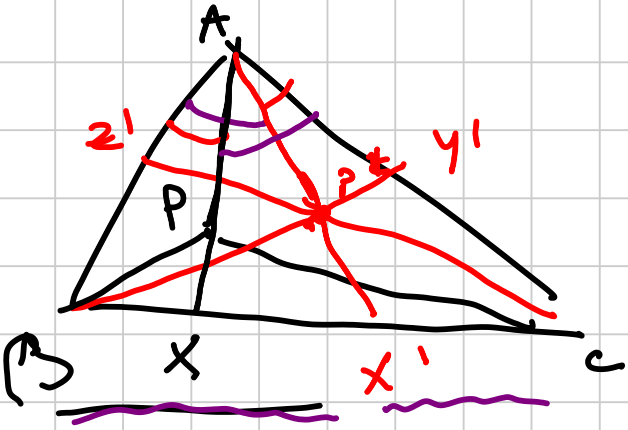
3) $[a^2 S_A : \dots]$

H, O coniugati isogonali



$P[u:v:w]$

P^c ?



Se $P\,[u : v : w]$

$X \overset{?}{=} [0 : v : w]$

$$X = [\Delta XBC : \Delta XCA : \Delta XAB]$$

$$[0 \;:\; \;:\;]$$

Def. $P \backsim [\Delta PBC : \Delta PCA : \Delta PAB]$

$$\frac{\Delta PCA}{\Delta PAB} = \frac{v}{w}$$

$$\parallel$$

$$\frac{\Delta XCA}{\Delta XAB} = \frac{XC}{XB} \Longrightarrow X \backsim [0 . v . w]$$

$$\frac{XC}{XB} = \frac{v}{w}$$

$$\frac{X'C}{X'B} = \,?$$

$$X' \backsim [\Delta X'BC : \Delta X'CA : \Delta X'AB]$$

$$\frac{X'C}{\sin {\color{red}\bullet}} = \frac{b}{\sin AX'C} \longrightarrow X'C = \frac{b}{\sin AX'C} \sin {\color{red}\bullet}$$

$$\frac{X'B}{\sin {\color{purple}\bullet}} = \frac{c}{\sin AX'B} \qquad X'B = \frac{c}{\sin AX'B} \sin {\color{purple}\bullet}$$

$$\boxed{\frac{X'C}{X'B} = \frac{b}{c}\,\frac{\sin {\color{red}\bullet}}{\sin {\color{purple}\bullet}}} \longrightarrow \frac{X'C}{X'B} = \frac{b^2}{c^2}\,\frac{XB}{XC} = \frac{b^2}{c^2}\,\frac{w}{v} = \frac{\frac{b^2}{v}}{\frac{c^2}{w}}$$

$$\frac{XC}{XB} = \frac{b}{c}\,\frac{\sin {\color{purple}\bullet}}{\sin {\color{red}\bullet}} \longrightarrow \frac{\sin {\color{red}\bullet}}{\sin {\color{purple}\bullet}} = \frac{b}{c}\,\frac{XB}{XC}$$

Abbiamo mostrato che

$$\frac{X'C}{X'B} = \frac{\frac{b^2}{v}}{\frac{c^2}{w}} \; ! \longrightarrow X' \backsim \left[0 : \frac{b^2}{v} : \frac{c^2}{w}\right]$$

$$y' \, x\left[\frac{a^2}{u} : 0 : \frac{c^2}{w}\right]$$

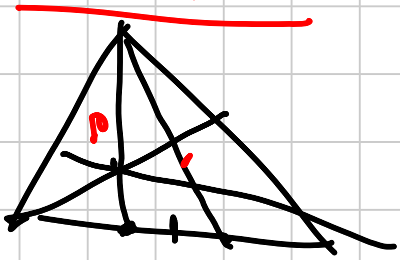
$$z' \, y\left[\frac{a^2}{u} : \frac{b^2}{v} : 0\right]$$

$$P'\left[\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right]$$

$$H\left[\frac{a^2}{a^2 S_A} : \dots\right] = H\left[\frac{1}{S_A} : \dots\right]$$

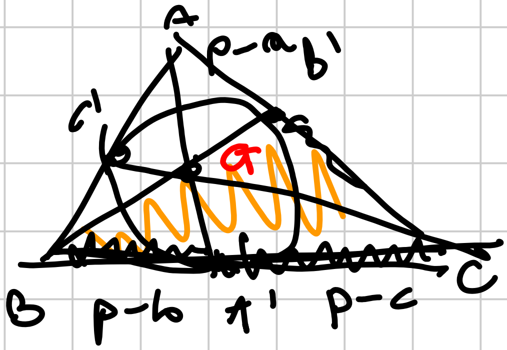
$$H\left[\tan\alpha : \dots\right]$$

ISOTOMIC



$$P[u:v:w] \longrightarrow P'\left[\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right]$$

G



$$G\left[\frac{1}{\ } \right]$$

$$A'[0 : p-c : p-b]$$

$$B'[p-c : 0 : p-a]$$

$$C'[p-b : p-a : 0]$$

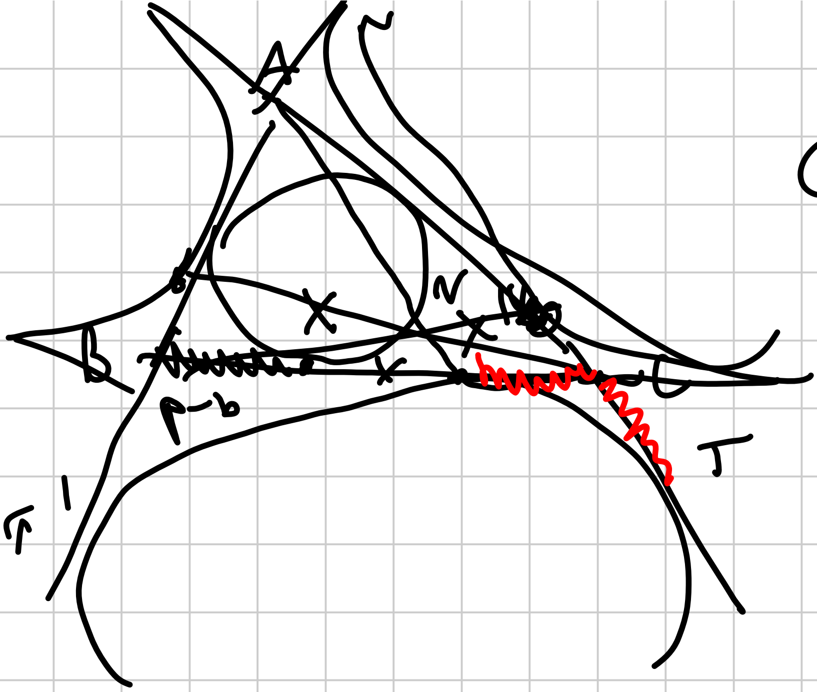
$$A'\left[0 : \frac{1}{p-b} : \frac{1}{p-c}\right]$$

$$B'\left[\frac{1}{p-a} : 0 : \frac{1}{p-c}\right]$$

$$C'\left[\frac{1}{p-a} : \frac{1}{p-b} : 0\right]$$

$$\longrightarrow G\left[\frac{1}{p-a} : \frac{1}{p-b} : \frac{1}{p-c}\right]$$

$$N[p-a : p-b : p-c]$$

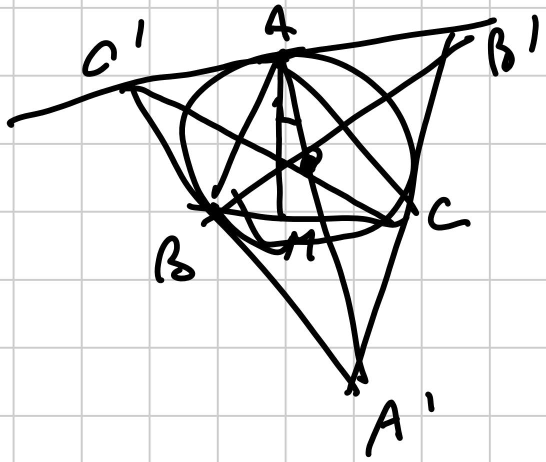


$CT = AT - AC = p-b$

$AT' + AT = AB + BT' + AC + C\ldots =$

$= AB + BX' + AC + CX' =$ Perimetro

$G[1:1:1]$ $\qquad$ $L[a^2 : b^2 : c^2]$



$O[a^2 S_A : b^2 S_B : c^2 S_C]$

$a^2 S_A + b^2 S_B + c^2 S_C = 16A^2$ $\qquad$ (*) Esame

Dimostra!

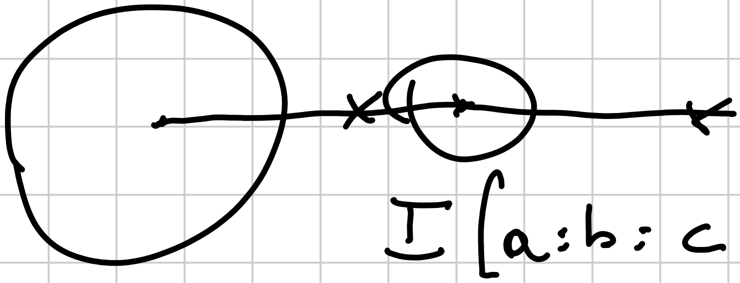
$-a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2$

$H[4S_B S_C : 4S_A S_C : 4S_A S_B]$

$S_B S_C + S_A S_C + S_A S_B = 4A^2$

$F : \left[\dfrac{a^2 S_A + 4 S_B S_C}{2} : \quad \right]$

Per esercizio



$I\,[a : b : c]$ $\quad 2p$

$O\,[a^2 S_A : b^2 S_B : c^2 S_C]$ $\quad 16A^2$

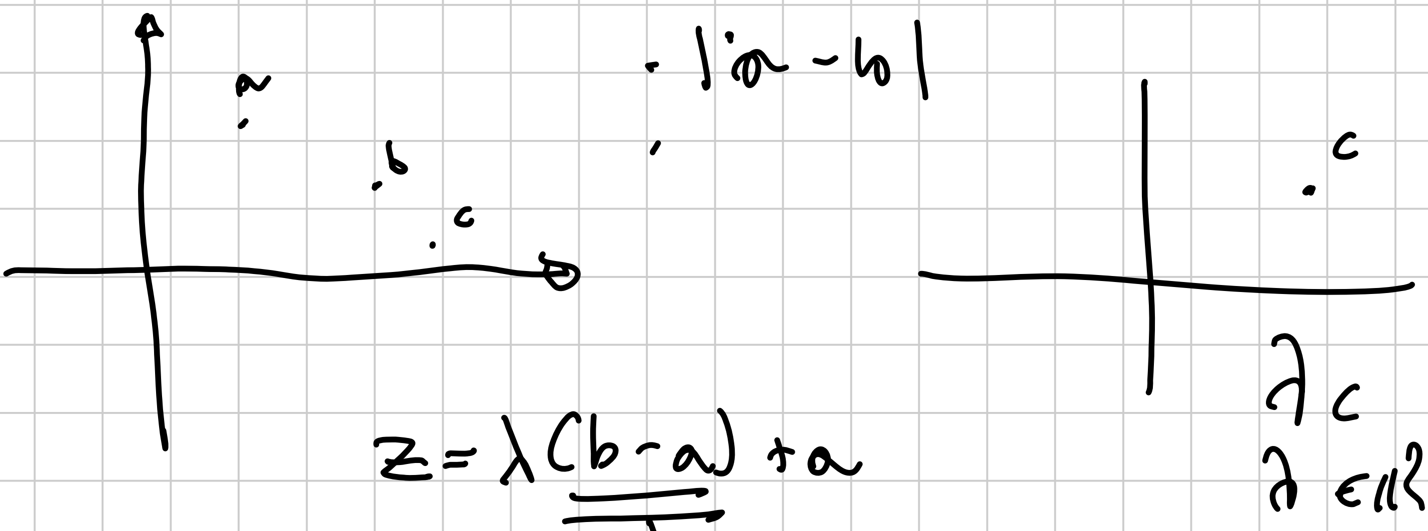
$\frac{AX}{XB} = \lambda \qquad \frac{\vec{A} + \lambda \vec{B}}{1+\lambda} =$

$$\frac{\vec{I} + \frac{r}{R}\vec{O}}{1 + \frac{r}{R}} = \ldots$$

$N\,[p-a : p-b : p-c]$ $\qquad N^c\left[\frac{a^2}{p-a} : \ldots\right]$

$G\left[\frac{1}{p-a} : \ldots\right]$ $\qquad G^c\,[a^2(p-a) : \ldots]$

$|a-b|$



$z = \underbrace{\lambda(b-a)}_{\lambda \in \mathbb{R}} + a$

λc

$\lambda \in \mathbb{R}$

$\exists \lambda \quad c = \lambda(b-a) + a \quad \lambda \in \mathbb{R}$ All.

$\frac{c-a}{b-a} \in \mathbb{R} \iff \boxed{\frac{c-a}{b-a} = \frac{\bar c - \bar a}{\bar b - \bar a}}$

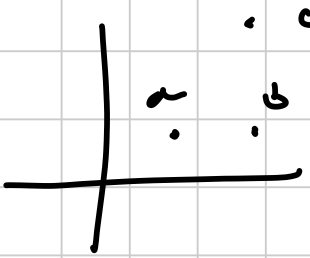
$\frac{z-a}{b-a} \in \mathbb{R} \iff \frac{z-a}{b-a} = \frac{\bar z - \bar a}{\bar b - \bar a}$ $\quad (b\bar a) - (\overline{b \bar a})$

1) $(\bar b - \bar a) z + (a-b)\bar z + \boxed{b\bar a - a\bar b} = 0$

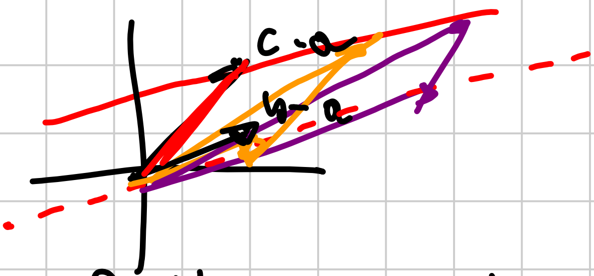
$\underbrace{i(\bar b - \bar a)}_{A} z + \underbrace{i(a-b)}_{\bar A}\bar z + \underbrace{i(b\bar a - a\bar b)}_{B} = 0$

2) $A z + \bar A \bar z + B = 0 \qquad B \in \mathbb{R}.$

Parallelismo



// ab per c



$// \; z = c - a + \lambda(b-a) \quad \lambda \in \mathbb{R}$

$z = c - \cancel{a} + \lambda(b-a) + \cancel{a}$

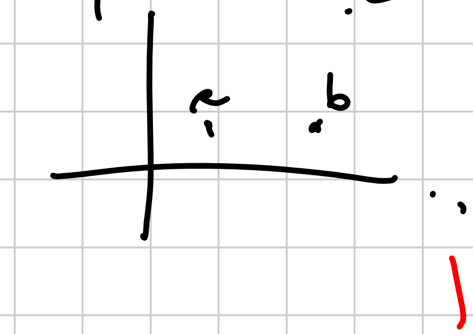
$z = c + \lambda(b-a)$

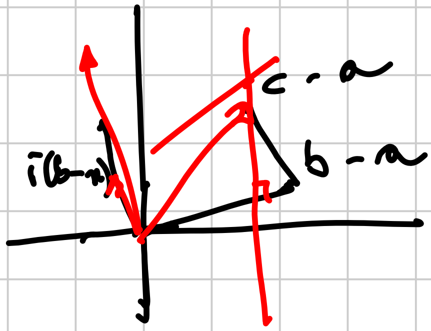
$\exists \lambda$ t.c. $d = c + \lambda(b-a) \quad \lambda \in \mathbb{R}$

$\frac{d-c}{b-a} \in \mathbb{R} \iff \boxed{\frac{d-c}{b-a} = \frac{\bar d - \bar c}{\bar b - \bar a}}$

? $\frac{z-c}{b-a} = \frac{\bar z - \bar c}{\bar b - \bar a}$ $\cdots \longrightarrow (\bar b - \bar a) z + (a-b)\bar z$
$+ c\bar a - a\bar c + \bar c b - \bar b c = 0$

Perpendicolari





$z = c - a + \lambda i (b-a) \quad \lambda \in \mathbb{R}$

$z = c + \lambda i (b-a) \quad \lambda \in \mathbb{R}$

$ab \perp cd$

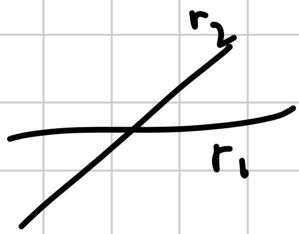
$\exists \lambda$ t.c. $d = c + \lambda i (b-a)$

$$\frac{d-c}{b-a} \in \text{Im} \iff \boxed{\frac{d-c}{b-a} = -\frac{\bar d - \bar c}{\bar b - \bar a}}$$

Retta

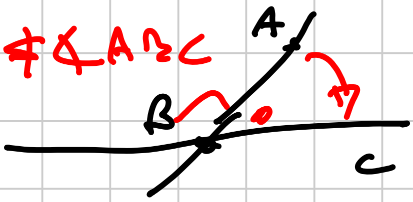
$$\frac{z-c}{b-a} = -\frac{\bar z - \bar c}{\bar b - \bar a} \longrightarrow (\bar b - \bar a) z + (b-a)\bar z + c\bar a + a\bar c - c\bar b - \bar c b = 0$$

Angoli ORIENTATI



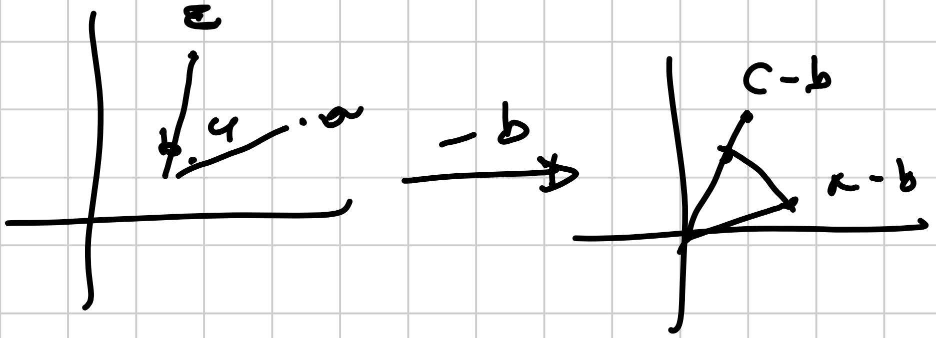
$\measuredangle(r_1, r_2)$ ruotare r_1 in senso orario per coinc. con r_2

$\measuredangle \underline{ABC} \neq \measuredangle ABC$



Att può non essere l'angolo classico

Att 2 = a meno di $k\pi$ $k \in \mathbb{Z}$.



$$c-b = (a-b)\,e^{i \measuredangle abc} \cdot \frac{|c-b|}{|a-b|}$$

EQ. ANGOLO $\dfrac{c-b}{a-b} = e^{i\varphi}\dfrac{|c-b|}{|a-b|}$ $\qquad \varphi = \measuredangle abc$

Supponiamo di voler mostrare che

$$\underset{\varphi}{\measuredangle ABC} \overset{?}{=} \underset{\theta}{\measuredangle xyz}$$

$$\frac{c-b}{a-b} = e^{i\varphi}\frac{|c-b|}{|a-b|} \quad (1)$$

$$\frac{z-y}{x-y} = e^{i\theta}\frac{|z-y|}{|x-y|} \quad (2)$$

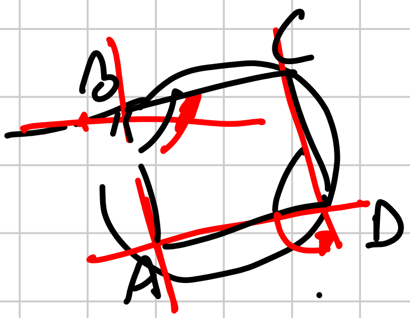
$$\frac{c-b}{a-b}\cdot\frac{x-y}{z-y} = e^{i(\varphi-\theta)}\overbrace{\frac{|c-b|}{|a-b|}\,\frac{|x-y|}{|z-y|}}^{\mathbb{R}}$$

$$\varphi \overset{1}{=} \theta$$

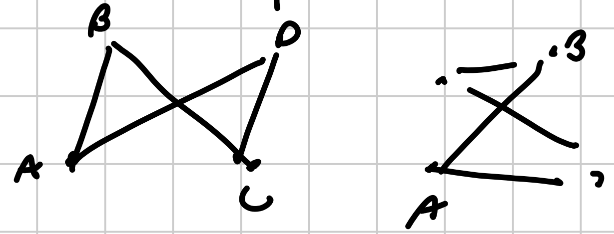
$$\Updownarrow$$

$$\frac{c-b}{a-b}\,\frac{x-y}{z-y} \in \mathbb{R} \iff \boxed{\frac{c-b}{a-b}\,\frac{x-y}{z-y} = \frac{\bar c-\bar b}{\bar a-\bar b}\,\frac{\bar x-\bar y}{\bar z-\bar y}}$$

Ciclici

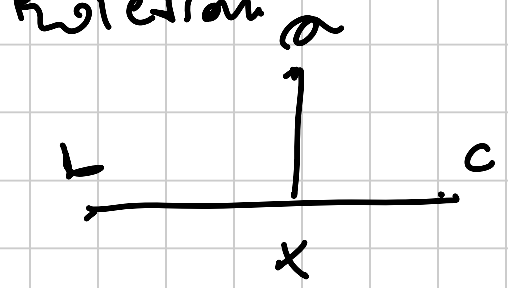


$$\measuredangle ABC = \measuredangle ADC$$



$$\frac{c-b}{a-b}\,\frac{a-d}{c-d} = \frac{\bar c-\bar b}{\bar a-\bar b}\,\frac{\bar a-\bar d}{\bar c-\bar d}$$

Proiezioni



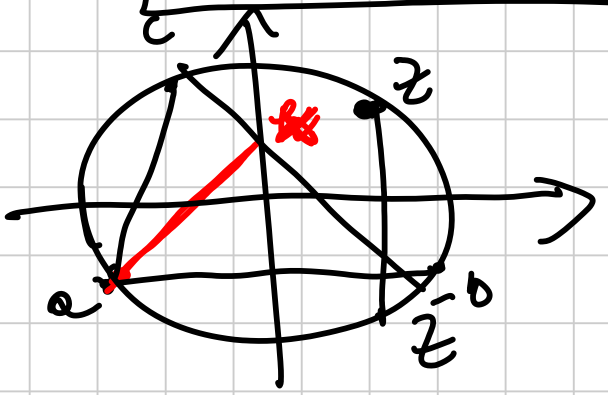
$x \in bc$

$$\frac{x-b}{c-b} = \frac{\bar x - \bar b}{\bar c - \bar b}$$

$ax \perp bc$

$$\frac{c-b}{x-a} = -\frac{\bar c - \bar b}{\bar x - \bar a}$$

$$\boxed{x = \frac{1}{2}\left(a + \frac{(b-c)\bar a + \bar b c - b\bar c}{\bar b - \bar c}\right)}$$



$$z\bar z = |z|^2 = 1$$

$\bar z = \frac{1}{z}$ luogo dei punti sulla circ. unitaria

$$x = \frac{1}{2}\left(a + \frac{(b-c)\frac{1}{a} + \frac{c}{b} - \frac{b}{c}}{\frac{c-b}{bc}}\right) =$$

$$= \frac{1}{2}\left(a + \frac{\frac{bc(b-c) + ac^2 - ab^2}{a\,\cancel{bc}}}{\frac{c-b}{\cancel{bc}}}\right) =$$

$$= \frac{1}{2}\left(a + \frac{\cancel{(b-c)}(bc - a(b+c))}{\cancel{c-b}}\right)$$

$$\frac{1}{2}\left(a - \frac{bc + ab + ac}{a}\right) =$$

$$= \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right)$$

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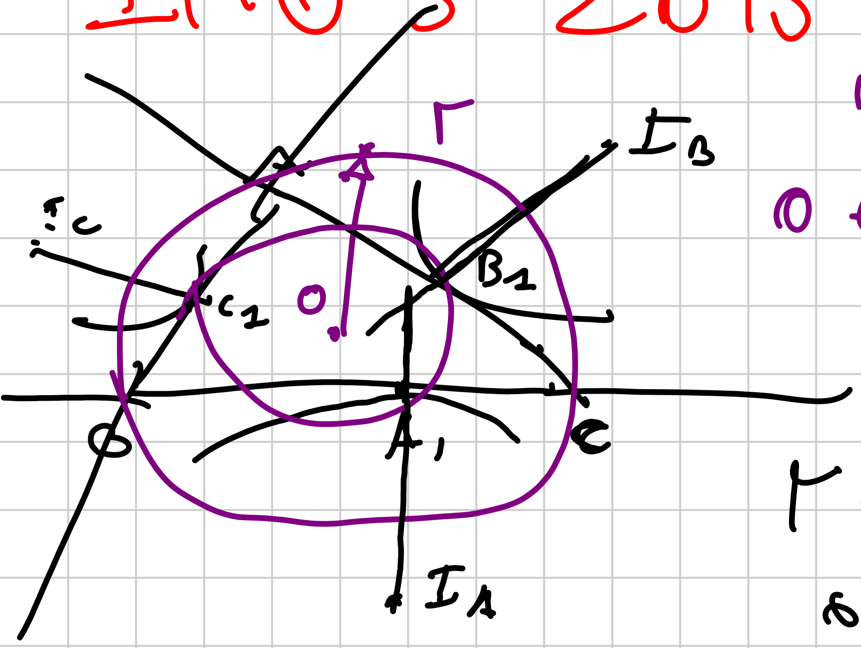
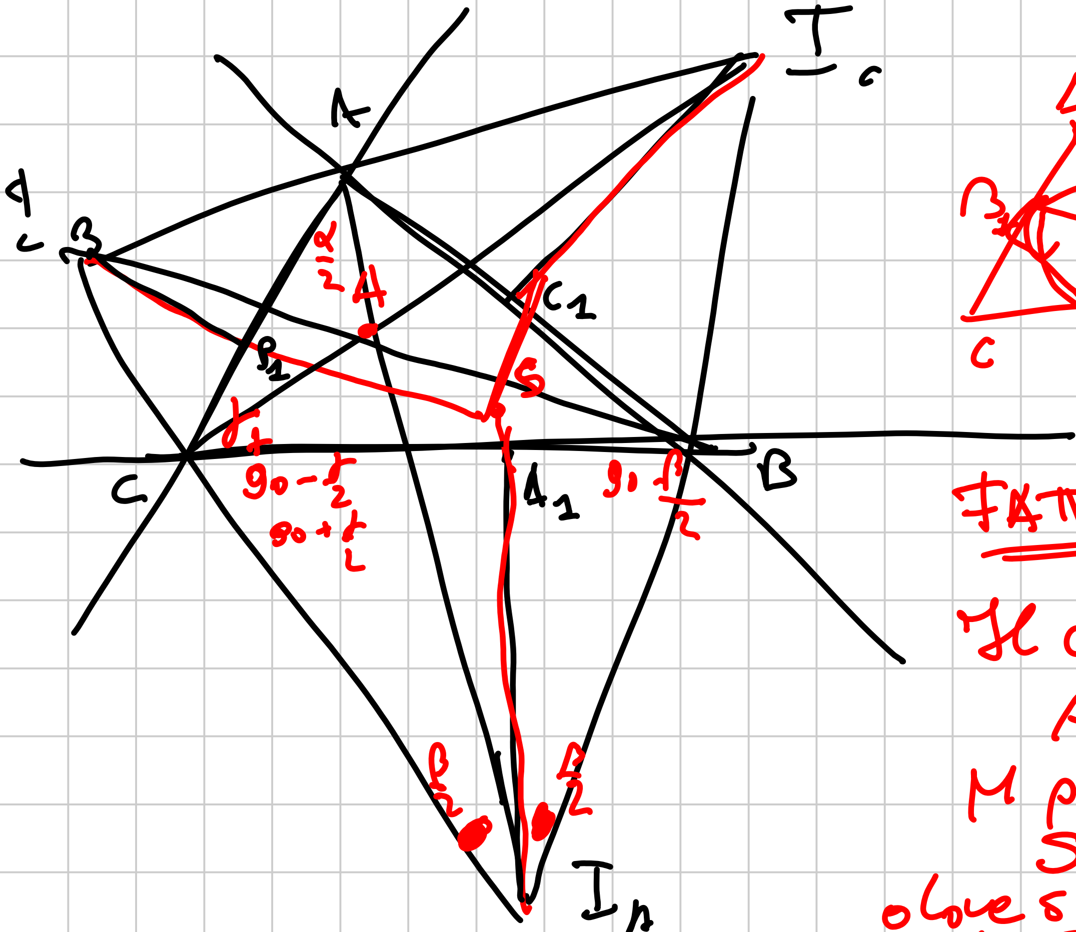


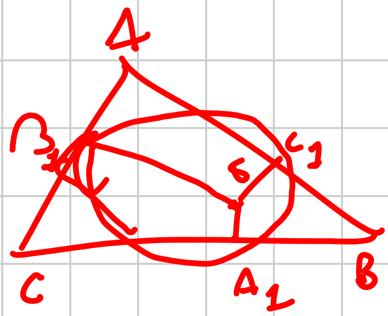
Hp

$O \in \Gamma \implies A\hat{B}C$ rettangolo

Γ unitaria

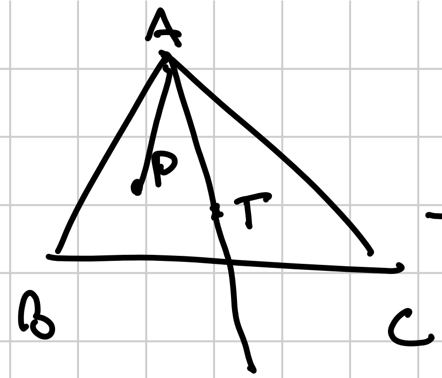
$o\bar{o} = 1$





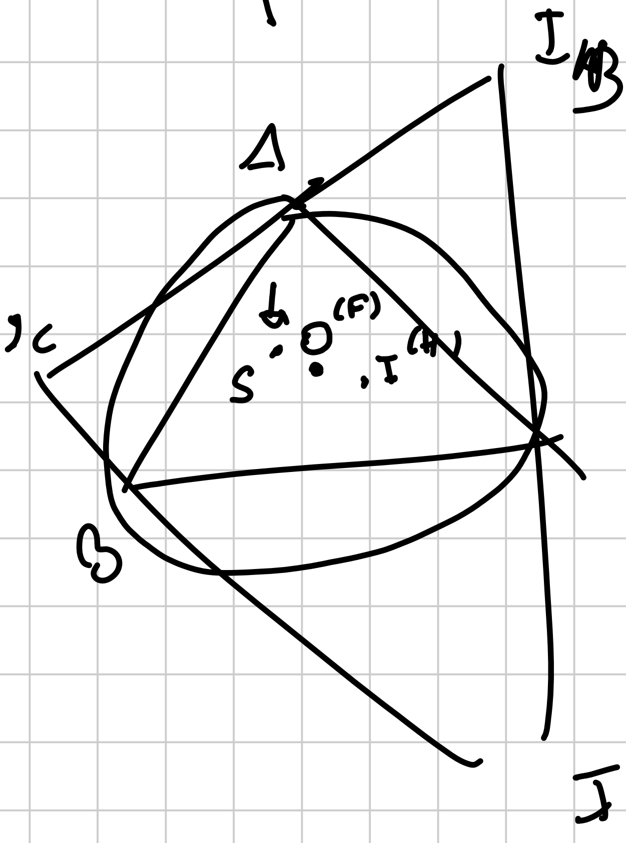
FATTO

Il circocentro di $A_1B_1C_1$ è M p.to medio di SS' dove S' è con. isogonale di S in ABC



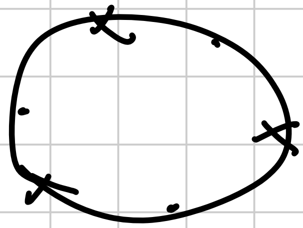
$\sphericalangle BAP = \sphericalangle TAC$ T: Lelo, il [illegible]

$$\rightarrow t = \frac{-p + a + b + c - \bar{p}(ab + bc + ca) + \bar{p}^2 abc}{1 - p\bar{p}}$$



b	F	H

c S è il simmetrico dell'incentro rispetto al circuito



<u>TRUCCO</u>   a, b, c

$a = u^2$
$b = v^2$ $\longrightarrow$ Punti $-uv, -vw, -uw$
$c = w^2$

i punti medi degli archi

$i = -(uv + vw + uw)$

$s = uv + vw + uw$ u^2, v^2, w^2

$$\rho = \frac{s + s^c}{2} =$$

$$\begin{cases} u + v + w = A \\ uv + vw + uw = B \\ uvw = C \end{cases} \quad \bar{A} = \frac{B}{C} \quad \bar{B} = \frac{A}{C} \quad \bar{C} = \frac{1}{C}$$

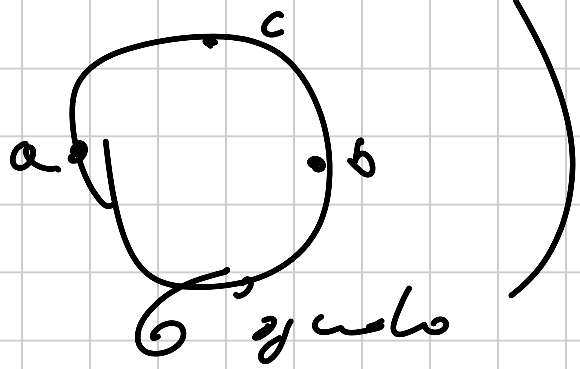
$O\bar{O} = 1 \longrightarrow \ldots$ 9° grado $\neq 0$

ABC rettangolo

$\Updownarrow$

$(a+b)(b+c)(c+a)=0$

$u^2 \quad v^2 \quad v^2 \quad w^2 \quad u^2+w^2$



cerchio