G1. Advanced LucaMac

Titolo nota 02/09/2015

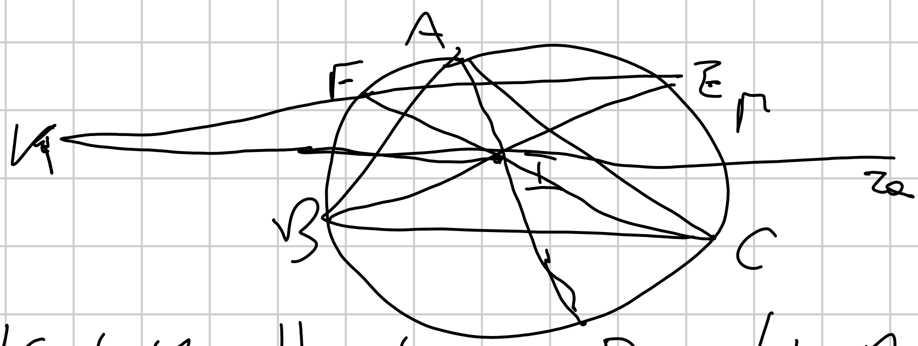
Parallele: data una retta ed un punto trovare la parallela a tale retta per il dato punto.

$x+y+z=0$

A, B, C $\qquad C_\infty \in AB$ $\qquad CC_\infty$

$AB: ux+vy+wz=0$ $\qquad x+y+z=0$

Esercizio BMO 2015.2



K, L, M allineati $\qquad$ (L, M analogamente definiti)

Sol

ABC

$I=(a,b,c)$

$\Gamma: a^2yz+b^2xz+c^2xy=0$

$AI: \det\begin{pmatrix} x & y & z \\ 1 & 0 & 0 \\ a & b & c \end{pmatrix}=0 \qquad AI: yc-zb=0$

$a^2cy^2+b^2cxy+c^2bxy=0$

$a^2y+b^2x+bcx=0$

$D=(-a^2, b(b+c), c(b+c))$

$E=(a(a+c), -b^2, c(a+c))$

$F=(a(a+b), b(a+b), -c^2)$

$BC: x=0 \qquad x+y+z=0$

$A_\infty=(0,1,-1)$

$r_a: x(b+c)-ya-za=0$

$$EF: -xbc + yc(a+c) + zb(a+b) = 0$$

$$K = (a(b-c), b^2, -c^2)$$

$$L = (-a^2, b(c-a), c^2)$$

$$M = (a^2, -b^2, c(a-b))$$

$$\det\begin{pmatrix} a(b-c) & b^2 & -c^2 \\ -a^2 & b(c-a) & c^2 \\ a^2 & -b^2 & c(a-b) \end{pmatrix} \overset{?}{=} 0$$

$$(b-c)(c-a)(a-b) \overset{?}{=} \sum_{cyc} bc(c-b)$$

$$\sum_{cyc} bc^2 - \sum_{cyc} b^2c \overset{?}{=} \sum_{cyc} bc^2 - \sum_{cyc} b^2c$$

$$P = (\alpha, \beta, \gamma) \qquad Q = \left(\frac{a^2}{\alpha}, \frac{b^2}{\beta}, \frac{c^2}{\gamma}\right)$$ Coniugato Isogonale!!

$P = (\alpha_1, \beta_1, \gamma_1)$ $Q = (\alpha_2, \beta_2, \gamma_2)$ $\qquad \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1$

$$\vec{PQ} = (x_1, y_1, z_1) = (\alpha_2 - \alpha_1, \beta_2 - \beta_1, \gamma_2 - \gamma_1)$$

$$PQ^2 = -(a^2 y_1 z_1 + b^2 x_1 z_1 + c^2 x_1 y_1) \leftarrow$$

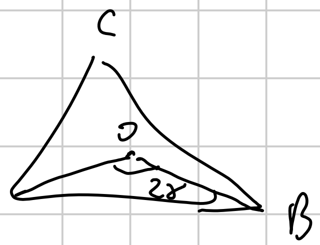
Dim

WLOG Origine in O (circocentro)

$$\vec{PQ} = x_1\vec{A} + y_1\vec{B} + z_1\vec{C}$$

$$PQ^2 = \|(x_1\vec{A} + y_1\vec{B} + z_1\vec{C})(x_1\vec{A} + y_1\vec{B} + z_1\vec{C})\|$$

$$\|\vec{A}\cdot\vec{A}\| = R^2 \qquad \|\vec{A}\cdot\vec{B}\| = R^2 - \frac{c^2}{2}$$



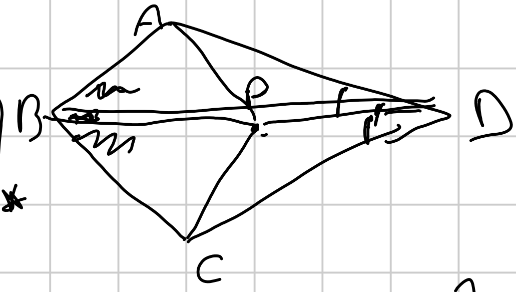
$$x_1 + y_1 + z_1 = 0 \quad \left(\text{perché } \sum \alpha_1 = 1 = \sum \alpha_2\right)$$

$$PQ^2 = R^2\underbrace{(x_1 + y_1 + z_1)}_{=0}{}^2 - \sum_{cyc} c^2 x_1 y_1 = -\sum_{cyc} a^2 y_1 z_1$$

Esercizio IMO 2004.5

BD ~~NON~~ è

$\boxed{\text{bisettrice né di } A\hat{B}C \text{ né di } A\hat{D}C \text{ !!}}$ *



* BDP non degenere!

$$P\hat{B}C = D\hat{B}A \quad P\hat{D}C = B\hat{D}A$$

$\boxed{\text{A e C coniugati isogonali in PBD}}$

ABCD ciclico $\stackrel{?}{\iff}$ $AP = CP$

PBD $(BD = a,\ PB = c,\ PD = b)$

$P = (1,0,0) \quad B = (0,1,0) \quad D = (0,0,1)$

$A = (\alpha, \beta, \gamma) \quad C = \left(\frac{a^2}{\alpha}, \frac{b^2}{\beta}, \frac{c^2}{\gamma}\right)$

$\Gamma: \sum_{cyc} a^2 yz = (x+y+z)(ux+vy+wz)$ per opportuni u, v, w

$B \Rightarrow v = 0$

$D \Rightarrow w = 0$

$A \Rightarrow \sum_{cyc} a^2\beta\gamma = \left(\sum_{cyc}\alpha\right) u \alpha \Rightarrow u = \frac{\sum_{cyc} a^2\beta\gamma}{(\sum\alpha)\alpha}$

$$a^2b^2c^2 \sum_{cyc} \frac{1}{\beta\gamma} = \left(\sum_{cyc}\frac{a^2}{\alpha}\right)\frac{a^2}{\alpha} u$$

$$\alpha b^2c^2 \sum_{cyc}\alpha = \sum_{cyc} a^2\beta\gamma \cdot u$$

$$\boxed{\left[\sum_{cyc} a^2\beta\gamma\right]^2 = \alpha^2 b^2 c^2 \left(\sum_{cyc}\alpha\right)^2} \iff \text{ABCD ciclico!}$$

$P = (1,0,0)$

$$A = \left(\frac{\alpha}{\sum_{cyc}\alpha}, \frac{\beta}{\sum\alpha}, \frac{\gamma}{\sum\alpha}\right) \quad C = \left(\frac{a^2}{\alpha\sum_{cyc}\frac{a^2}{\alpha}}, \frac{b^2}{\beta\sum\frac{a^2}{\alpha}}, \frac{c^2}{\gamma\left(\sum\frac{a^2}{\alpha}\right)}\right)$$

$$\vec{AP} = \left(\frac{\beta+\gamma}{\sum\alpha}, \frac{-\beta}{\sum\alpha}, \frac{-\gamma}{\sum\alpha}\right) \quad \vec{CP} = \left(\frac{\frac{b^2}{\beta}+\frac{c^2}{\gamma}}{\sum\frac{a^2}{\alpha}}, -\frac{\frac{b^2}{\beta}}{\sum\frac{a^2}{\alpha}}, -\frac{\frac{c^2}{\gamma}}{\sum\frac{a^2}{\alpha}}\right)$$

$$\frac{1}{(\sum\alpha)^2}\left[a^2\beta\gamma - (\beta+\gamma)(c^2\beta + b^2\gamma)\right] = \frac{1}{\left(\sum\frac{a^2}{\alpha}\right)^2}\cdot\left[\frac{a^2b^2c^2}{\beta\gamma} - \left(\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right)\cdot(b^2c^2)\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\right]$$

$$\left(\sum \frac{a^2}{\alpha}\right)^2\left[a^2\beta\gamma-(\beta+\gamma)(c^2\beta+b^2\gamma)\right]=\left(\sum\alpha\right)^2\left[b^2c^2\right]\left[\frac{a^2}{\beta\gamma}-\left(\frac{b^2}{\beta}+\frac{c^2}{\gamma}\right)\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\right]$$

$$\left[\sum a^2\beta\gamma\right]^2\underbrace{\left[a^2\beta\gamma-(\beta+\gamma)(c^2\beta+b^2\gamma)\right]}_{\neq 0}=\left(\sum\alpha\right)^2 b^2c^2\alpha^2\cdot\left[a^2\beta\gamma-(\beta+\gamma)(b^2\gamma+c^2\beta)\right]$$

$$\left(\sum a^2\beta\gamma\right)^2=\left(\sum\alpha\right)^2 b^2c^2\alpha^2 \iff AP=CP$$

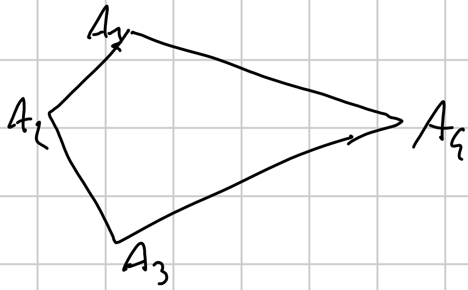
Fine!

Potenze

$$Pot_P(\Gamma)=OP^2-R^2=-\sum a^2yz-R^2=-\sum a^2yz+\left(\sum x\right)\left(\sum ux\right)$$

Es ISL 2011 G2

NON ciclico!



O_1, r_1 centro e raggio di $\odot A_2A_3A_4=\Gamma_1$ e cyc

$$\sum_{i=1}^{4}\frac{1}{O_iA_i^2-r_i^2}\overset{?}{=}0$$

$$\sum_{i=1}^{4}\frac{1}{Pot_{A_i}(\Gamma_i)}$$

$A_1A_2A_3$ $\quad A_2A_3=a \quad A_1A_3=b \quad A_1A_2=c$

$A_4=(\alpha,\beta,\gamma)$ $\quad \sum a^2\beta\gamma\neq 0$

$$\Gamma_4:\ -\sum a^2yz=0$$

$$Pot_{A_4}(\Gamma_4)=-\frac{\sum a^2\beta\gamma}{\left(\sum\alpha\right)^2}$$

$$\Gamma_3:\ -\sum a^2yz+(x+y+z)(wz)=0$$

$$-\sum a^2\beta\gamma+\left(\sum\alpha\right)\gamma w \qquad w=\frac{\sum a^2\beta\gamma}{\gamma\sum\alpha}$$

$$Pot_{A_3}(\Gamma_3)=\frac{\sum a^2\beta\gamma}{\gamma\sum\alpha}$$

$$Pot_{A_2}(\Gamma_2)=\frac{\sum a^2\beta\gamma}{\beta\sum\alpha} \qquad Pot_{A_1}(\Gamma_1)=\frac{\sum a^2\beta\gamma}{\alpha\sum\alpha}$$

$$-\frac{(\sum\alpha)^2}{\sum\alpha^2\beta\gamma}+\sum\left(\alpha\ \frac{\sum\alpha}{\sum\alpha^2\beta\gamma}\right)\stackrel{?}{=}0$$

$$-\frac{(\sum\alpha)^2}{\sum\alpha^2\beta\gamma}+\frac{(\sum\alpha)^2}{\sum\alpha^2\beta\gamma}\stackrel{?}{=}0 \quad \text{vero!}$$

Parallele e Perpendicolari:

$\vec{PQ}=(x_1,y_1,z_1)$ $\quad\vec{RS}=(x_2,y_2,z_2)$ $\quad x_1+y_1+z_1=$
$=x_2+y_2+z_2=0$

$PQ\perp RS \iff T=\sum_{cyc}a^2(y_1z_2+y_2z_1)=0$

Non c'è bisogno però che sia $x_1+y_1+z_1$ che $x_2+y_2+z_2$ siano 0! Ne basta 1.

$PQ\perp RS \iff (x_1\vec{A}+y_1\vec{B}+z_1\vec{C})(x_2\vec{A}+y_2\vec{B}+z_2\vec{C})=0$

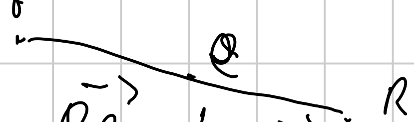
WLOG origine in O! $\quad R^2(x_1+y_1+z_1)(x_2+y_2+z_2)-\frac{T}{2}=0$.

es1

WLOG origine in O! $\quad \vec{H}=(1,1,1)=\vec{A}+\vec{B}+\vec{C}$

$\vec{M}=(\frac{1}{2},\frac{1}{2},0)$

$\vec{MH}=(\frac{1}{2},\frac{1}{2},1)=(1,1,2)$

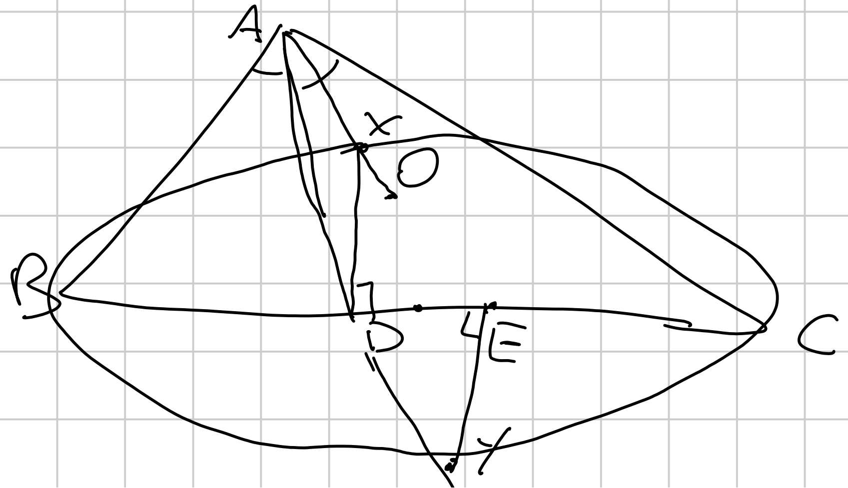


$\vec{PQ}=k\vec{PR}$

$PQ\perp ST \iff PR\perp ST$

Esercizio IMO SL 2012 G4

$AB\neq AC$



$O=(a^2 S_A, b^2 S_B, c^2 S_C)$ $\qquad S_A=\dfrac{b^2+c^2-a^2}{2}$ e cyc

$D=(0,b,c)$

$M=(0,1,1)$ $\qquad$ Con somme finale $\dfrac{E+D}{2}=M$

$E=(0,c,b)$

$AD: yc-zb=0 \qquad AO: yc^2S_C-zb^2S_B=0$

$H_A=(0,S_C,S_B) \qquad AH_A: yS_B-zS_C=0 \qquad AH_A // DX // EY$

$x+y+z=0$

$R=(-a^2,S_C,S_B)$

$$DX: \det\begin{pmatrix} x & y & z \\ 0 & b & c \\ -a^2 & S_C & S_B\end{pmatrix}=0$$

$$DX: x(bS_B-cS_C)-ya^2c+za^2b=0$$

$$EY: \det\begin{pmatrix} x & y & z \\ 0 & c & b \\ -a^2 & S_C & S_B\end{pmatrix}=0$$

$$EY: x(cS_B-bS_C)-ya^2b+za^2c=0$$

$$X=(a^2bc,\ b^2S_B,\ c^2S_C)$$

$$Y=(-2a^2(b+c),\ b((b+c)^2+a^2),\ c((b+c)^2+a^2))$$

Γ

$B\in\Gamma \Rightarrow v=0 \qquad C\in\Gamma \Rightarrow w=0$

$$\sum a^2yz=\left(\sum x\right)ux$$

$$\cancel{a^2bc}\left[S_BS_C+a^2bc\right]=\left(a^2bc+b^2S_B+c^2S_C\right)\cancel{a^2bc}\,u$$

$$u=\frac{(S_BS_C+a^2bc)bc}{a^2bc+b^2S_B+c^2S_C}=\frac{bc((b+c)^2+a^2)}{2(b+c)^2}$$

$$a^2bc\left((b+c)^2+a^2\right)^{\cancel{2}}-2a^2(b+c)\left[bc^2\cancel{((b+c)^2+a^2)}+b^2c\cancel{((b+c)^2+a^2)}\right]=$$

$$\overset{?}{=}\cancel{(b+c)}\left((b+c)^2-a^2\right)\left(-\cancel{2}a^2\cancel{(b+c)}\right)\frac{bc\cancel{((b+c)^2+a^2)}}{\cancel{2(b+c)^2}}$$

$$a^2bc\left[\cancel{(b+c)^2}+a^2-\cancel{2}(b+c)^2\right]\overset{?}{=}-\left((b+c)^2-a^2\right)bca^2$$

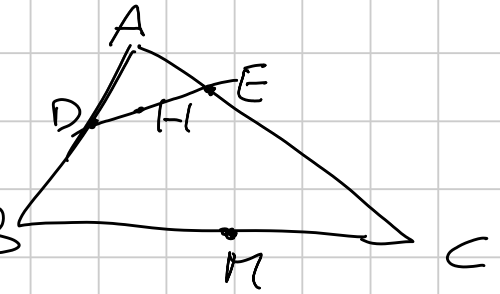
VERO!

Es IMO SL 2005 G5

$AB \neq AC$ $\quad$ $HM \perp$ asse radicale tra ⊙ABC e ⊙ADE.

$AD = AE$

D, H, E allineati



$AD = AE = \ell$

$D = (c-\ell, \ell, 0) \qquad E = (b-\ell, 0, \ell)$

$H = \left(\frac{1}{S_A}, \frac{1}{S_B}, \frac{1}{S_C}\right)$

$$\det\begin{pmatrix} c-\ell & \ell & 0 \\ b-\ell & 0 & \ell \\ 1/S_A & 1/S_B & 1/S_C \end{pmatrix} = 0$$

$\ell \neq 0 \qquad \ell\left(\sum \frac{1}{S_A}\right) = \frac{b}{S_C} + \frac{c}{S_B}$

$$\ell = \frac{S_A(cS_C + bS_B)}{\sum S_A S_B} = \frac{(b^2+c^2-a^2)(c(a^2+b^2-c^2)+b(a^2+c^2-b^2))}{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$$

$$\ell = \frac{(b^2+c^2-a^2)(b+c)}{(a+b+c)(b+c-a)}$$

Γ: ⊙ADE

$u = 0 \qquad \sum a^2 yz = \left(\sum x\right)(vy + wz)$

$c^2\ell(c-\ell) = c(v)\ell \qquad v = c(c-\ell)$

$w = b(b-\ell)$

$\tau: c(c-\ell)y + b(b-\ell)z = 0 \qquad A \in \tau$

$\vec{P} = (0, b(b-\ell), -c(c-\ell))$

$\vec{AP} = (c(c-\ell) - b(b-\ell), b(b-\ell), -c(c-\ell))$

Come trovato sopra (scambio A con C)

$\vec{MH} = (2, 1, 1)$

$$a^2(b(b-\ell) - c(c-\ell)) - b^2(c(c-\ell) + b(b-\ell)) + c^2(c(c-\ell) + b(b-\ell)) \overset{?}{=} 0$$

$$b(b-l)\left[a^2+c^2-b^2\right] \stackrel{?}{=} c(c-l)\left[a^2+b^2-c^2\right]$$

$$b-l = \frac{c(a^2+b^2-c^2)}{(a+b+c)(b+c-a)} \qquad c-l = \frac{b(a^2+c^2-b^2)}{(a+b+c)(b+c-a)}$$

Ok fine!

Poli e Polari

data un circonferenza ed un punto si cerca la tangente in quel punto si trova con la formula dello sdoppiamento

Se faccio, invece, sdoppiamento su un qualsiasi punto del piano ottengo la polare di quel punto.

$$\Gamma : \sum_{cyc} a^2 yz = \left(\sum x\right)\left(\sum u x\right) \qquad P(x_0, y_0, z_0)$$

$$\sum u x^2 + \sum (u+v) xy$$

$Pol_\Gamma(P)$:

$$\sum_{cyc} a^2 \frac{y_0 z + y z_0}{2} = \sum u x_0 x + \sum (u+v) \frac{x y_0 + x_0 y}{2}$$

Γ è la circoscritta ad ABC!

$$\sum_{cyc} a^2 (y_0 z + y z_0) = 0$$

$$\sum_{cyc} x\left(c^2 y_0 + b^2 z_0\right) = 0$$

$$\tau : ux + vy + wz = 0$$

$$\begin{array}{l} a^2 \\ b^2 \\ c^2 \end{array} \begin{cases} c^2 y_0 + b^2 z_0 = u & (1) \\ c^2 x_0 + a^2 z_0 = v & (2) \\ a^2 y_0 + b^2 x_0 = w & (3) \end{cases}$$

$$(2)+(3)-(1) \qquad 2b^2c^2 x_0 = \frac{c^2 w + b^2 v + u a^2}{1}$$

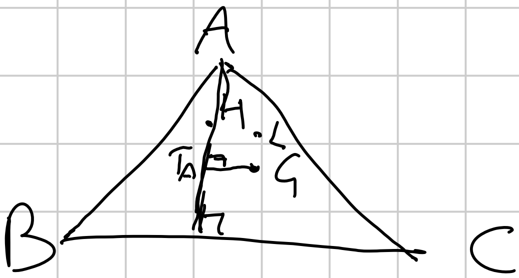
$$x_0 = \frac{c^2 w + b^2 v - a^2 u}{2b^2c^2}$$

$$\mathrm{Pol}_\Gamma(z) \doteq \left(\frac{c^2u + b^2v - a^2v}{2b^2c^2}, \frac{c^2u + a^2v - b^2v}{2a^2c^2}, \frac{a^2v + b^2v - c^2u}{2a^2b^2} \right)$$
$$= \left(a^2(c^2u + b^2v - a^2v), b^2(c^2u + a^2v - b^2v), c^2(a^2v + b^2v - c^2u) \right)$$

Baricentriche + "Bunching and Schur"
Trovare la circonferenza per 3 punti / di diametro 2 in modo furbo.

<u>Esercizio</u> G baricentro, H ortocentro, L lemoine.
Dimostrare che L è interno alla circonferenza di diametro GH.

<u>Soluzione</u>



Diametro $GH \Rightarrow$ Trovare altri punti sulla circonferenza.

$G\hat{P}H = 90°$ $\quad T_A$ proiezione di G su AH

$GT_A \perp AH \Leftrightarrow GT_A \parallel BC$

T_B, T_C analogamente

Vogliamo $\odot T_AT_BT_C$

A_∞ il pt all'infinito della retta BC $\quad A_\infty = (0,-1,1)$

$AH: yS_B - zS_C = 0$ $\quad G = (1,1,1)$

$$GT_A: \det\begin{pmatrix} x & y & z \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = 0$$

$GT_A: 2x = y + z$ $\quad \alpha = S_A \quad \beta = S_B \quad \gamma = S_C$

$T_A \doteq (a^2, 2S_C, 2S_B) \doteq (\beta+\gamma, 2\gamma, 2\beta)$

$\sum a^2 yz = (\sum x)(\sum ux)$ $\quad T_B = (2\gamma, \alpha+\gamma, 2\alpha)$

$\sum(\beta+\gamma)yz = (\sum x)(\sum ux)$ $\quad T_C = (2\beta, 2\alpha, \alpha+\beta)$

$$4\cancel{(\beta+\gamma)}\beta\gamma + 2(\alpha+\gamma)\cancel{(\beta+\gamma)}\beta + 2(\alpha+\beta)\cancel{(\beta+\gamma)}\gamma$$
$$=$$
$$3\cancel{(\beta+\gamma)}\left(u(\beta+\gamma)+2v\gamma+2w\beta\right)$$

$\beta+\gamma = a^2 > 0$

$$2(\alpha\beta+\alpha\gamma+4\beta\gamma) = 3\left(u(\beta+\gamma)+2v\gamma+2w\beta\right) \quad (1)$$

e cyc

Ma non voglio ~~risolvere~~ il sistema....

OSS La circonferenza per T_A, T_B, T_C è unica! Allora u, v, w sono unici

OSS u, v, w DEVONO essere simmetrici.

OSS u, v, w hanno grado 1 (in α, β, γ)

$u = n\alpha + m\beta + m\gamma \quad v = m\alpha + n\beta + m\gamma \quad w = m\alpha + m\beta + n\gamma$

coeff $\alpha\beta$ in (1)

$$2 = 3n + 6m$$

coeff $\beta\gamma$ in (1)

$$8 = 6m + 12n$$

$$9n = 6 \quad n = \frac{2}{3} \quad m = 0$$

$u = \frac{2}{3}\alpha \quad v = \frac{2}{3}\beta \quad w = \frac{2}{3}\gamma$ sono simmetrici $\Rightarrow$ visto che la soluzione c'è allora la risolvono 3!

Ritorniamo a S_A, S_B, S_C!

$$\omega: -\sum a^2 yz + \frac{2}{3}\left(\sum x\right)\left(\sum S_A x\right) = 0$$

Vogliamo $L = (a^2, b^2, c^2)$ interno ad ω

Potenza non positiva!

A meno di moltiplicare per $2\sum(a^2+b^2+c^2)^2$ posso usarle anche NON normalizzate

$$-3a^2b^2c^2 + \frac{1}{3}\left(\sum a^2\right)\left(\sum a^2(b^2+c^2-a^2)\right) \overset{?}{\leq} 0$$

$$9a^2b^2c^2 \overset{?}{\geq} \left(\sum a^2\right)\left(2\sum a^2b^2 - \sum a^4\right)$$

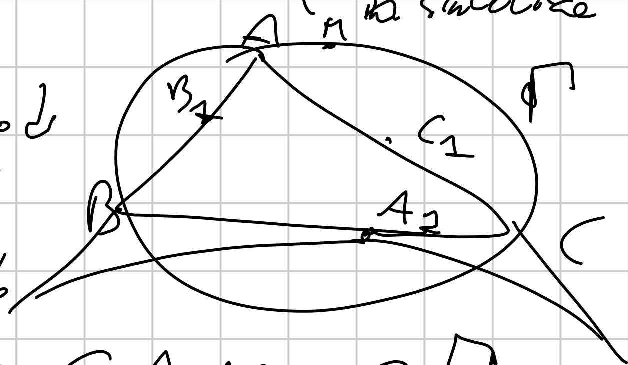
$$9a^2b^2c^2 \overset{?}{\geq} 2\sum_s a^4b^2 + 6a^2b^2c^2 - \sum_s a^4b^2 - \sum_c a^6$$

$\sum a^6 + 3a^2b^2c^2 \geq \sum_s a^4b^2$

SCHUR su (a^2, b^2, c^2) quindi vera!!!

Es IMO 2013.3 (con un (abbastanza grande) aiuto da sintetica)

Se circocentro di $A_1B_1C_1 \in \Gamma$ allora ABC è rettangolo



$\triangle A_1B_1C_1 \subseteq \triangle ABC \subseteq \Gamma$

Circocentro di $A_1B_1C_1$ è esterno al triangolo $\Rightarrow$ ottusangolo

wlog $B_1\hat{A}_1C_1 > 90°$

Il centro è sull'arco $\overset{\frown}{BC}$ contenente A.

M pt medio di tale arco

$BB_1M \cong CC_1M$ (segmenti + angoli)

$MB_1 = MC_1$ ed esiste un unico punto su tale arco $\overset{\frown}{BC}$ tale che $PB_1 = PC_1$

M è il centro

Sia A_2 il punto il l'inscritta tange BC

$BA_1M \cong CA_2M \Rightarrow A_1M = A_2M$

$B_1M = C_1M = A_1M = A_2M$

A_1, B_1, C_1, A_2 è ciclico!

se $b = c$ il problema si conclude facilmente notando che $A \equiv M$!

se $b \neq c$

Baricentriche! ABC riferimento

$B_1 = (b+c-a, 0, a+b-c)$ $C_1 = (b+c-a, a+c-b, 0)$

$A_1 = (0, a+c-b, a+b-c)$ $A_2 = (0, a+b-c, a+c-b)$

(Conto di segmenti)!

$\sum a^2yz = (\sum x)(\sum ux)$

per A_1 e A_2 LHS sono uguali -

$b\neq c$ $\quad v(a+c-b)+w(a+b-c) = v(a+b-c)+w(a+c-b)$

$$\Downarrow b\neq c$$

$$v = w = \frac{(a+c-b)(a+b-c)}{4}$$

$$\cancel{b}(b+c-a)(a+b-c) = 2\cancel{b}\left[u(b+c-a) + \frac{(a+b-c)^2(a+c-b)}{4}\right]$$

per simmetria $c(b+c-a)(a+c-b) = 2\left[u(b+c-a) + \frac{(a+b-c)(a+c-b)^2}{4}\right]$

$$2u(b+c-a)(ab+b^2-ac-c^2) = \cancel{2}(a+b-c)(a+c-b)(b-c)$$

$$(b-c)(b^2+c^2-a^2) = 0$$

$b\neq c \quad a^2=b^2+c^2 \Rightarrow$ retto!

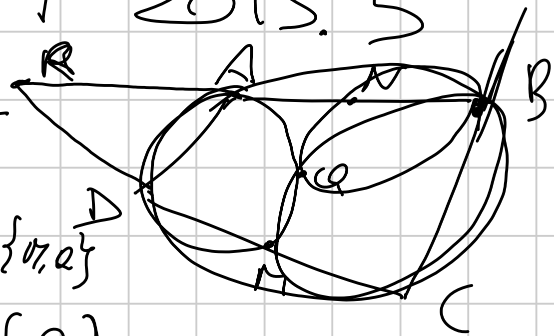
Es BST 2015.5

$\frac{CM}{CD} = \frac{BN}{AB}$

ABCD ciclico

$\odot AMD \cap \odot BMC = \{M, Q\}$

$\odot BNQ$ tange BC!



$AB \cap CD = \{R\}$

(se sono parallele è all'infinito)

RBC $\quad BC=a \quad BR=c \quad CR=b \quad AB=\alpha \quad CD=\beta$

$BN=\varphi \quad CM=\gamma$

$A=(\alpha, c-\alpha, 0) \quad D=(\beta, 0, b-\beta) \quad N=(\varphi, c-\varphi, 0)$

$M=(\gamma, 0, b-\gamma)$

$$\boxed{\alpha\gamma=\beta\varphi}$$

$B\in\omega \Rightarrow v=0$

$C\in\omega \Rightarrow w=0$

$A\in\omega \Rightarrow c\alpha(c-\alpha) = \alpha u \qquad D\in\omega \Rightarrow u = b(b-\beta)$

$u = c(c-\alpha)$

$$\boxed{c(c-\alpha)=b(b-\beta)}$$

Le circonferenze $\odot ABCD$, $\odot ADM$, $\odot BCM$ hanno come assi radicali

AD, BC, MQ $\Rightarrow$ concorrono

$AD: x(c-\alpha)(b-\beta) - y\alpha(b-\beta) - z\beta(c-\alpha) = 0$

$BC: x=0$

$AD \cap BC = \{T\}$ $\quad T = (0, \beta(c-\alpha), -\alpha(b-\beta))$

$$MQ: \det\begin{pmatrix} x & y & z \\ \gamma & 0 & b-\gamma \\ 0 & \beta(c-\alpha) & -\alpha(b-\beta) \end{pmatrix} = 0$$

$$MQ: x\beta(c-\alpha)(b-\gamma) - y\alpha\gamma(b-\beta) - z\beta\gamma(c-\alpha) = 0$$

Se un punto diverso da M si trova sia su MQ che su $\odot BCM$ allora si trova $\odot ADM$.

Q' tale che $\odot BNQ'$ tange BC

$Q' \in \odot BCM$

$$\boxed{\text{Tesi} \iff Q' \in QM}$$

Sia Γ_T la circonferenza per B, N, che tange BC.

$B \in \Gamma_T \Rightarrow v = 0$

$$\sum a^2 yz = \left(\sum x\right)(ux + wz)$$

$$a^2 yz = (y+z)wz$$

ha una sola sol $\iff w = a^2$

perché $w\left(\frac{z}{y}\right)^2 + w\frac{y}{z} - a^2\frac{y}{z} = 0$

$$\Delta = 0$$

$$(w-a^2)^2 = 0$$

$$w = a^2$$

$N \in \Gamma_T \quad c^2\varphi(c-\varphi) = \cancel{c}(\cancel{\varphi}u)$

$$u = c(c-\varphi)$$

$\odot BNQ': \sum a^2 yz = \left(\sum x\right)(za^2 + xc(c-\varphi))$

$\odot BCM: \sum a^2 yz = \left(\sum x\right)b(b-\gamma)x$

asse radicale: $za^2 = x(b(b-\gamma) - c(c-\varphi))$

$$a^2 y\left(x(b(b-\gamma) - c(c-\varphi))\right) + b^2 x\left(b(b-\gamma) - c(c-\varphi)\right) + a^2c^2xy$$

$$= b(b-\gamma)x\left[a^2x + a^2y + x(b(b-\gamma) - c(c-\varphi))\right]$$

$$(b(b-\gamma) - c(c-\varphi))(a^2y + b\cancel{\gamma}x) + a^2c^2y = a^2(x+y)b(b-\gamma)$$

$$a^2 y\left[\cancel{b(b-\gamma)} - c(c-\varphi) + c_1^2 - \cancel{b(b-\gamma)}\right] = x[b]\left[a^2(b-\gamma) - b\gamma(b-\gamma) + c\gamma(c-\varphi)\right]$$

$$a^2 c\varphi y = xb\left((b-\gamma)(a^2-b\gamma) + c\gamma(c-\varphi)\right)$$

$$Q' = \left(a^2 c\varphi,\ b\left(c\gamma(c-\varphi) - b\gamma(b-\gamma) + a^2(b-\gamma)\right),\ c\varphi\left(b(b-\gamma) - c(c-\varphi)\right)\right)$$

$$\cancel{a^2 c\varphi \beta(c-\alpha)}(b-\gamma) - \cancel{b\alpha}\gamma\cancel{(b-\beta)}\left(c\gamma(c-\varphi) - b\gamma(b-\gamma) + a^2(b-\gamma)\right) +$$
$$- \cancel{c\varphi\beta}\gamma\cancel{(c-\alpha)}\left(b(b-\gamma) - c(c-\varphi)\right) \overset{?}{=} 0$$

$$a^2(b-\gamma) - c\gamma(c-\varphi) + b\gamma(b-\gamma) - a^2(b-\gamma) - b\gamma(b-\gamma) + c\gamma(c-\varphi) \overset{?}{=} 0$$

fine!!!