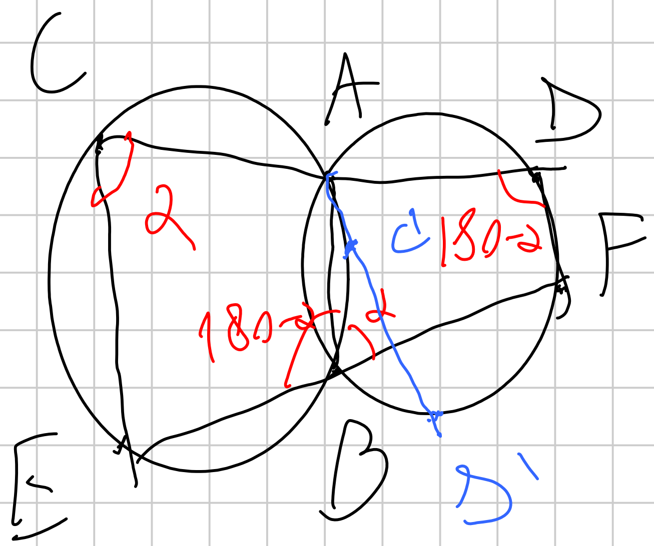
S16 M-G3

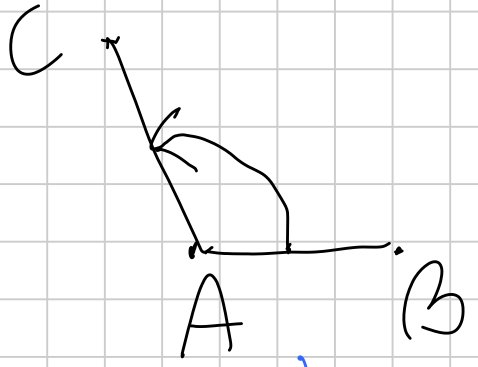
Note Title 9/6/2016

PROBLEMA 1 (TEO REIM)



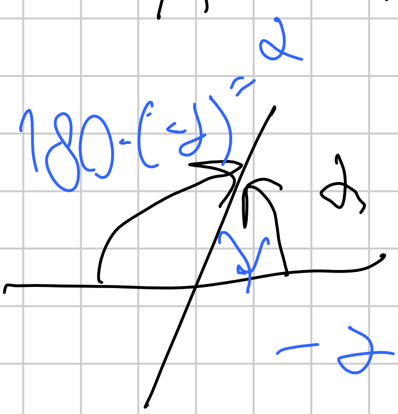
Tn $CE \parallel DF$

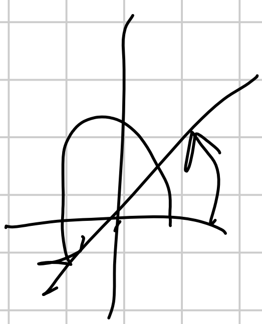
ANGOLI ORIENTATI



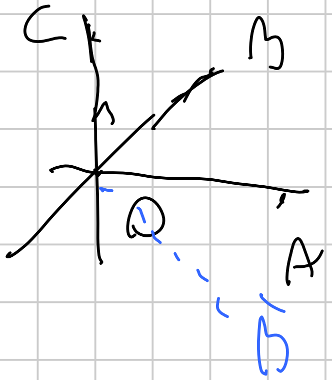
$\measuredangle BAC = (B\hat{A}C) = \measuredangle (BA, AC)$

$\measuredangle BAC = -\measuredangle CAB$

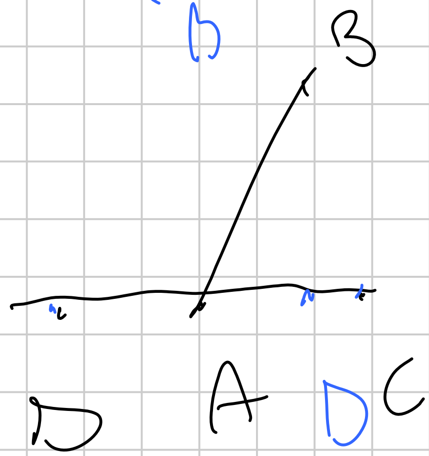




1) $\measuredangle BAC = -\measuredangle CAB$

2) 

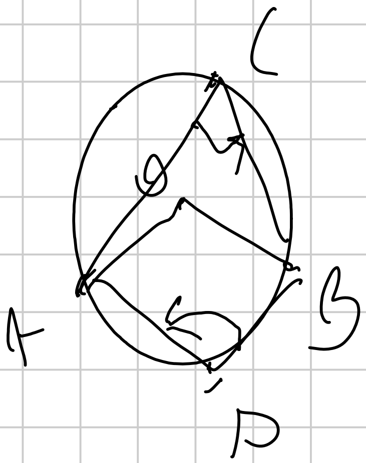
$\measuredangle AOC = \measuredangle AOB + \measuredangle BOC$

3) 

$\measuredangle_{1} A_{1} D$ alterni

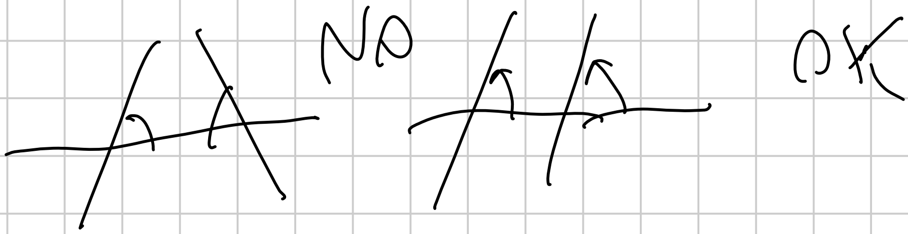
$\Updownarrow$

$\measuredangle BAC = \measuredangle BAD$

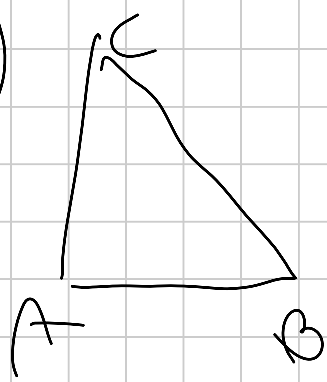
4) 

$\measuredangle AOB = 2\measuredangle ACB = 2\measuredangle ADB$

5) $l_1 // l_2 \iff \measuredangle(l_1, l_3) = \measuredangle(l_2, l_3)$



6) $l_1 \perp l_2 \iff \measuredangle(l_1, l_2) = \measuredangle(l_2, l_1)$

7) 

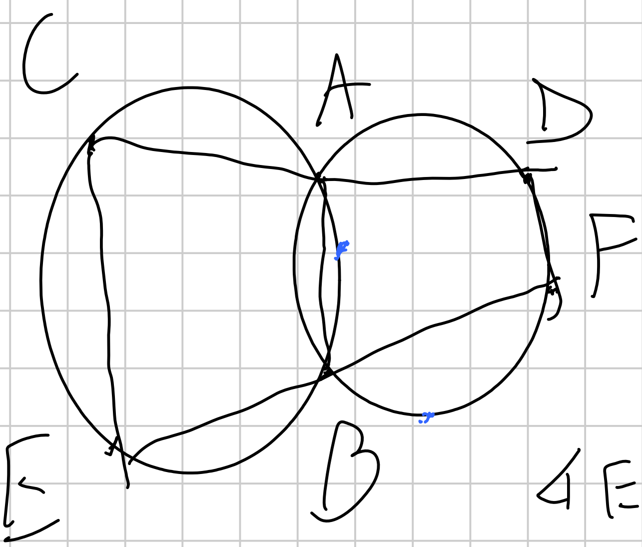
$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = \pi \ (180)$$

8) 

$$\triangle ABC \sim \triangle DEF$$
$$\Updownarrow$$
$$\measuredangle BAC = \measuredangle EDF$$
$$\measuredangle ACB = \measuredangle DFE$$
$$\measuredangle CBA = \measuredangle FED$$

- Scrivo tutto normalmente, ma sto attento a nominare gli angoli in senso antiorario
- Scrivo USANDO le proprietà di sopra

— o — o —



$$\measuredangle ECA = \measuredangle EBA$$
$$\measuredangle EBA = \measuredangle FBA$$
$$\measuredangle FBA = \measuredangle FDA$$
$$\measuredangle ECD = \measuredangle FDC$$

ESERCIZI X CASA

1) DIMOSTRARE TUTTE LE PROPRIETÀ

2) 4 Γ_i $1 \le i \le 4$ $\Gamma_i \cap \Gamma_{i+1}$ A_i, B_i

Th: $A_1A_2A_3A_4$ ciclico $\iff$ $B_1B_2B_3B_4$ è ciclico

3) P punto, A, B, C allineati $\bigcirc_{BCP}, \bigcirc_{ACP}, \bigcirc_{ABP}$
(centri in) X, Y, Z

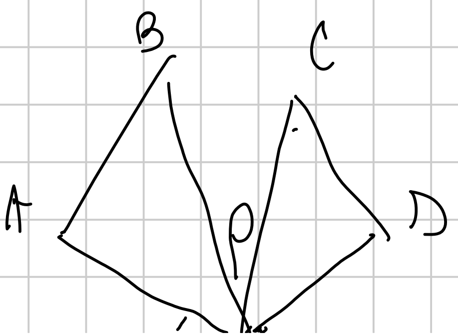
Th: $XYZP$ è ciclico

4) P punto, A, B, C $\triangle$ $\bigcirc_{BCP}$ ha centro in X e ciclicamente.

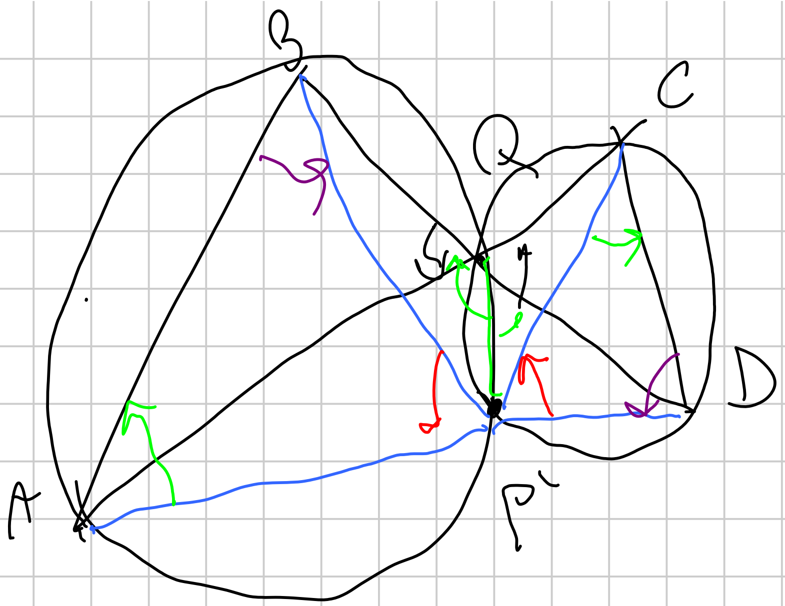
$\bigcirc_{XBC}, \bigcirc_{ZAB}, \bigcirc_{ACY}$ si intersecano in Q

— o — o —

LEMMA ROTOMOTETIA



$\exists!\, P$ t.c. $\triangle PAB \sim \triangle PCD$



$\bigcirc_{ABQ}$ $\quad$ $\bigcirc_{CQD}$

$(Q = AC \cap BD)$

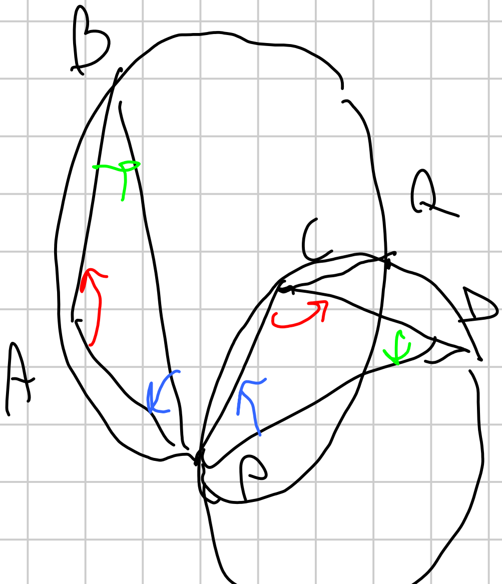
Considero $P'AB$, $P'CD$

$\measuredangle BP'A = \measuredangle BQA = \measuredangle DQC = \measuredangle DP'C$

$\measuredangle P'AB = \measuredangle P'QB = \measuredangle P'QD = \measuredangle P'CD$

$\Delta P'AB$, $\Delta P'CD$ hanno angoli uguali $\Rightarrow$ sono simili

2^a FRECCIA il punto P è unico



$\bigcirc_{ABP} \cap \bigcirc_{PCD} = Q$

$\measuredangle CDP = \measuredangle CQP$

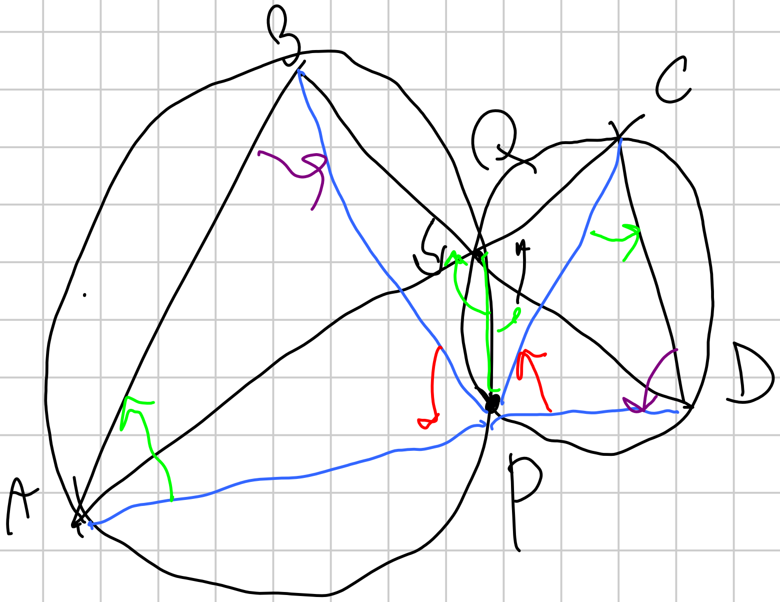
$\shortparallel$

$\measuredangle ABP = \measuredangle AQP$

$\Rightarrow A, Q, C$ allineati

Analogamente $B, Q, D \Rightarrow P = P'$

$\Rightarrow \exists P$ ed è unico. C.V.D.



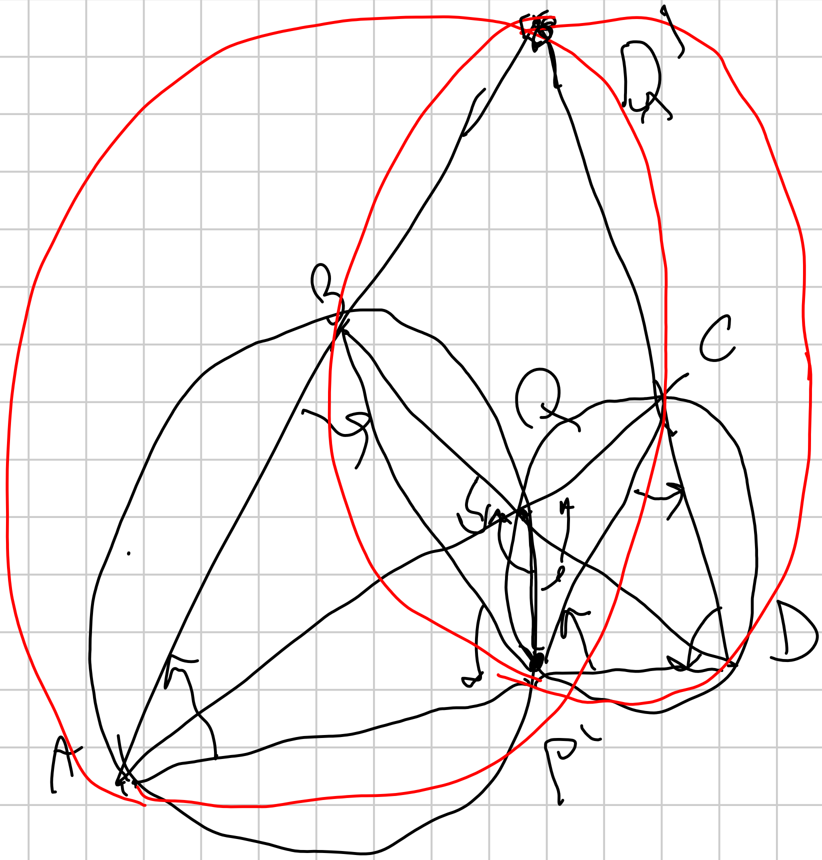
OSS. ANCHE

$\triangle PAC \sim \triangle PBD$

$$\frac{PA}{PB} = \frac{PC}{PD} \iff \frac{PA}{PC} = \frac{PB}{PD}$$

$$\measuredangle CPA = \measuredangle CPB + \measuredangle BPA = \measuredangle CPB + \measuredangle DPC$$
$$= \measuredangle DPB \qquad \text{C.V.D.}$$

OSS. 2. APPLICO il lemma inverso della rotomotetia a $\triangle PAC$, $\triangle PBD$



A, C, BD $\exists P$ t.c.

$\Downarrow$

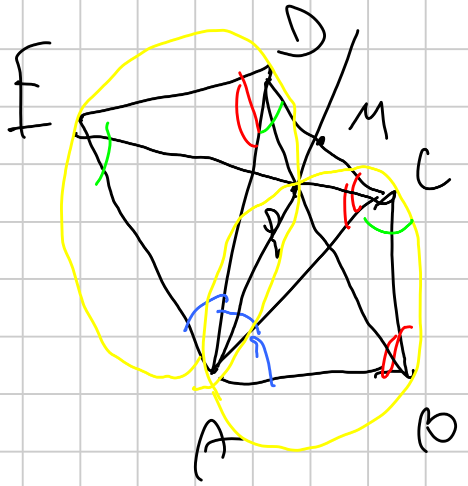
$\exists Q = AB \cap CD$

t.c. $QACP$, $QBPD$ sono ciclici

PROBLEMA 2

$\angle BAC = \angle CAD = \angle DAE$

$\angle CBA = \angle DCA = \angle EDA$

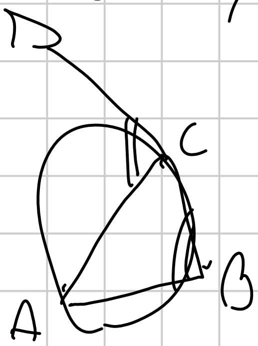


$P = BD \cap EC$

$AP \cap DC = M$

Th: $CM = DM$

Per il lemma della rotomotetia, il centro di rotomotetia che manda $BC \to DE$ è l'intersezione delle circonferenze a BCP, DPE $\Rightarrow A \in \odot_{BCP}, \odot_{DEP}$



$\Rightarrow DC$ tang $\odot_{BCA}$

Analogamente DC tang $\odot_{DEA}$

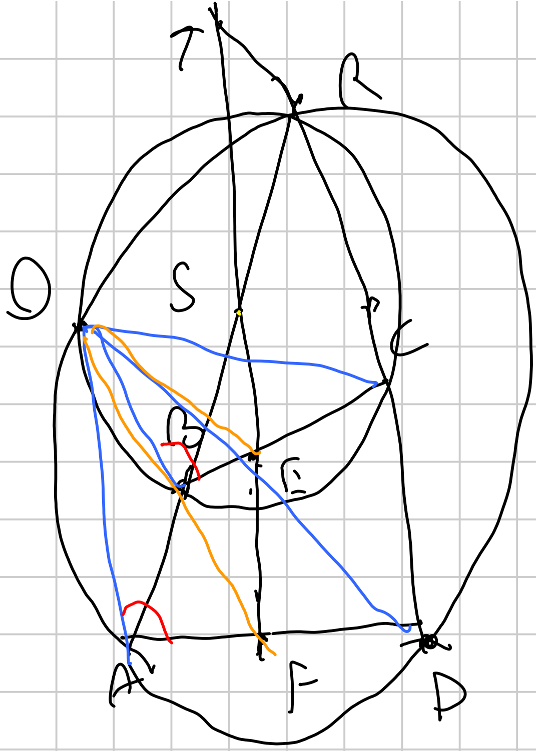
$MC^2 = MP \cdot MA = MD^2 \Rightarrow MC = MD$ C.V.D.

— o — o —

PROBLEMA 3

$R = BA \cap CD$, $T = CD \cap EF$

$S = AB \cap EF$



$E \in BC$, $F \in AD$

t.c. $\dfrac{BE}{EC} = \dfrac{AF}{FD}$ $\left(\dfrac{a}{b} = \dfrac{c}{d} \Downarrow \dfrac{a}{a+b} = \dfrac{c}{c+d} \right)$

Th $\bigcirc_{SEB}$, $\bigcirc_{SAF}$, $\bigcirc_{TEC}$, $\bigcirc_{TFD}$ concomm.

$\Delta OBC \sim \Delta OAD$

$\Downarrow$

$\dfrac{OB}{BC} = \dfrac{OA}{AD}$

Ipotesi $\dfrac{BE}{BC} = \dfrac{AF}{AD}$

$\longrightarrow \dfrac{OB}{BE} = \dfrac{OA}{AF}$ $\left(\dfrac{OB}{OA} = \dfrac{BC}{AD} = \dfrac{BE}{AF} \right)$

$\Downarrow$

$\angle OBE = \angle OAF \longrightarrow \underline{\Delta OBE \sim \Delta OAF}$

Il centro di rotomotetia $BE \to AF$ è l'intersezione di $\bigcirc_{BES}$, $\bigcirc_{AFS}$

$\parallel$

$O \longrightarrow \bigcirc_{BESO}$, $\bigcirc_{AFSO}$ cicl.

Per analogia (scambio B e C, A e D)

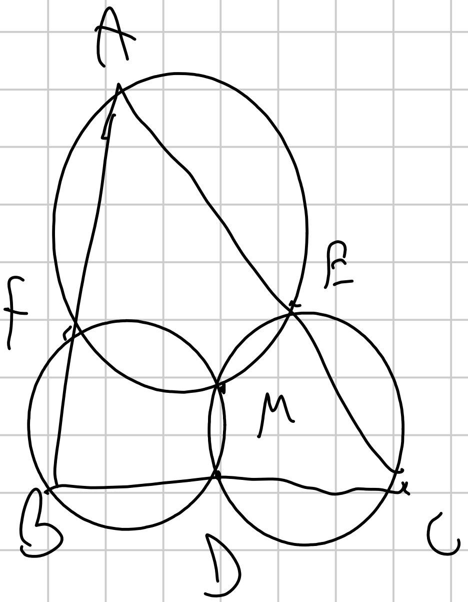
$\bigcirc_{CETO}$, $\bigcirc_{DFTO}$ cicl.

$\bigcirc_{BES}$, $\bigcirc_{AES}$, $\bigcirc_{CET}$, $\bigcirc_{DFT}$ concorrono in O

C.V.D.

— o — o —

MIQUEL



$D, E, F \in BC, AC, AB$ a caso

$\bigcirc_{CDE}$, $\bigcirc_{AFE}$, $\bigcirc_{FBD}$

Th concorrono in punto M

DIM $M' = \bigcirc_{CED} \cap \bigcirc_{BFD}$

Voglio $M' \in \bigcirc_{AFE}$

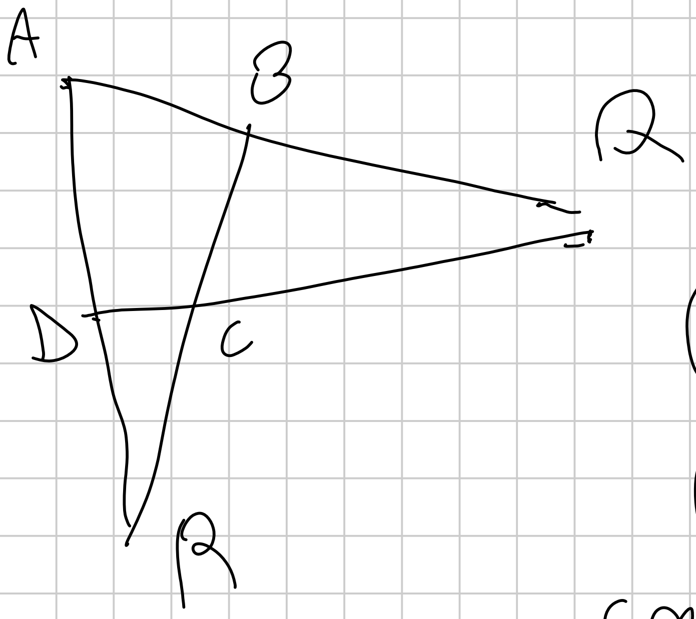
$\measuredangle EM'F = \measuredangle EM'D + \measuredangle DM'F = \measuredangle ECD + \measuredangle DBF =$
(per le ciclicità)

$= 180 - \measuredangle BAC = -\measuredangle BAC = \measuredangle CAB = \measuredangle EAF$

$\Longrightarrow$ EMAF ciclico

— o —

MIQUEL ++



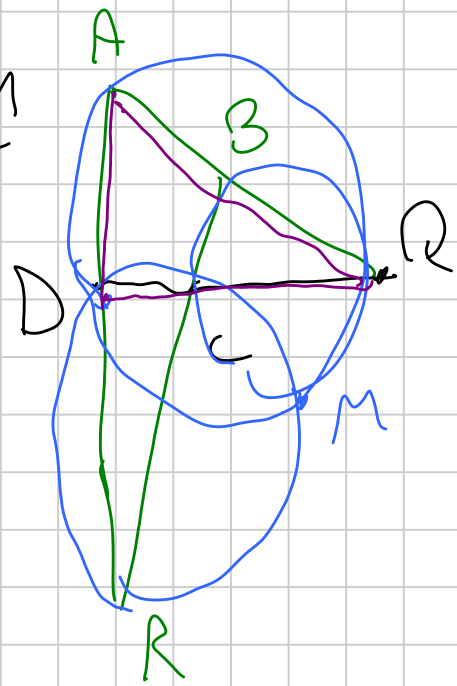
$Q = AB \cap CD$

$R = AD \cap BC$

$\bigcirc_{DCR}, \bigcirc_{ABR}, \bigcirc_{ADQ}, \bigcirc_{BCQ}$

concorrono in M

DIM



Per Miquel $\triangle$ su ABR

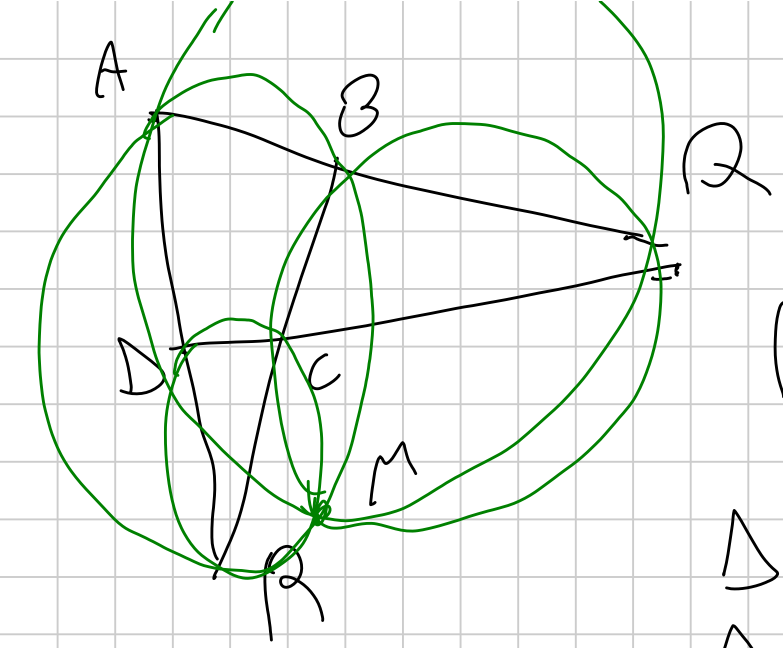
$\bigcirc_{RDC}, \bigcirc_{BCQ}, \bigcirc_{ADQ}$

concorrono in M

Miquel su $ADQ \Rightarrow \bigcirc_{BCQ}, \bigcirc_{ABR}, \bigcirc_{DCR}$

concorrono in M'

Unendo i due fatti, $M = M' \Rightarrow$ le 4 circonferenze concorrono.



RD, BQ

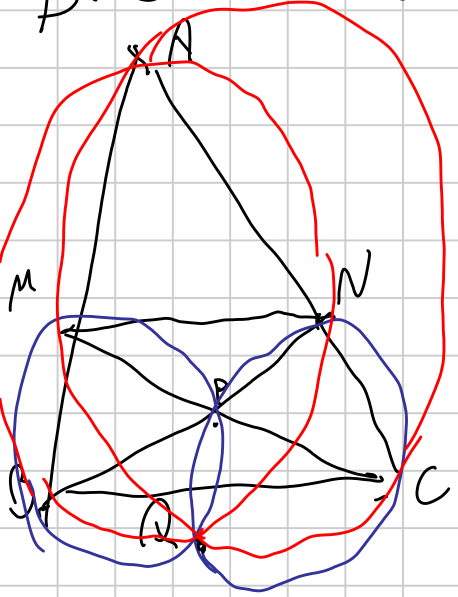
Per il lemma della rotomotetia

$\triangle MDR \sim \triangle MQB$

$\triangle MRB \sim \triangle MDQ$ e simili

— o —

BALKAN 2009/2



$MN \parallel AC \quad MC \cap BN = P$

$\odot_{BMP} \cap \odot_{PNC} = Q$

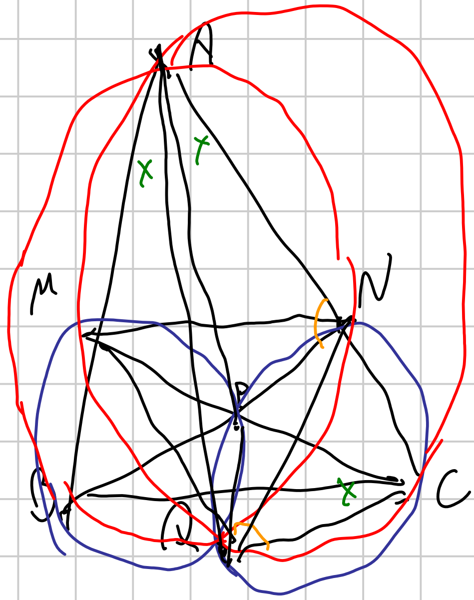
Th: $\angle BAQ = \angle PAC$

1) Inversione + simmetria: centro A, raggio $\sqrt{AN \cdot AB}$, bisettrice di BAC
(X STASERA)

2) Notiamo che è la configurazione del quadrilatero completo

$\angle ABQ = \angle MPQ = \angle CPQ = \angle CNQ = \angle ANQ$

$\Rightarrow ABQN, AMQC$ ciclici



$\measuredangle BAQ = \measuredangle BNQ = \measuredangle PCQ$

$\measuredangle PAC = \measuredangle PNC = \measuredangle PNA$

Se Teor. è vero, $\Delta PQC \sim \Delta PNA$

$\Downarrow$

provo a dimostrarlo

Mi basta $\frac{PN}{NA} = \frac{PQ}{QC} \Rightarrow \frac{PN}{PQ} = \frac{AN}{QC}$

$\parallel$

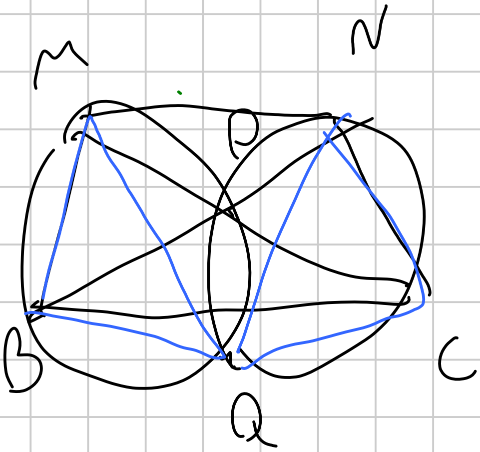
$\frac{AM}{MQ}$

Per la teor, mi basta $\frac{AM}{MQ} = \frac{AN}{QC} \Leftrightarrow \frac{MQ}{QC} \overset{?}{=} \frac{AM}{AN} \overset{MN // BC}{=} \frac{AB}{AC} = \frac{MB}{NC}$

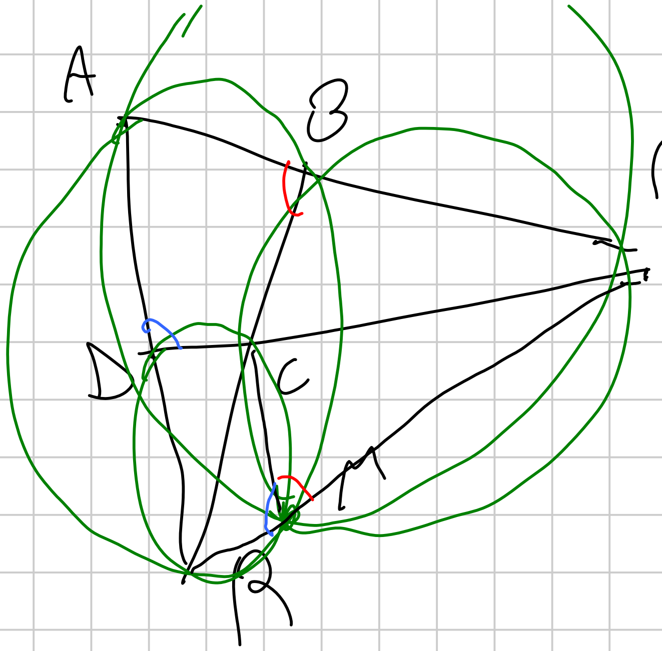
$\frac{MB}{MQ} = \frac{NC}{CQ}$ Guardo ΔMBQ, ΔNCQ, è la configurazione della mistilinea $\Rightarrow \Delta MBQ \sim \Delta NCQ$

$\Downarrow$

Teor



— ◻ —



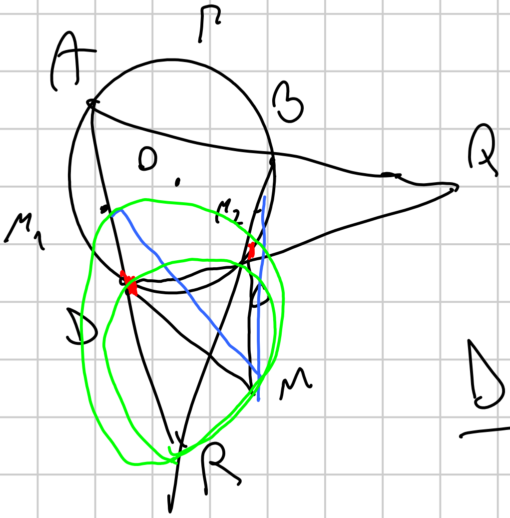
Quand'è che R, Q, M sono allineati?

$\angle CMR = \angle CDR = \angle CDA$

$\angle CMQ = \angle CBQ = \angle CBA$

$M \in RQ \iff \angle CDA = \angle CBA \iff ABCD$ è ciclico

Da adesso in poi, $ABCD$ è ciclico.



O il centro di Γ

Th: $OM \perp QR$

Dim

Sia M_1 pt medio di AD

Sia M_2 ...... BC

ΔMAD, ΔMBC sono simili (per rotomotetia)

$\angle MDA = \angle MCB$ $\quad \dfrac{AD}{DM} = \dfrac{BC}{CM}$ $\quad \left(\dfrac{AD}{M_1D} = \dfrac{BC}{M_2C} = 2\right)$

Segue $\dfrac{M_1D}{DM} = \dfrac{M_2C}{CM}$ e $\angle M_1DM = \angle M_2CM$

Sono simili e hanno M in comune $\Rightarrow \exists$ rotomotetia di centro M che manda

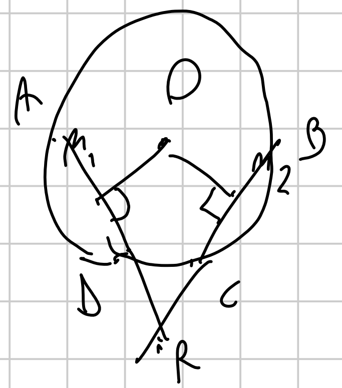
$M_1D \to M_2C \qquad M_1M_2 \to DC$

$\exists$ rotomotetia di centro M: $M_1M_2 \to DC \Rightarrow R = M_1D \cap M_2C$

sta sulle circonferenze

$\bigcirc_{MRDC}$, $\bigcirc_{MRM_1M_2}$

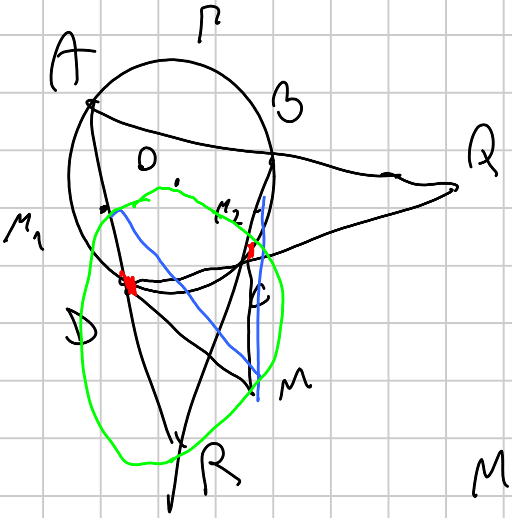
Supponiamo M_1M_2RM ciclico.



MA! M_1M_2RO è ciclico $\Rightarrow \bigcirc_{M_1M_2R}$ passa per O, M

$\Downarrow$

M_1M_2ROM ciclico



$\angle OMR = \angle OM_2R = 90°$

$\angle OMR = 90°$

$M \in RQ \Rightarrow OM \perp RQ$

— o — o —

OSS: $ACOM$, $BODM$ sono ciclici

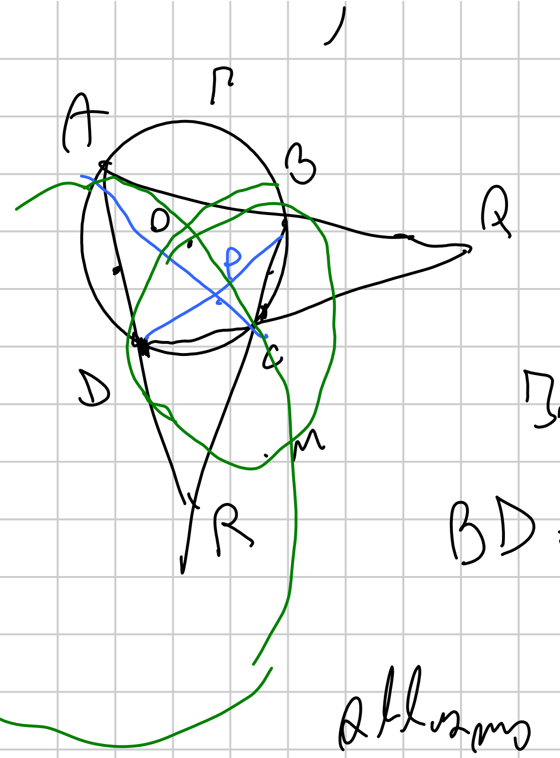
$\frac{1}{2}\angle AOC \qquad \frac{1}{2}\angle AOC$

$\angle AOC = 2\angle ABC = 2\angle ADC$

$\angle AMC = \angle AMR + \angle RMC = \angle ABR + \angle RDC = \angle ABC + \angle ADC = \angle AOC$

Siccome $ABMR$ è ciclico $\Rightarrow \angle AMR = \angle ABR$

" " $CMDR$ " " $\Rightarrow \angle RMC = \angle RDC$



P, M inversi rispetto a Γ

DIM

Inverto in Γ

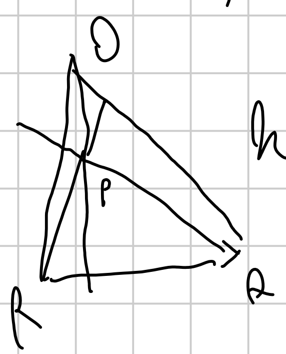
$BD \Rightarrow \bigcirc_{BOD}$ $AC \Rightarrow \bigcirc_{AOC}$

abbiamo dimostrato prima che ADCM, BODM sono ciclici

$$P' = (AC \cap BD)' = (AC)' \cap (BD)' = \bigcirc_{ACO} \cap \bigcirc_{BDO} = M$$

[CONSEGUENZA: OPM allineati $OM \perp RQ \Rightarrow OP \perp RQ$

$pol_\Gamma(P) = RQ$, $pol_\Gamma(Q) = RP$, $pol_\Gamma(R) = PQ$ e da $OR \perp PQ$, $OQ \perp RP$



nel $\triangle PRQ$, O è ortocentro]

ES x CASA

IMO 1985/5 ; CHINA 1992/4, RUSSIA 1997/GRADE 9 ES 7

TST 15/2 ABCD ciclico $AC \cap BD = P$ $AD \cap BC = E$

$\bigcirc_{APD} \cap \bigcirc_{BPC} = Q$ M pt medio di AD

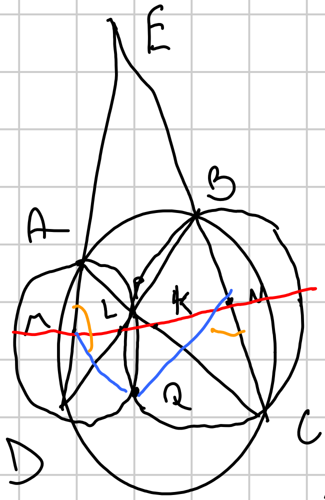
N ---- BC

$MN \cap AC = K \quad MN \cap BD = L$

Th: a) $EMNQ$ ciclico

b) $AMKQ$, $BLQN$ ciclici

c) $\bigcirc_{AMK}$, $\bigcirc_{BLC}$, AB concorrono



Dim: $APBE \Rightarrow Q$ è Miquel

$\Downarrow$

$\triangle ADQ \sim \triangle CBQ \Rightarrow \triangle AMQ \sim \triangle CNQ$

$\measuredangle EMQ = \measuredangle AMQ = \measuredangle CNQ = \measuredangle ENQ \Rightarrow EMNQ$ ciclico

a) ✓

b) $\triangle AMQ \sim \triangle CNQ \Rightarrow \measuredangle AQM = \measuredangle CQN$

HOPE: $\measuredangle AKM = \measuredangle AQM = \measuredangle CQN$

$\shortparallel$

$\measuredangle CKN$

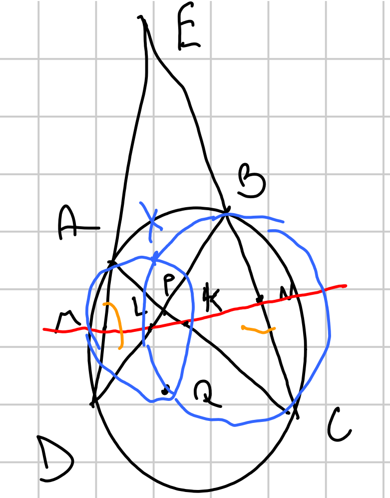
Applichiamo la rotomotetia su AM, $NC \Rightarrow$ centro di rotomotetia $=$

$\bigcirc_{AMK} \cap \bigcirc_{KNC} = Q \Rightarrow AKMQ$, $NKQC$ ciclici

$BLNQ$

c) $\bigcirc_{AKM}$, $\bigcirc_{BLC}$, AB concorrono

$\bigcirc_{AKM} \cap \bigcirc_{BLC} = X$



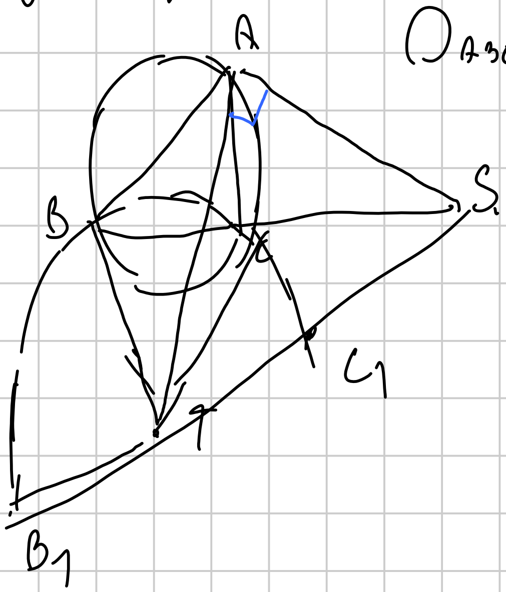
$\measuredangle AXQ = \measuredangle AKQ = \measuredangle CKQ = \measuredangle CNQ = \measuredangle BNQ$

$= \measuredangle BXQ$ perché $BLNQ$ ciclico

$\measuredangle AXQ = \measuredangle BXQ \Rightarrow AXB$ allineati

— O —

USA TST 2007



$\bigcirc_{ABC} = \Gamma$, BT, CT tangenti Γ. Sia $S \in BC$ t.c. $\angle SAT = 90°$

ω la circ. di centro T e raggio BT

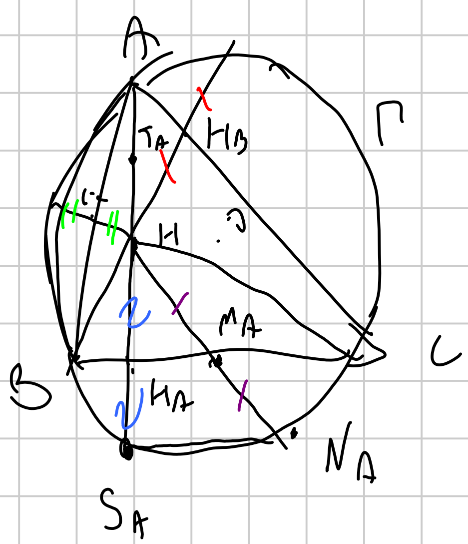
$B_1 = ST \cap \omega$ $\quad C_1 = ST \cap \omega$

$\triangle ABC \sim \triangle AB_1C_1$

(x CASA)

— O — O —

Un po' di cose note (si spera)



Omotetia di fattore $\frac{1}{2}$ con H

$S_A \to H_A$ $\quad \Gamma \to$ circonferenza per

$N_A \to M_A$

$A \to T_A$

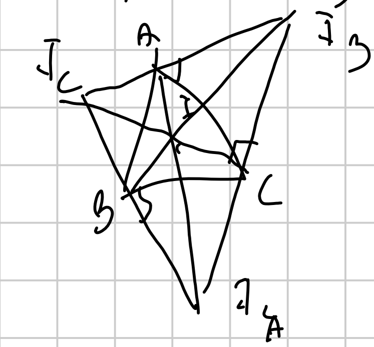
- punti medi dei lati
- piedi delle altezze
- punti medi AH e simili

ed è Feuerbach.

$\Rightarrow$ Centro $\in$ retta di Eulero $=$ pt MEDIO DI OH

Feuerbach tange incerchio (nel punto di Feuerbach) e gli excerchi.

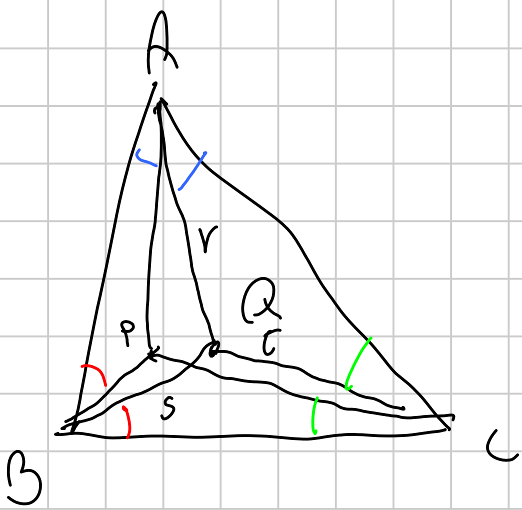
(SAPEVATELO!)



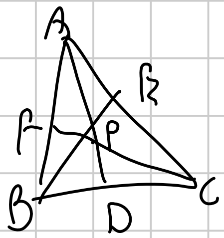
→ $I_C C \perp I_B I_A$. $I_B I_C I_A$ ha ortocentro I.

Quindi la sua Feuerbach è proprio Γ (passa per i piedi delle altezze, ovvero A, B, C)

— O — O —



Γ Ceva:



$$\frac{BD}{DC} \cdot cyc = 1$$

$$\frac{\sin BAD}{\sin DAC} \cdot cyc = 1$$

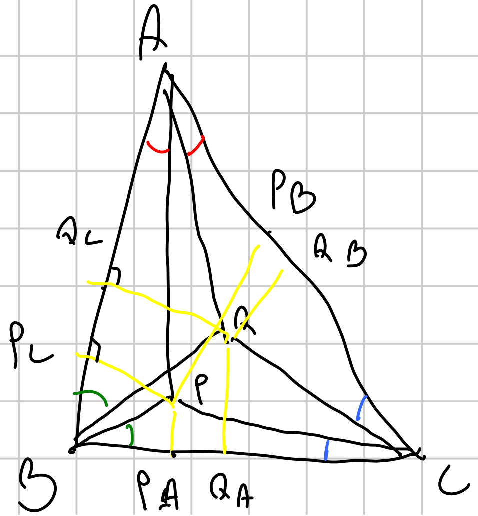
$$\frac{[BAD]}{[DAC]} \cdot cyc = 1$$

$\Downarrow$

AP, BP, CP concorr.

$$\frac{\sin BAP}{\sin PAC} = \frac{\sin(r, AC)}{\sin(BA, r)} \Rightarrow \text{per Ceva } r, s, t \text{ concorrono}$$

P e Q vengono detti CONIUGATI ISOGONALI



P_A, P_B, P_C proiezioni di P sui lati

Q_A, Q_B, Q_C ... Q ...

Th $P_A P_B P_C Q_A Q_B Q_C$ sono conciclici

$\triangle BPP_A \sim \triangle BQQ_C$ $\quad \measuredangle PBP_A = \measuredangle Q_CBQ$ e

$\measuredangle QQ_CB = \measuredangle PP_AB = 90$ $\Rightarrow$ Sono simili

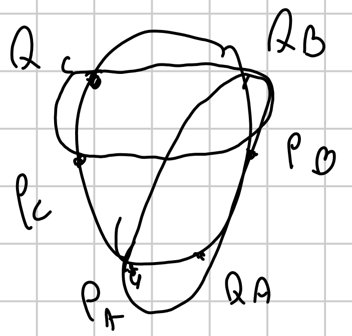
Analogamente $\triangle BQQ_A \sim \triangle BPP_C$

$$\frac{BP}{BP_A} = \frac{BQ}{BQ_C} \; ; \quad \frac{BP}{BP_C} = \frac{BQ}{BQ_A} \Rightarrow \frac{BP_A}{BQ_C} = \frac{BP}{BQ} = \frac{BP_C}{BQ_A}$$

$\Rightarrow BP_A \cdot BQ_A = BP_C \cdot BQ_C \Rightarrow P_AQ_AP_CQ_C$ è ciclico ω_B

Rifaccio il ragionamento per A, C $\Rightarrow$ $P_AQ_AP_BQ_B$ è ciclico ω_C

$P_BQ_BQ_CP_C$ è ciclico ω_A



Presuppongo la tesi falsa $\Rightarrow$ sono 3 circonferenze distinte.

Asse rad. $\omega_A \omega_B = AB$

" $\omega_A \omega_C = AC$

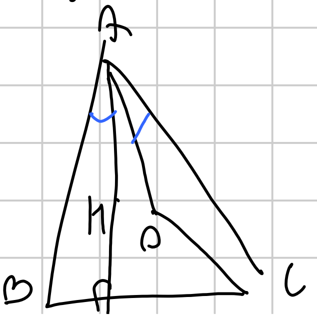
" $\omega_B \omega_C = BC$

Gli assi radicali di 3 circonferenze concorrono $\Rightarrow$ ASSURDO

$\Rightarrow P_AQ_AP_BQ_BP_CQ_C$ è ciclico

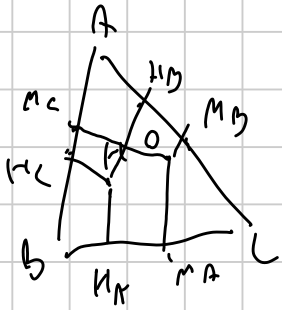
OSS.

H e O sono coniugati isogonali.



$\angle BAH = 90 - \beta$

$\angle ACO = \frac{1}{2}(180 - \angle AOC) = 90 - \frac{1}{2}\cdot(2\angle ABC) = 90 - \beta$

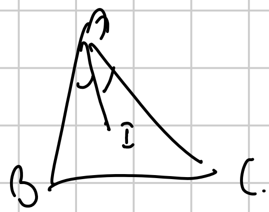


Per il teorema dei coniugati isogonali

$H_A M_A H_B M_B H_C M_C$ è ciclico

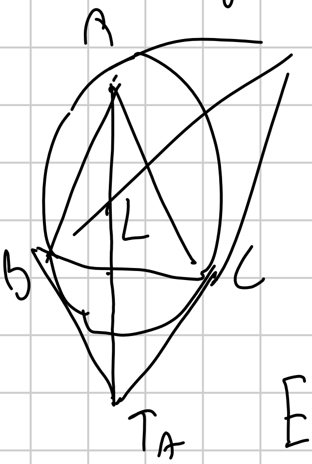
M_O è Feuerbach.

— O —



I è coniugato isogonale di se stesso

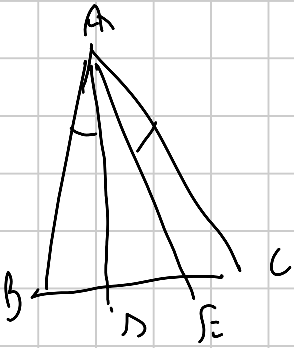
G è coniugato isogonale di Lemoine (pt di concorrenza delle simediane)



AT_A è simediana.

Le simediane concorrono in L.

ES X CASA (TEOREMA DEI RAPPORTI DI STEINER)

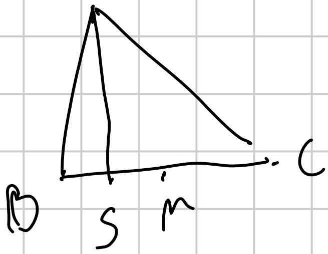


ABC △, D, E ∈ BC $\angle BAD = \angle EAC$

AD, AE coniugate

Th: $\frac{AB^2}{AC^2} = \frac{BD \cdot BE}{CD \cdot CE}$ (DIM: teo dei seni su ABD, ACD, ABE, ACE)

CASO PARTICOLARE

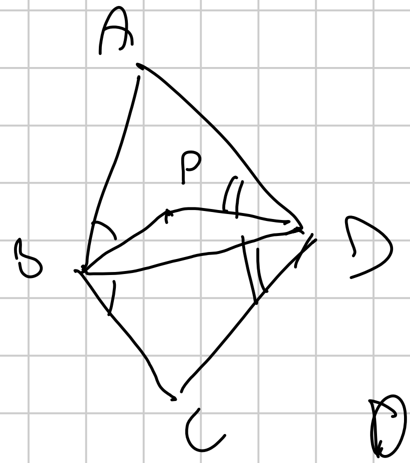


$\frac{BS}{CS} = \frac{AB^2}{AC^2}$

Fine digressione

— O — O —

IMO 2004/5



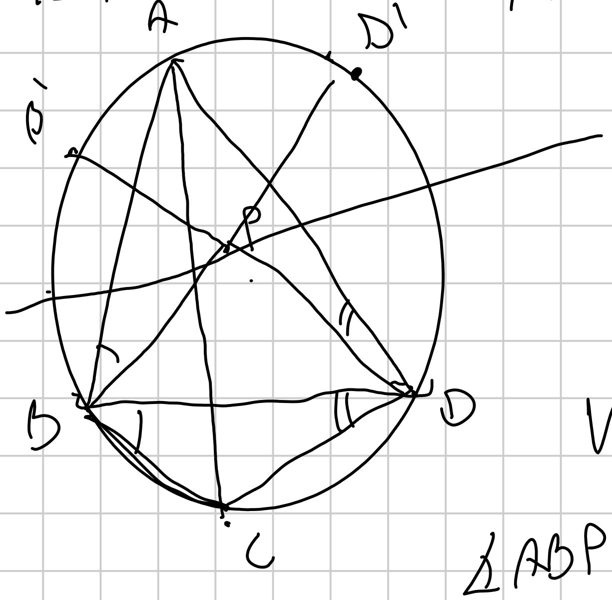
P all'interno del quadrilatero ABCD tale che

$\angle ABP = \angle DBC$ e $\angle CDB = \angle PDA$

Tesi: ABCD ciclico $\iff AP = CP$

OSS: in $\triangle BPD$, A e C sono coniugati isogonali

1ª Freccia ciclico $\Rightarrow AP = CP$



Tesi $\iff P \in$ asse di AC

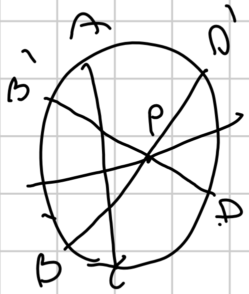
B', D' simmetrici di B, D rispetto all'asse di AC

Voglio dimostrare BPD' allineati (C ↔ A)

$\measuredangle ABP = \measuredangle DBC = \measuredangle AB'D' =$

$= \measuredangle ABD' \Rightarrow BPD'$ allineati

Analogamente $B'PD$ allineati

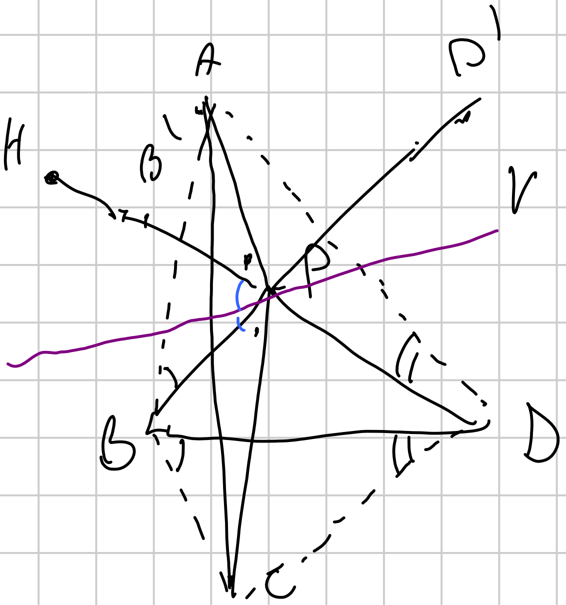


Faccio la simmetria rispetto all'asse di AC

$B \to B'$ $D \to D'$ $\Rightarrow BD' \to B'D$

$P \to P' \Rightarrow P \in$ asse di AC ✓

2ª FRECCIA. $AP = CP \Rightarrow$ ciclico



Poiché A, C sono coniugati in $\triangle BPD$

$\Downarrow$

$\angle APH = \angle BPC$

Considero la bisettrice esterna di $\angle BPD$

$\Rightarrow$ è bisettrice (interna) di APC

MA APC è isoscele $\Rightarrow$ bisettrice è anche altezza $\Rightarrow$ è $\perp AC$ $\Rightarrow$ è il suo asse

$r \perp AC$

B', D' simmetrici di B, D rispetto a r $\Rightarrow$ $B' \in PD$, $D' \in PB$

$\angle ABD' = \angle ABP = \angle DBC = \angle AB'D'$ (per la simmetria)

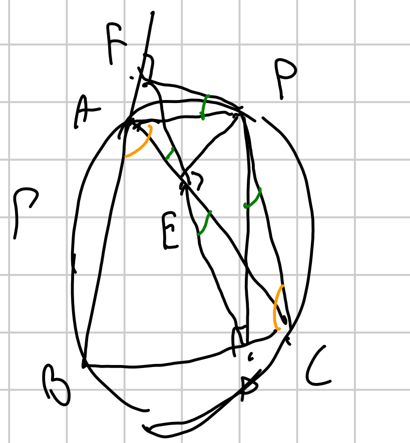
$ABB'D$ è ciclico

Analogamente $ADD'B$ è ciclico $\Rightarrow$ $ABB'DD'$ ciclico

Faccio la simmetria rispetto a r $\Rightarrow$ la circonferenza passa anche per C $\Rightarrow$ $ABCD$ ciclico $\Rightarrow$ Tesi C.V.D.

— o — o —

Retta di Simson



$P \in \Gamma = O_{ABC}$

D, E, F proiezioni di P sui lati

Allora D, E, F sono allineati.

DIM. Notiamo che $\angle PDC = \angle PEC = 90° \Rightarrow PEDC$ è ciclico

$\angle PEA = \angle PFA = 90 \Rightarrow PAFE$ è ciclico

$\angle PDB = \angle PFB = 90° \Rightarrow PDBF$ è ciclico

Per la tesi mi basta $\angle CED \overset{?}{=} \angle AEF$

$$\angle CED = \angle CPD = 180 - \underbrace{\angle PDC}_{=90°} - \angle DCP = 90 - \angle DCP = 90° - \angle BCP =$$

$$= 90° - \angle BAP = 90° - \angle PAF = 90° - (180 - 90 - \angle FPA) = \angle FPA = \angle FEA$$

$\Rightarrow DEF$ sono allineati

C.V.D.

— o — o —

SIMSON 1,5

D, E, F t.c. $\angle PDB = \angle PEC = \angle PFA$ con $D \in AB$ ecc...

$\Rightarrow D, E, F$ sono allineati