A mista (Advanced)

Titolo nota 03/09/2017

Scambust!

SL 97/6 $\qquad n=100 \qquad v \in \mathbb{R}^n \qquad \|v\| = 1$

$$a_1^2 + a_2^2 + \dots + a_{100}^2 = 1 \Rightarrow \underbrace{a_1^2 a_2 + a_2^2 a_3 + \dots + a_{100}^2 a_1}_{S} < \frac{12}{25}$$

$a_i^2 , \quad a_{i+1}$

$$S^2 \le \left(\cancel{\sum a_i^2}\right)^{\to 1}\left(\sum a_i^4\right) \le 1 \qquad \boxed{S \le 1}$$

$a_i , \quad a_i a_{i+1}$

$$S^2 \le \left(\cancel{\sum a_i^2}\right)^{\to 1}\left(\sum a_i^2 a_{i+1}^2\right) \le \left(\sum_{i \text{ pari}} a_i^2\right)\left(\sum_{i \text{ dispari}} a_i^2\right) \le \frac{1}{4} \Rightarrow \boxed{S \le \frac{1}{2}}$$

$$\left(a_i , \quad a_{i-1}^2 + 2a_i a_{i+1}\right)$$

$$\frac{\partial f}{\partial a_i} = \lambda \frac{\partial g}{\partial a_i} \qquad \begin{aligned} f &= a_1^2 + \dots + a_{100}^2 \\ g &= a_1^2 a_2 + \dots + a_{100}^2 a_1 \end{aligned}$$

$$\underline{2a_i} = \lambda \underbrace{\left(2a_i a_{i+1} + a_{i-1}^2\right)}$$

$$(3S)^2 \le \left(\cancel{\sum a_i^2}\right)^{\to 1}\left(\sum \left(a_{i-1}^2 + 2a_i a_{i+1}\right)^2\right)$$

$$= \sum a_i^4 + 4\sum a_i^2 a_{i+1} a_{i+2} + 4\sum a_i^2 a_{i+1}^2$$

$$\left[\sum a_i^4 \le \left(\sum a_i^2\right)^2\right] \qquad a_i^2 \stackrel{\text{def}}{=} x_i$$

$$\sum x_i = 1$$

$$\le \sum a_i^4 + 2\sum a_i^2 a_{i+1}^2 + 2\sum a_i^2 a_{i+2}^2$$

$$+ 4\sum a_i^2 a_{i+1}^2$$

$$\le 1 + 4\sum a_i^2 a_{i+1}^2$$

$$\overset{!}{\le} 1 + 4\Big(\sum_{i\ \text{pari}} a_i^2\Big)\Big(\sum_{i\ \text{dispari}} a_i^2\Big) \le 2 = 1 + 4\cdot\frac{1}{4}$$

$$S \le \frac{\sqrt{2}}{3} < \frac{12}{25}$$

Se $n \le 4$ $\qquad a_i = \frac{1}{\sqrt{n}}$

$$\sum a_i^2 = 1, \quad \sum b_i^2 = 1, \quad \sum a_i b_i = 0$$

$a_1, \dots, a_n$

$b_1, \dots, b_n$

$n \ge 2$

$$\Rightarrow \Big(\sum a_i\Big)^2 + \Big(\sum b_i\Big)^2 \le n$$

$$\|a\| = 1, \quad \|b\| = 1, \quad (a,b) = 0$$

$u = (1, \dots, 1)$ $\qquad (u,u) = n$

$\sum a_i \cdot 1$

$$(a,u)^2 + (b,u)^2$$

$(a,u) = \alpha$

$(b,u) = \beta$

$$\vec{u} = \vec{r} + \mu_1 \vec{a} + \mu_2 \vec{b}$$

$r_i = 1 - \mu_1 a_i - \mu_2 b_i$

$\mu_1 = \alpha, \ \mu_2 = \beta$

$$\alpha = (u,a) = (r+\alpha a+\beta b, a) = \underbrace{(r,a)}_{0} + \underbrace{\alpha(a,a)}_{\alpha} + \underbrace{\beta(b,a)}_{0}$$

$$(r,a) = 0$$

$$(u,a)^2 = \alpha^2$$

$$(u,a)^2 + (u,b)^2 \overset{?}{\leq} (u,u)$$

$$(u,u) = \alpha^2 + \beta^2 + (r,r) + \cancel{2\alpha(a,r)} + \cancel{2\beta(b,r)} + \cancel{2\alpha\beta(a,b)}$$

$$\geq \alpha^2 + \beta^2$$

Algebra Lineare....

$abc = 1$

$$\sum_c \frac{a}{b} \geq \sum_c a$$

1) Saggio

$$\sum a^2 c \overset{!}{\geq} \sum a^{5/3} b^{2/3} c^{2/3}$$

2) Artistica

$$\sum_c \frac{a^2}{ab}$$

$$\Rightarrow \boxed{\sum a \geq \sum ab}$$

IMO SL 16/A7

TST 08/6

$f: \mathbb{R}^+ \to \mathbb{R}^+$ $\quad \boxed{f(x+y) \ge f(x) + y f(f(x))}$

$\Rightarrow \nexists f$

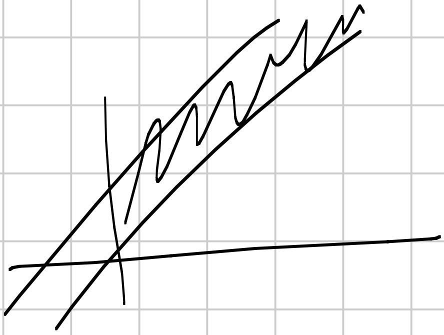
1) $\boxed{f(x+y) \ge y f(f(x))}$ $\quad x = \varepsilon$?

$x+y = f(x)$ $\quad y = f(x) - x$

$\underbrace{f(f(x))}_{\in \mathbb{R}^+} (f(x) - x - 1) \le 0$

$f(x) \le x+1$

- Se $f(x) > x \Rightarrow f(x) \le x+1$



$\lim_{x \to +\infty} f(x) \to +\infty$ $\quad x = 1$

$f(y+1) \ge \underbrace{y f(f(1))}$

$y = f(x) - 1, \ x = 1$

$$\underbrace{f(f(x))}_{+\infty} \geq \underbrace{(f(x)-1)\, f(f(1))}_{+\infty}$$

$f(xy) \geq y \underbrace{f(f(x))}\ > 5y \qquad f(f(x)) \geq 5$

$> 4(y+x)$

$\Rightarrow \boxed{f(y) > 4y}$

IMO 09/5 $\quad \boxed{f: \mathbb{N}^+ \to \mathbb{N}^+}$

$a, \ f(b), \ \underbrace{f(b+f(a)-1)}$

$a = 1$ $\qquad 1, x, y \Rightarrow x \neq y$

$f(b) = f(b+f(1)-1) \qquad 1 + x+1 > y+1 > x$

$\boxed{f(1) = 1}$ $\qquad x+1$

$\boxed{a = f(f(a))} \Rightarrow f$ è iniettiva e surgettiva

$a + f(b) > f(b+f(a)-1)$

$$f(a)+f(b) > f(a+b-1)$$

$$a+b > f(f(a)+f(b)-1)$$

$a=b=2$

$$f(2f(2)-1) < 4$$

$$\| \quad k \qquad \boxed{k \le 3}$$

$$2f(2)-1 = f(k) \Rightarrow k=3$$

$a=2,\ b=3$ $\quad f(f(2)+f(3)-1) < 5$

$$f(3f(2)-2) = k \qquad k \le 4$$

$$3f(2)-2 = f(k)$$

$k=1,2,\dots$

$$f(3) = 2f(2)-1$$
$$f(4) = 3f(2)-2$$
$$\vdots$$

$f(2)=1$

$k=4-$

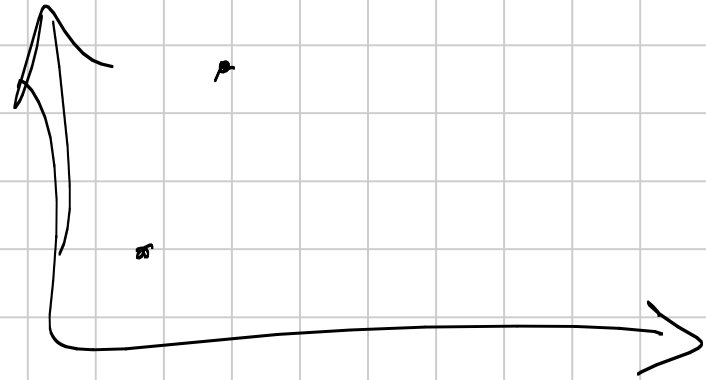
$a=2$ $\quad \boxed{f(n) = (n-1)f(2) - (n-2)} \quad =$

$b=n$ $\quad f(f(2)+f(n)-1) = k$

$k \le n+1$

$$nf(2)-(n-1) = f(k)$$

Se $\overline{k \le n}$ $\quad \boxed{n(f(2)-1) + \cancel{c}} = (k)(f(2)-1) + \cancel{c}$



IMO 13/5

i) $f(x)f(y) \geq f(xy)$

ii) $f(x+y) \geq f(x) + f(y)$ $\Rightarrow$ $f(x) = x$ $\qquad f: \mathbb{Q}_{>0} \to \boxed{\mathbb{R}}$

iii) $\exists a > 1 \quad f(a) = a \qquad a \in \mathbb{Q}_{>0}$

ii) $\boxed{f(nx) \geq n f(x)}$ $\qquad f(1) \geq 1$

$f(n) \geq n f(1) \geq n$

$f(n) \geq n$, $n \in \mathbb{N}^*$

$x = 1, y = a$

i) $f\left(\frac{p}{q}\right) \underbrace{f(q)}_{>0} \geq \underbrace{f(p)}_{>0} \Rightarrow f(r) > 0$

$f: \mathbb{Q}^+ \to \mathbb{R}^+$

ii) $f(x+y) \geq f(x) + f(y) > f(x)$

f è crescente

$\exists n : f(n) > n \qquad f(n) = n + \varepsilon$

$f(mn) \geq n f(n) = mn + m\varepsilon$

$f(n) = n$

$[f(x)]^n \geq f(x^n)$

$x = a \qquad f(a^n) \leq a^n$

i) $x = a \qquad a f(y) \geq f(ay)$

y

$y = a$

$f(a^2) \leq a^2$

$f(n) \geq n$

$f(a^n) \leq a^n$

$b_n = \lfloor a^n \rfloor$

$$\lfloor a^n \rfloor \le f(\lfloor a^n \rfloor) \le f(a^n) \le a^n < \lfloor a^n \rfloor + 1$$

$$\boxed{b_n \le f(b_n) \le b_n + 1}$$

Se ci fosse un n_0 t.c. $f(n_0) \ge n_0 + 1$ NO

$$f(x+n_0) \ge f(x) + f(n_0) \ge f(x) + n + 1$$

$$f(x+y) \ge x+y+1$$

$$\Rightarrow \quad f(n) \ge n+1 \qquad \text{con } n \ge n_0$$

$$f(n) < n+1$$

$$\boxed{f(n) \ge n + \frac{1}{m}}$$

$$f(nm) \ge nm + m \cdot \frac{1}{m} = nm + 1$$

$$\Rightarrow \quad f(n) = n + \varepsilon$$

$$\boxed{f(n) = n}$$

i) $f(\frac{p}{q}) f(q) \ge f(p)$

$$f\left(\frac{p}{q}\right) \ge \frac{p}{q}$$

ii) $f(nx) \ge n f(x) \qquad x = \frac{m}{n}$

$$f\left(\frac{m}{n}\right) \le \frac{m}{n}$$ ✓

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TST USA SL 2004 $f: \mathbb{R}_{\ge 1} \to \mathbb{R}_{\ge 1}$

$$\begin{cases} \text{i)} & f(x) \le 2(x+1) \\ \text{ii)} & x f(x+1) = f(x)^2 - 1 \end{cases}$$

IMO 2011/3

$f: \mathbb{R} \to \mathbb{R}$ $f(x+y) \le y f(x) + f(f(x))$ $\Rightarrow \boxed{\forall x \le 0 \quad f(x) = 0}$

— o — o —

TST USA SL 2004

i) $f(x) \leq 2(x+1)$

ii) $xf(x+1) = f(x)^2 - 1$!

$f: \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1}$

$f(x) \leq 2(x+1)$ $\leadsto$ $f(x) \leq x+1$?

$$f(x)^2 = xf(x+1) + 1 \leq 2x^2 + 4x + 1$$
$$< 2(x+1)^2$$
$$\Rightarrow f(x) < \sqrt{2}(x+1)$$

:)

$$f(x)^2 = xf(x+1) + 1 \leq x\sqrt{2}(x+2) + 1$$
$$< \sqrt{2}(x+1)^2$$

$$f(x) \leq \sqrt[4]{2}(x+1)$$

$$\boxed{f(x) \leq \sqrt[2^n]{2}(x+1)}$$

$\Rightarrow$ $f(x) \leq x+1$

$f(x_0) = x_0 + 1 + \varepsilon$

$$\boxed{f(x) > x+1}$$

$$f(x)^2 = xf(x+1) + 1 \geq x+1 > x$$!

$f: \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1}$

$f(x) > \sqrt{x}$

$f(x)^2 = x\sqrt{x+1}+1 > x^{\frac{3}{2}}$

$\boxed{f(x) > x^{\frac{3}{5}}} \Rightarrow f(x) \geq x^{1-\frac{1}{2^n}}$

$f(x) \geq x$

$f(x)^2 = xf(x+1)+1 \geq x(x+1)+1$

$> (x+1/2)^2$

$f(x) > x+\frac{1}{2}$

$f(x) > x+1-\frac{1}{2^n}$

$\boxed{\Rightarrow f(x) \geq x+1}$

IMO 2011/3

$f: \mathbb{R} \to \mathbb{R}$

$$f(x+y) \leq y f(x) + f(f(x))$$

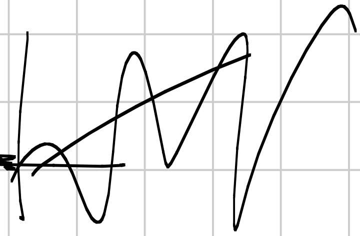
$\Longrightarrow$ $f(x) = 0$ $\forall x \leq 0$

$f(x) \equiv 0$ ✓

$f(a) \neq 0$

$f(x)\left(\ldots\right) \geq 0$

$x \leq 0$



$f(a) < 0$

$$f(x) = f(a + x - a) \leq (x-a)\underbrace{f(a)}_{<0} + \underbrace{f(f(a))}_{\text{c.t.}}$$

$\lim_{x \to +\infty} f(x) = -\infty$ $\quad$ $\lim_{x \to -\infty} f(x) \leq +\infty$

$\boxed{f(a) > 0}$? $\quad \lim_{x \to -\infty} f(x) = -\infty \Longrightarrow \exists a' : f(a') < 0$!

$$\lim_{x \to +\infty} f(x) = -\infty$$

$$f(x+y) \leq y f(x) + f(f(x))$$

$x \to +\infty \qquad f(x) \to -\infty$

$x \to -\infty \qquad f(x) \to -\infty$

$x \to +\infty \;\| \; f(f(x)) \to -\infty$

$$f(0) = f(-y+y) \le y f(-y) + f(f(-y)) = -\infty$$

$$\lim_{y \to +\infty} f(0) = -\infty \quad !!!$$

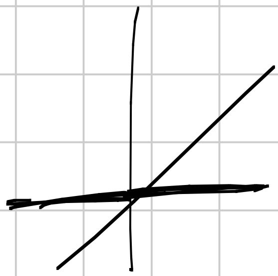
$\boxed{f(a) \le 0}$ $\quad \forall a \in \mathbb{R}$ $\qquad f(a) = 0 \quad \forall a \le 0$

$\cancel{x}+y = f(x) \qquad y = f(x) - x$

usiamo che $f(x) \neq 0$

$$\big(f(x) - x\big) \underbrace{f(x)} \ge 0$$

Supponiamo per assurdo che $f(b) = 0$ $\quad \boxed{f(x) < x}$



$$f(0) = f(x - x) \le -x f(x) + f(f(x)) \le -x^2$$

assurdo $\quad \nexists b : f(b) = 0$

$x+y = b \;\| \; 0 = f(b) = f(b+0) \le \cancel{f(b)} + f(f(b)) = f(0)$

$f(0) \ge 0 \quad \Rightarrow \quad \boxed{f(0) = 0}$

$x+y = 0$

$y = -x \;\| \; 0 = f(0) = f(x - x) \le -x f(x) + f(f(x)) \le -x f(x)$

$f(x) \geq 0 \rightarrow$ OK

TST Vietnam '03

IMO SL 09/5