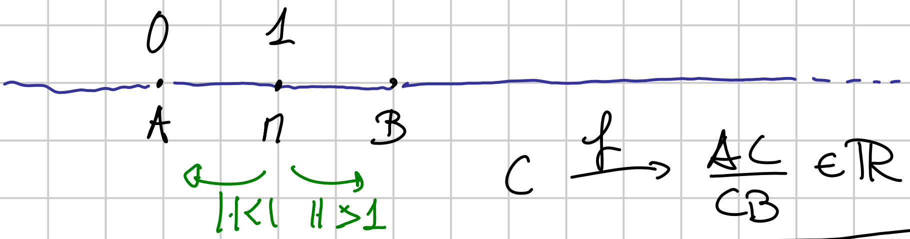
G2M - Sam - Metodi Proiettivi Sintetici

Note Title 9/4/2016

0. Rapporti con segno

$\frac{AB}{BC}$ B interno al segmento AC $\iff > 0$

B esterno al segmento AC $\iff < 0$



$C \xrightarrow{f} \frac{AC}{CB} \in \mathbb{R}$

$\frac{AC}{CB} = \lambda$

$AC = \lambda CB$ $\qquad CB = \frac{a}{1+\lambda}$ $\qquad \boxed{\lambda > 0}$

$AC + CB = a$

⊛ $f : r \setminus \{B\} \to \mathbb{R} \setminus \{-1\}$ è bijettiva.

$\lambda < 0$ distinguo i due pezzi ok

1. Birapporti

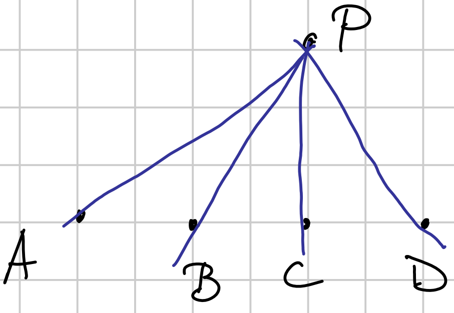
$$(A,B;C,D) = \frac{AC}{CB} \Big/ \frac{AD}{DB}$$

Oss: $(A,B;C,D) = (A,B;C,E) \implies D = E$

Dim: $\frac{A\cancel{C}}{\cancel{C}B} \Big/ \frac{AD}{DB} = \frac{A\cancel{C}}{\cancel{C}B} \Big/ \frac{AE}{EB} \implies \frac{AD}{DB} = \frac{AE}{EB}$

$\overset{⊛}{\implies} D = E.$

Prop:



$(A,B;C,D)$ dipende solo dagli angoli formati in P (P qualsiasi punto fuori dalla retta per A,B,C,D).

Dim:

$$AC = \frac{CP}{\sin C\hat{A}P} \cdot \sin A\hat{P}C \,, \qquad CB = \frac{CP}{\sin C\hat{B}P} \cdot \sin C\hat{P}B$$

Teo dei seni nel tri ACP

Attenzione: uso angoli orientati!

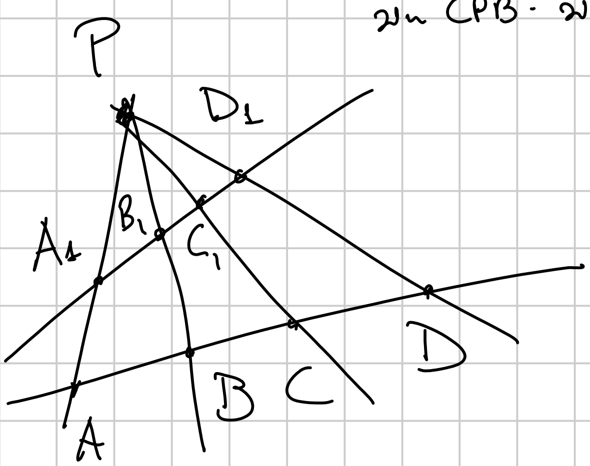
$$AD = \frac{DP \cdot \sin A\hat{P}D}{\sin D\hat{A}P} \qquad DB = \frac{DP}{\sin D\hat{B}P} \cdot \sin D\hat{P}B$$

$$\frac{AC}{CB} = \frac{\cancel{CP} \cdot \dfrac{\sin A\hat{P}C}{\sin C\hat{A}P}}{\cancel{CP} \cdot \dfrac{\sin C\hat{P}B}{\sin C\hat{B}P}} \qquad \frac{AD}{DB} = \frac{\cancel{DP} \cdot \dfrac{\sin A\hat{P}D}{\sin D\hat{A}P}}{\cancel{DP} \cdot \dfrac{\sin D\hat{P}B}{\sin D\hat{B}P}}$$

$$\frac{AC}{CB} \Big/ \frac{AD}{DB} = \frac{\sin A\hat{P}C}{\sin C\hat{P}B} \cdot \frac{\cancel{\sin C\hat{B}P}}{\cancel{\sin C\hat{A}P}} \cdot \frac{\sin D\hat{P}B}{\sin A\hat{P}D} \cdot \frac{\cancel{\sin D\hat{A}P}}{\cancel{\sin D\hat{B}P}} =$$

$$= \frac{\sin A\hat{P}C \cdot \sin D\hat{P}B}{\sin C\hat{P}B \cdot \sin A\hat{P}D} \,. \quad \blacksquare$$

Cor:



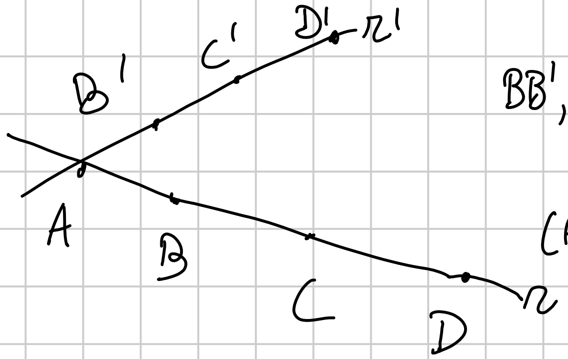
$$(A,B;C,D) = (A_1,B_1;C_1,D_1)$$

Def: r_1, r_2, r_3, r_4 rette concorrenti in P

$$(r_1, r_2; r_3, r_4) = (r_1 \cap l, r_2 \cap l; r_3 \cap l, r_4 \cap l)$$

l non per P e non parallela a nessuna di loro.

Prop:



BB', CC', DD' concorrono

sse

$$(A,B;C,D) = (A,B';C',D')$$

dim: ($\Leftarrow$) sia $P = BB' \cap CC'$,

interseco con r' $\qquad$ $r' \cap PD$

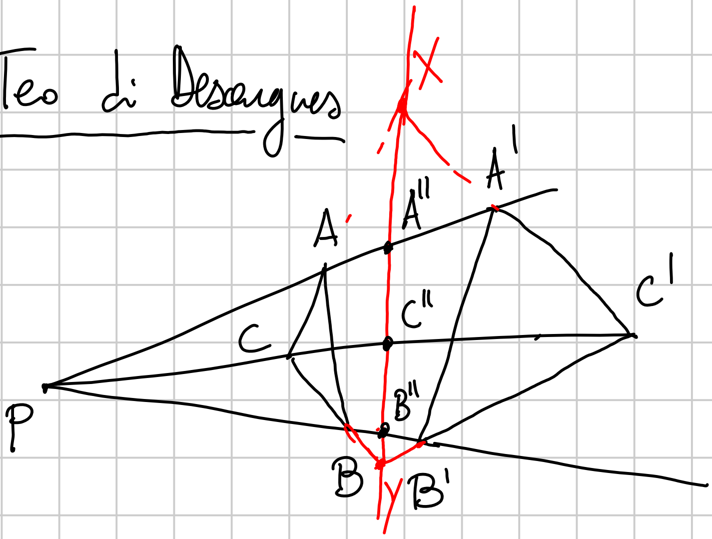
allora $\quad (A,B;C,D) = (PA,PB;PC,PD) = (A,B';C',X)$

$\quad \Vert \quad$ ipotesi $\qquad$ x def.

$(A,B';C',D')$

$$(A,B';C',D') = (A,B';C',X) \Rightarrow D' = X \quad \blacksquare$$

Teo di Desargues



AA', BB', CC' concorrono

$$\Downarrow$$

$AB \cap A'B'$
$BC \cap B'C'$ $\quad$ sono allineati
$AC \cap A'C'$

dim: $AC \cap A'C' = X \qquad CB \cap C'B' = Y$

Voglio dim che XY, AB, $A'B'$ concorrono

$$(P,C;C',C'') = (P,A;A',A'') \qquad XY$$

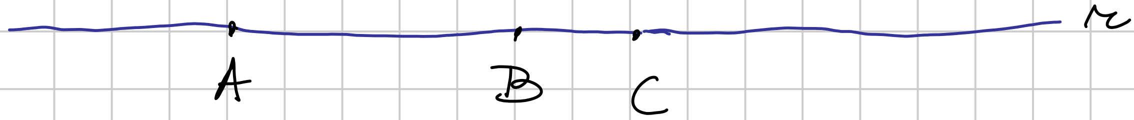
proiett. da X $\qquad \Vert$

proiett da Y $\to \Vert \qquad \Rightarrow AB, A'B', A''B''$ concorrono. $\blacksquare$

$$(P,B;B',B'')$$

Es: $AB \cap A'B'$, $AC \cap A'C'$, $BC \cap B'C'$ allineati $\Rightarrow AA'$, BB', CC' concorrenti.

Oss:



$r \setminus \{A\} \ni D \longrightarrow (A,B;C,D) \in \mathbb{R}$

$C = D \rightsquigarrow 1$ positivo se C, D sono entrambi esterni o entrambi interni a AB.

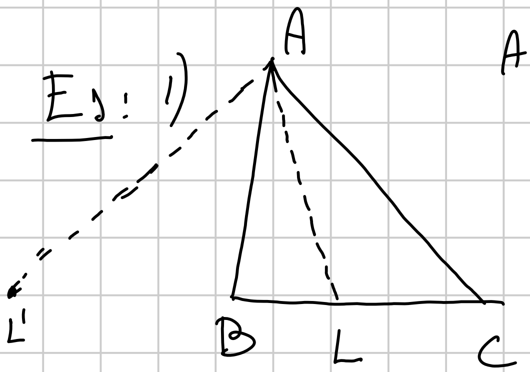
$B = D \rightsquigarrow 0$

$\boxed{\frac{AC}{CB}} \Big/ \frac{AD}{DB}$

$r \setminus \{A\} \to \mathbb{R} \setminus \left\{ -\frac{AC}{CB} \right\}$

biiettivo

Es: 1)

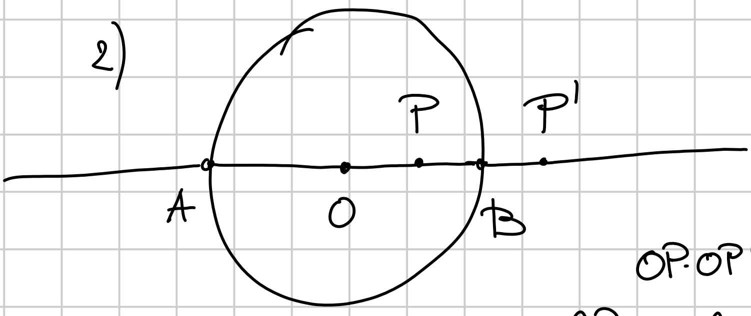


AL bisett. int. AL' bisett. esterna

$(B,C;L,L') = -1$

$\frac{BL}{LC} = \frac{AB}{AC} \qquad \frac{BL'}{L'C} = -\frac{AB}{AC}$

2)



P' inverso circolare di P AB diametro.

$(A,B;P,P')$

$OP \cdot OP' = R^2$

$\frac{AP}{PB} = \frac{AO+OP}{PO+OB} \qquad \frac{AP'}{P'B} = \frac{AO+OP'}{P'O+OB}$

segmenti orientati

$OP \cdot OP' = AO \cdot OB \qquad \frac{AP'}{P'B} = \frac{AO + \frac{AO \cdot OB}{OP}}{\frac{AO \cdot OB}{PO} + OB} =$

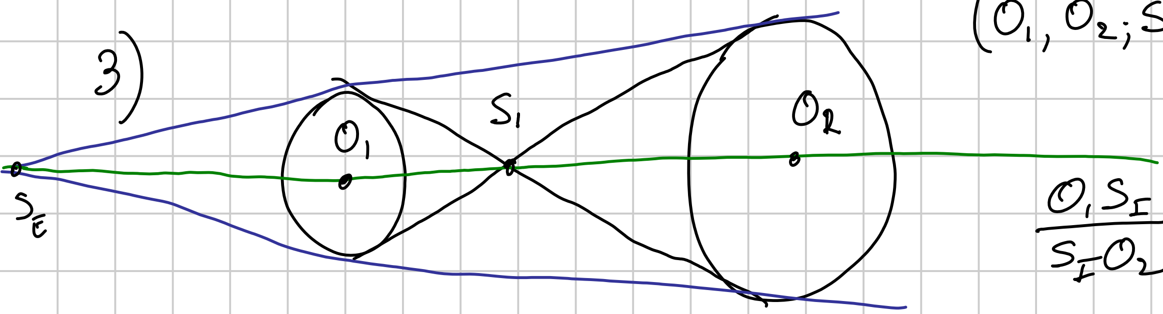
$= - \cancel{\frac{AO}{OB}} \left(\frac{OP+OB}{AO+PO} \right)$

$$\frac{AP}{PB} \Big/ \frac{AP'}{P'B} = \frac{AO+OP}{PO+OB} \Big/ -\left(\frac{OP+OB}{AO+PO}\right) = \qquad AO = OB$$

$$= - \frac{AO+\cancel{OP}}{PO\cancel{+}OB} \Big/ \frac{OP+\cancel{AO}}{O\cancel{B}+PO} = -1$$

3)

$(O_1, O_2; S_I, S_E) = ?$

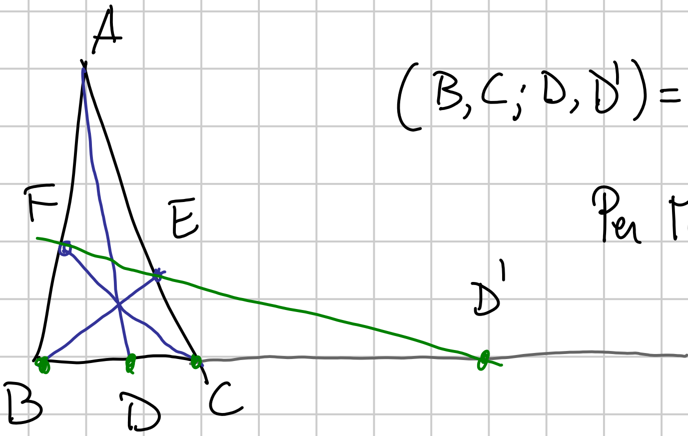


$$\frac{O_1S_I}{S_IO_2} = \frac{R_1}{R_2}$$

$$\frac{O_1S_E}{S_EO_2} = -\frac{R_1}{R_2}$$

$$(O_1, O_2; S_I, S_E) = -1$$

4)



$(B, C; D, D') = ?$

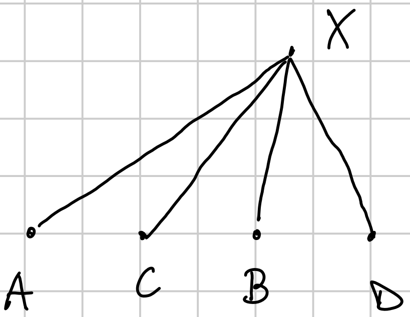
Per Menelao, F, E, D' allineati

$$\Rightarrow \frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD'}{D'B} = -1$$

Per Ceva $\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD}{DB} = 1$

$$\Rightarrow (B, C; D, D') = -1.$$

Prop:



Due delle seguenti implicano la terza

(i) $(A, B; C, D) = -1$

(ii) XC biseca $A\hat{X}B$

(iii) $XC \perp XD$

dim: (ii) + (iii) $\Longrightarrow$ $(A, B; C, D) = -1$ (i)

$\uparrow$ bisett int/est.

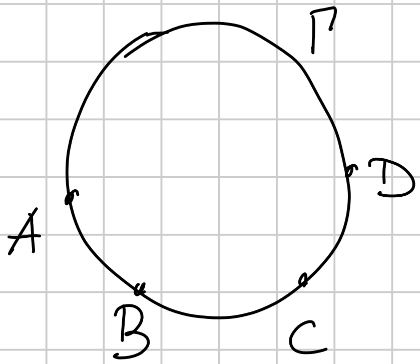
$(i)+(ii) \Rightarrow (iii)$ ovvia

$(i)+(iii) \Rightarrow (ii)$ per esercizio

<u>Oss:</u> $(A,B;C,D) = \lambda \qquad (A,C;B,D) = 1-\lambda$

$$\left(\lambda,\ \frac{1}{\lambda},\ 1-\lambda,\ \frac{\lambda}{1-\lambda},\ \frac{1-\lambda}{\lambda},\ \frac{1}{1-\lambda}\right)$$

Birapporto su una circonferenza

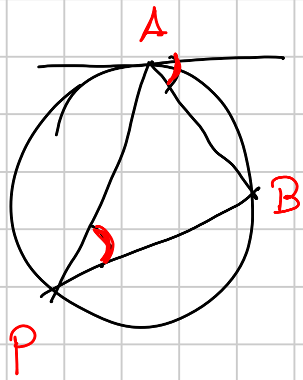


$(A,B;C,D)_\Gamma = (PA,PB;PC,PD)_P$

$P \in \Gamma$ qualsiasi

<u>notazione</u>: AA = tg a Γ in A

ben definito perché dipende solo dai seni degli angoli in P

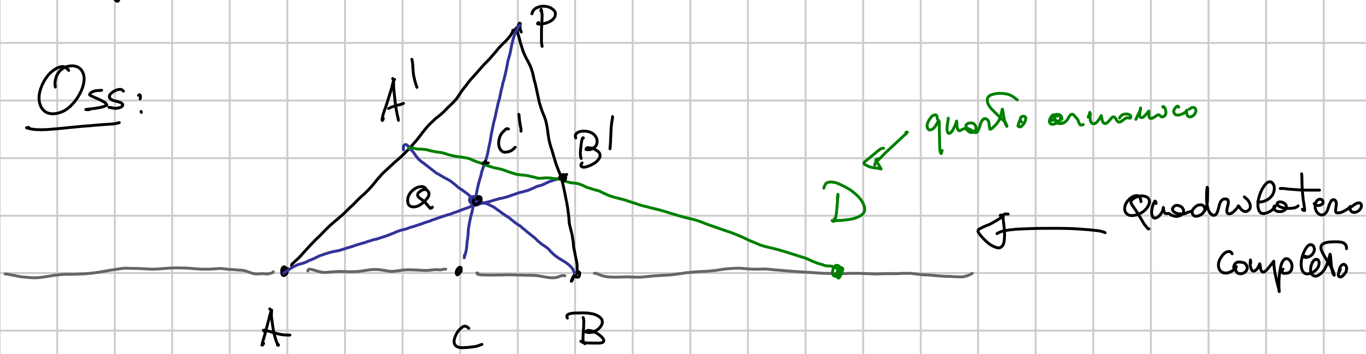


PA, PB

AA, AB

<u>Def</u>: $(A,B;C,D) = -1$ $\quad A,B,C,D$ sono una <u>quaterna armonica</u>

<u>Oss:</u>



$$(A,B;C,D) \underset{\text{proiett. da } P}{=} (A',B';C',D) \underset{\text{proiett. da } Q}{=} (B,A;C,D) = \frac{1}{(A,B;C,D)}$$

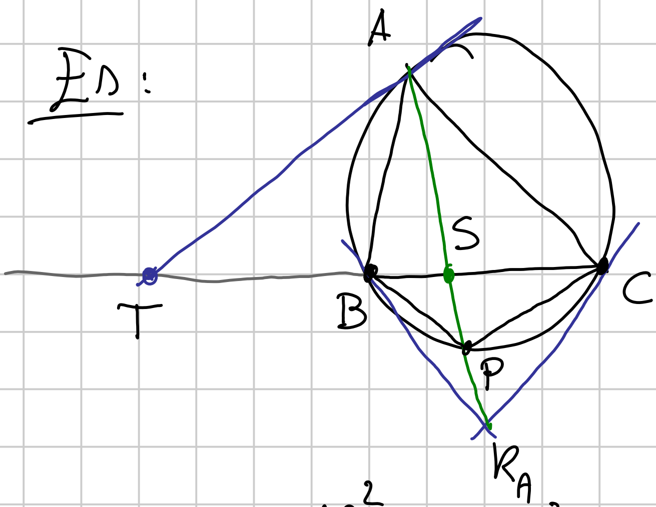
$(A,B;C,D) =$ → $+1$ solo se $C=D$ impossibile
→ -1

Prop: A, B, C, D quaterna armonica, $O =$ pt. medio di AB

1) $\dfrac{2}{AB} = \dfrac{1}{AC} + \dfrac{1}{AD}$ 2) $CA \cdot CB = CO \cdot CD$

3) $OC \cdot OD = OA^2 = OB^2$ 4) $\dfrac{OC}{OD} = \left(\dfrac{AC}{AD}\right)^2 = \left(\dfrac{BC}{BD}\right)^2$

Es:



$(B,C;S,T) = -1$

$$\frac{BS}{SC} = \frac{AB^2}{AC^2} = \frac{BP^2}{PC^2}$$

$$\left|(B,C;P,A)_\Gamma\right| = \left|\frac{BP}{PC} \Big/ \frac{BA}{AC}\right| = 1$$

$(B,C;P,A)_\Gamma = -1$

$\parallel$ ← proietto da A

$(AB, AC; AP, AA)$

$\parallel$ ← interseco con BC

$(B,C;S,T) = -1$ ■

Divagazione: AS è la simmediana

dim: Inversione di centro A e raggio $\sqrt{AB \cdot AC}$ + simm. nella bisettrice di $B\hat{A}C$.

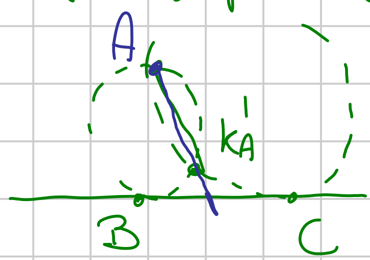
$B \to C$

$C \to B$

$\Gamma \to BC$

$BK_A \to$ circ. per B, A tg a BC

$CK_A \to$ circ per C, A tg a BC

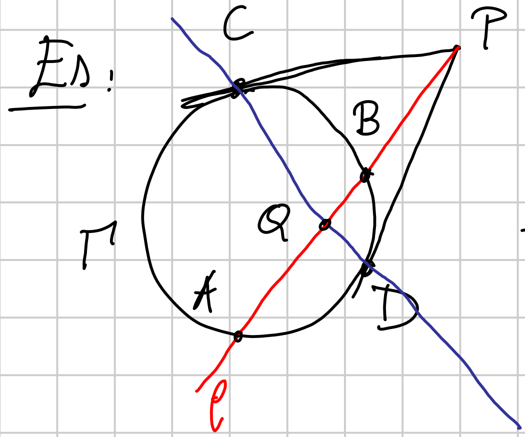


$AS \to AS'$ simm di AS risp alla bisettrice

$AS' = AK_A'$. □

Def: $ACBD$ ciclico con $(A,B,C,D)_\Gamma = -1$ si dice quadrilatero armonico.

Es:



$(A,B;C,D)_\Gamma = -1$

dim: $\triangle PDB \sim \triangle PAD$ $\qquad$ $\triangle PBC \sim \triangle PCA$

$$\frac{BD}{AD} = \frac{PB}{PD} \qquad \frac{BC}{AC} = \frac{PB}{PC}$$

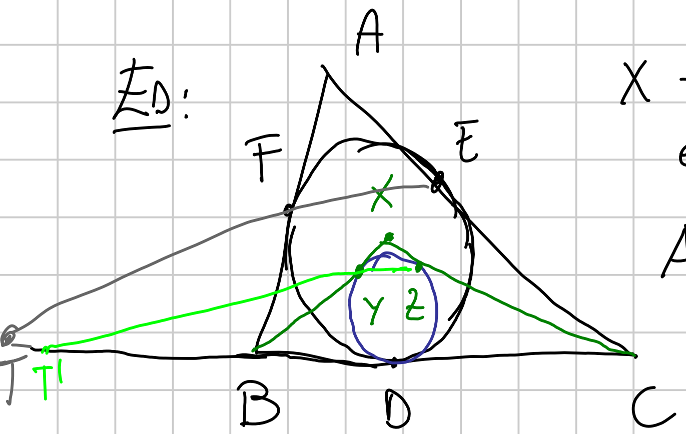
$$\frac{BD}{AD} = \frac{BC}{AC} \Longrightarrow (A,B;C,D)_\Gamma = -1.$$

↑ perché i pt sono nell'ordine giusto

Cor: Proietto da C $\Rightarrow (CA, CB; CC, CD) = -1$

interseco con $\ell \Rightarrow (A, B; P, Q) = -1$.

Es:



$TE \cdot TF = TD^2$

$T'Z \cdot T'Y = T'D^2$

X t.c. la cfr inscritta in XBC tange BC in D e XB, XC in Y, Z.

Allora EZYF è ciclico.

dim: 1) AD, BE, CF concorrono nel pt di Gergonne di ABC

$\Rightarrow (B,C;D,T) = -1$

2) XD, BZ, CY concorrono nel pt di Gergonne di BCX

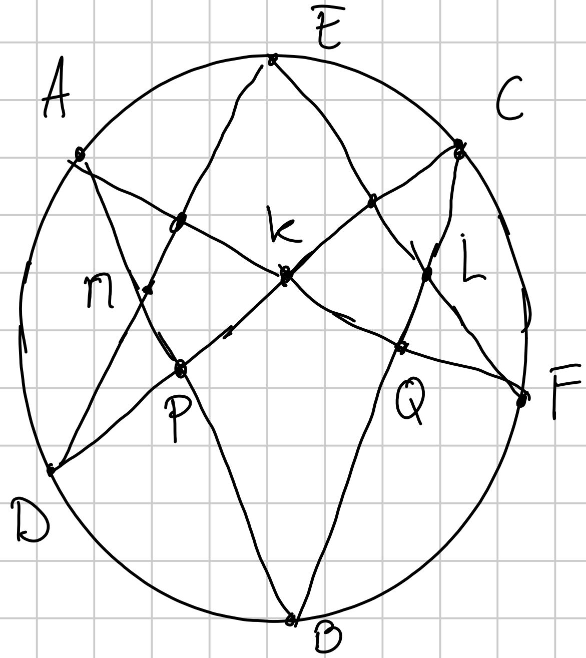
$\Rightarrow (B,C;D,T') = -1$

3) $\Rightarrow T = T' \Rightarrow$ ok.

Teo di Pascal

Γ cfr, ABCDEF esagono ciclico

$\Rightarrow \underset{\parallel \atop M}{AB \cap DE}, \ \underset{\parallel \atop L}{BC \cap EF}, \ \underset{\parallel \atop K}{CD \cap FA}$ sono allineati



Dim: Voglio Π, K, L allineati.

$$(C, L; Q, B) \underset{F}{=} (C, E; A, B)_{\Pi} =$$

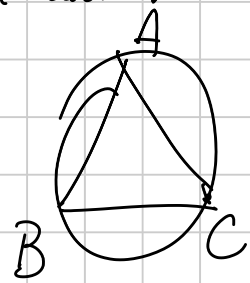
$$\underset{D}{=} (P, \Pi; A, B) \underset{K}{=} (C, \Pi K \cap CB; Q, B)$$

$\Rightarrow \Pi K \cap BC = L$

Oss 1: Posso permutare i 6 punti ottenendo 720 diverse terne ordinate di pt. allineati. $\Rightarrow$ 120 rette diverse.

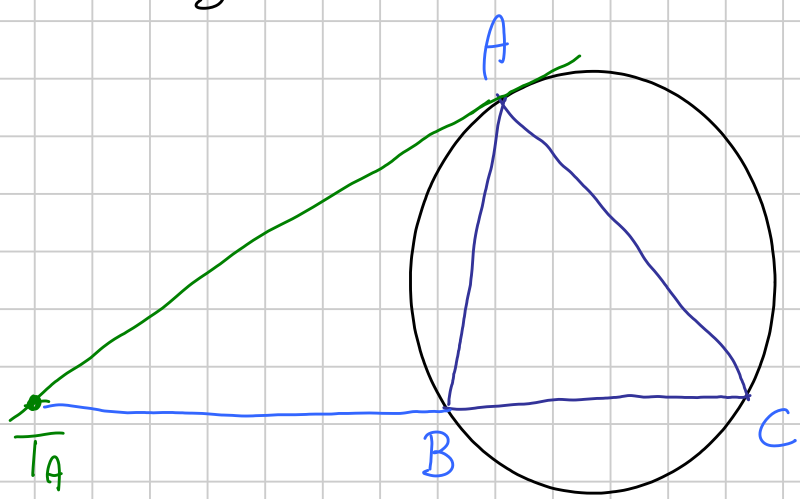
Oss 2: Si può usare pascal anche con punti coincidenti. In quel caso si considerano le tangenti.

Es:



AABBCC

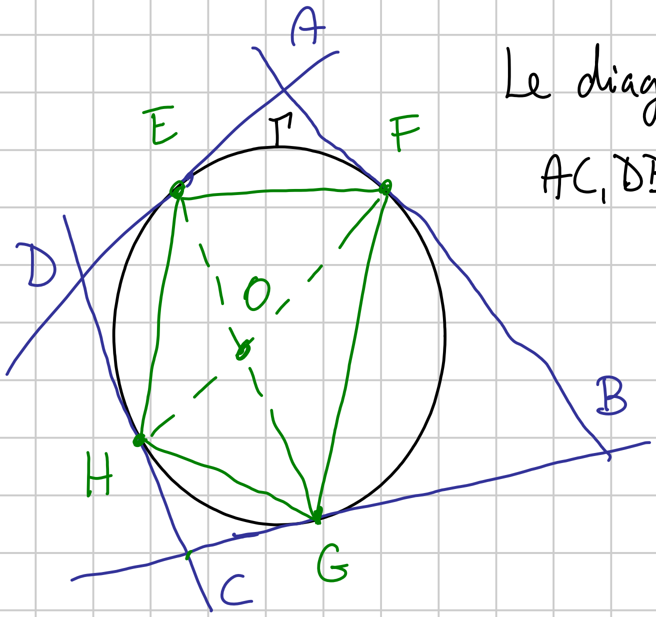
$AA \cap BC$
$AB \cap CC$ sono allineati
$BB \cap CA$



T_A, T_B, T_C allineati.

(asse di Lemoine)

Teo di Newton



Le diagonali concorrono AC, DB, EG, FH

dim: $O = EG \cap FH \quad X = EH \cap GF$

Pascal su $EGGFHH \rightsquigarrow$ $EG \cap FH = O$, $GG \cap HH = C$, $GF \cap HE = X$ allineati

Pascal su $EEHFFG \rightsquigarrow$ $EE \cap FF = A$, $EH \cap FG = X$, $HF \cap EG = O$ allineati.

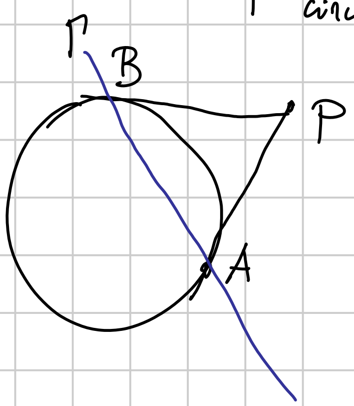
A, O, C, X allineati $\Rightarrow O \in AC$

$Y = EF \cap HG \rightsquigarrow D, O, B, Y$ allineati $\Rightarrow O \in BD$. ☒

Teo (Brianchon) BRIANCHON: $ABCDEF$ esagono circoscritto a Γ. Allora AD, BE, CF concorrono.

2) Poli e Polari

Γ circonferenza, P pt. esterno



$\text{pol}_\Gamma(P)$ = retta AB polare di P rispetto a Γ

PA, PB tg a Γ

equivalente: ⊙ $pol_\Gamma(P)$ = retta $\perp PO$ che passa per l'inverso di P in Γ.

⊙ funziona anche per P interno o per $P \in \Gamma$.

Oss: $P \in \Gamma \Rightarrow pol_\Gamma(P) = tg$ a Γ in P

Def: Γ cfr. r retta $pol_\Gamma(r) = P$ polo di r rispetto a Γ

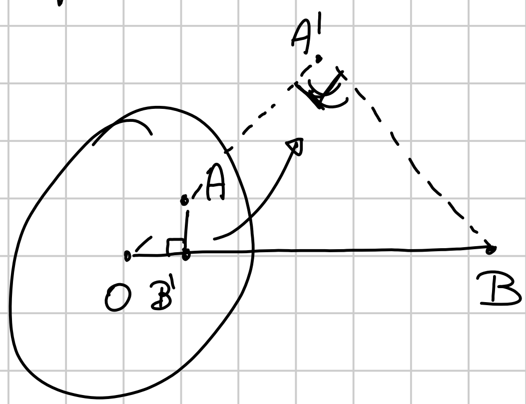
t.c. $OP \perp r$ e $OP \cap \{r\} = P'$ con P' inverso di P in Γ

Prop: 1) $A \in pol_\Gamma(B) \Longleftrightarrow B \in pol_\Gamma(A)$

2) $pol_\Gamma(r \cap s)$ = retta per $pol_\Gamma(r)$, $pol_\Gamma(s)$

3) $pol_\Gamma(A) \cap pol_\Gamma(B) = pol_\Gamma(AB)$

dim: 1)



$A\hat{B'}B = \frac{\pi}{2} \Longleftrightarrow A \in pol_\Gamma(B)$

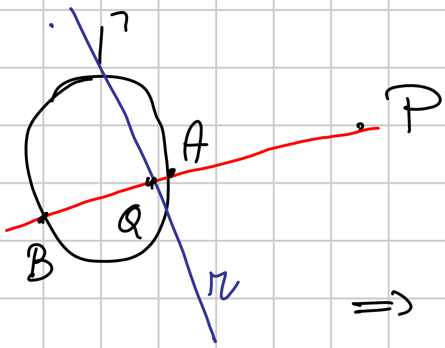
$\Updownarrow$

$O\hat{B'}A = \frac{\pi}{2} \Longleftrightarrow O\hat{A'}B = \frac{\pi}{2} \Longleftrightarrow B \in pol_\Gamma(A)$

2,3) offaw vostro

Oss: $pol_\Gamma(P) = \{pol_\Gamma(r) \mid P \in r\}$

Prop:



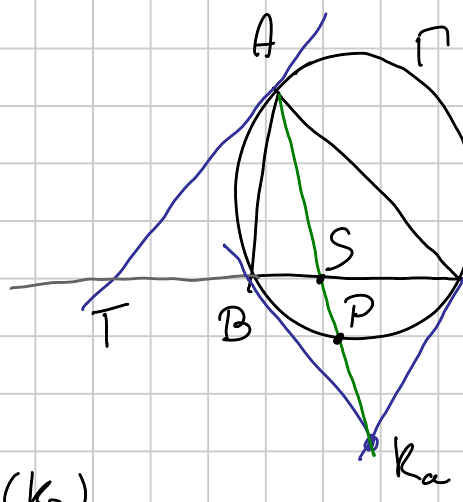
$r = pol_\Gamma(P)$

l = retta per P

$A, B = l \cap \Gamma \quad Q = l \cap r$

$\Rightarrow (A, B; P, Q) = -1.$

Es:



$(B,C;S,T)$

$BC = pol_\Gamma(K_a)$ $T \in BC$

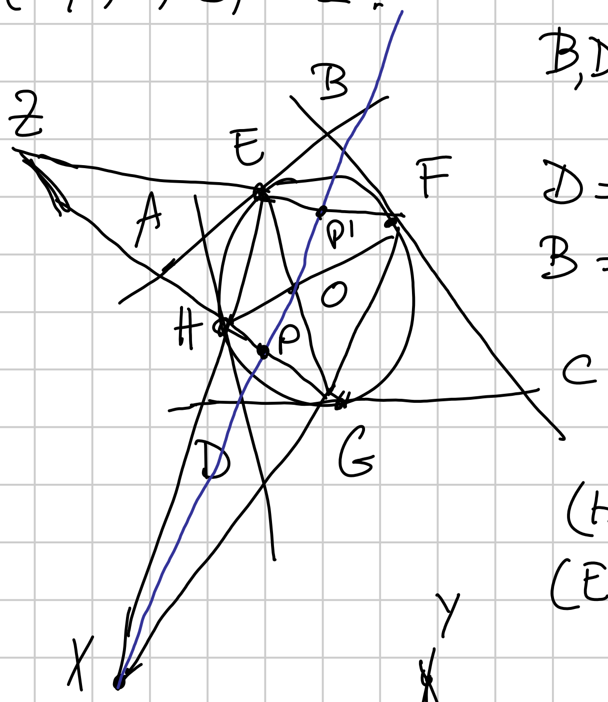
$K_a \in pol_\Gamma(T)$
$A \in pol_\Gamma(T)$ $\Rightarrow K_aA = pol_\Gamma(T)$

$\Rightarrow (B,C;S,T) = -1$

$BC = pol_\Gamma(K_a)$

$\Rightarrow (A,P;S,K_a) = -1$.

Es:



B, D, O, X sono allineati

$D = pol_\Gamma(HG)$

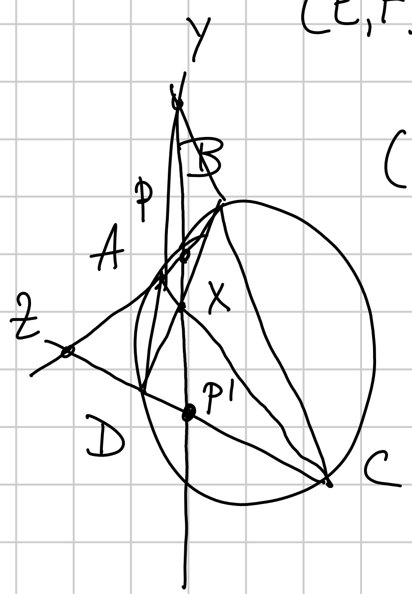
$B = pol_\Gamma(EF)$

$Z = pol_\Gamma(BD)$

$(H,G;Z,P) = -1$

$(E,F;Z,P') = -1$

Ovvero:



$(A,B;Z,P) =$

$= (C,D;Z,P') = -1$.

Oss: $\Gamma_A = \{ P : PB/PC = \frac{AB}{AC} \}$. N_A centro di Γ_A

$\Rightarrow N_A$ è pt. medio di LL' L = piede bisett. interna
L' = " " esterna

$N_A = pol_\Gamma(AK_A)$

3) L'infinito

Proiettivo = Piano + centri dei fasci impropri = r_∞

$$\cup$$
$$A_\infty, B_\infty$$

- PA_∞ = retta per P che appartiene al fascio di centro A_∞
- $r \cap s = A_\infty \Rightarrow r, s \in$ fascio di centro $A_\infty \Rightarrow r \parallel s$
- $A_\infty B_\infty = ??$ è l'insieme di tutti questi centri $= r_\infty$
- $r \cap r_\infty$ = centro del fascio improprio che contiene r

Posso definire il birapporto in tutto il proiettivo

$$(A, B; C, D_\infty) = (PA, PB; PC, PD_\infty)$$

$D_\infty \in AB$

A, B, C, D_∞ sono allineati.

Posso definire $pol_\Gamma(r_\infty) = O$, $pol_\Gamma(A_\infty) = r$

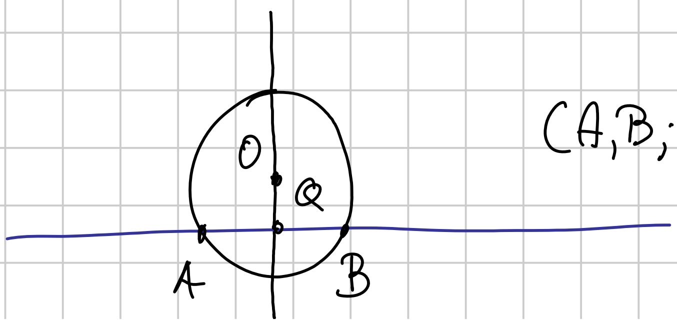
$r \perp OA_\infty$

$O \in r$

Oss: $A \quad M \quad B$ M pt. medio di AB

$(A, B; M, P_\infty) = -1$ P_∞ = pt. all'infinito di AB.

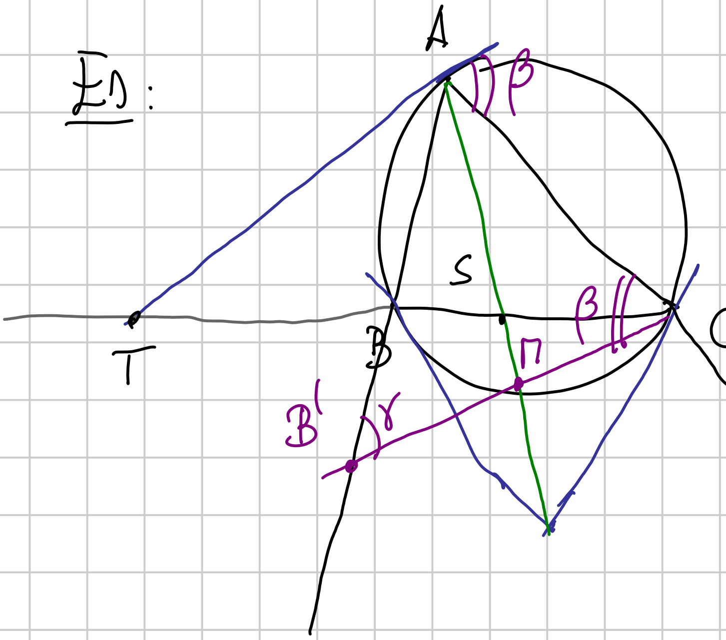
Oss:



$(A, B; Q, A_\infty) = -1$ $\Rightarrow A_\infty$

↑ lemma della polare

Es:



$\Rightarrow CB' \parallel TA$

P_∞ = pt all'inf di $B'C$

$(C, B'; M, P_\infty) = -1$

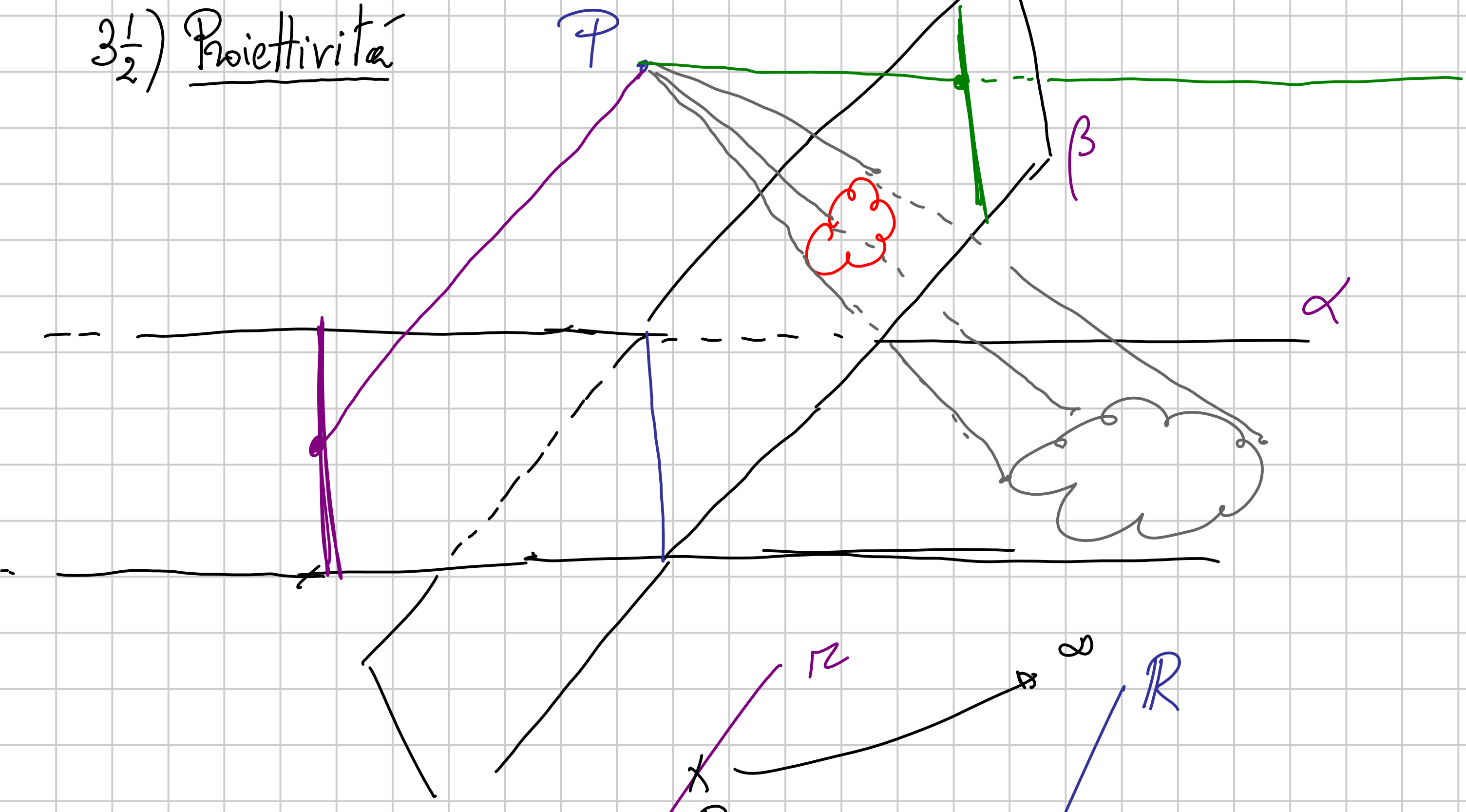
$\parallel$ proietto da A su BC

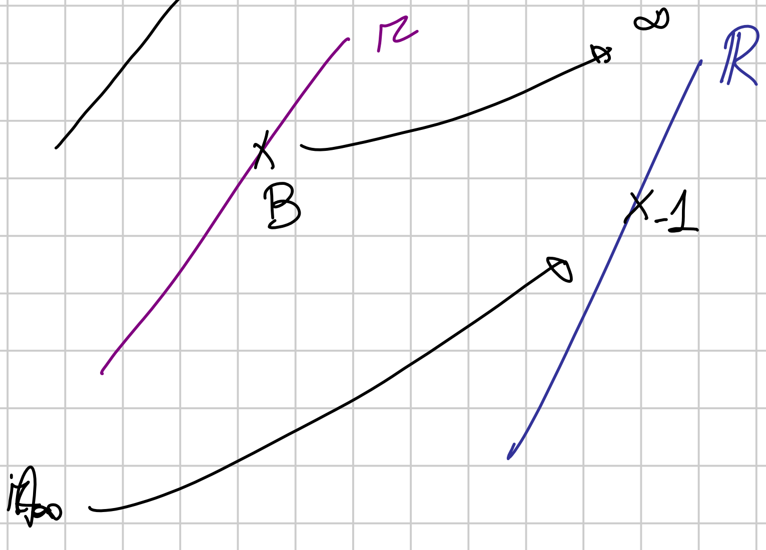
$(C, B; S, T) = -1$

$\parallel$ 1

$\overline{(B, C; S, T)} = -1$

3½) Proiettività





Aggiungendo le rette all'infinito, ho una trasf.

Proiettivo → Proiettivo bigettiva

- retta in rette
- conserva l'incidenza
- preserva i birapporti
- permette di definire polo e polare × le coniche