P advanced - Un po' di analisi

Titolo nota 02/09/2017

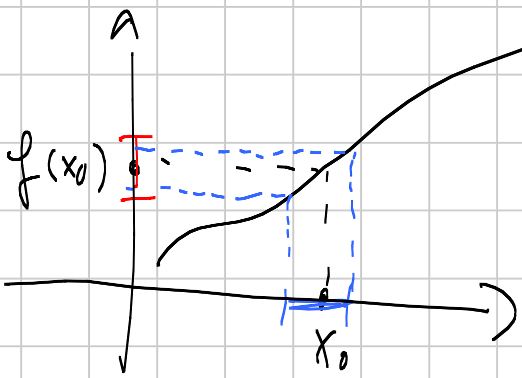
- Continuità

$f : \mathbb{R} \to \mathbb{R}$

$(*)$ $f(x+y) = f(x) + f(y) \implies f$ $\mathbb{Q}$-lineare

- basi di Hamel

$(*)$ + continuità $\implies f$ $\mathbb{R}$-lineare
monotonia
$\exists$ $\square$ nel piano che non interseca grafico di f
⋮



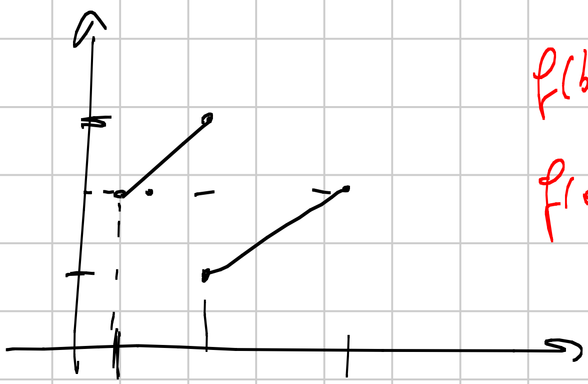
$\forall \varepsilon > 0$ $\exists \delta > 0$ t.c.

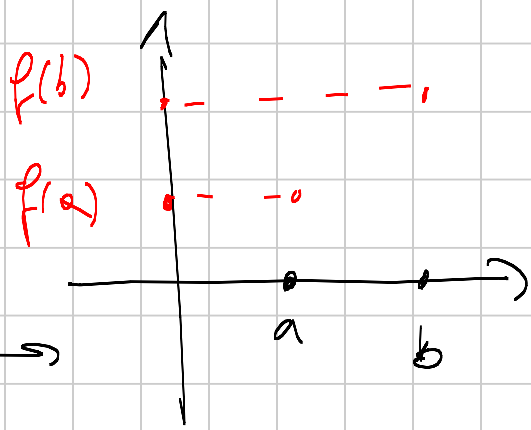
$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

$\Updownarrow$

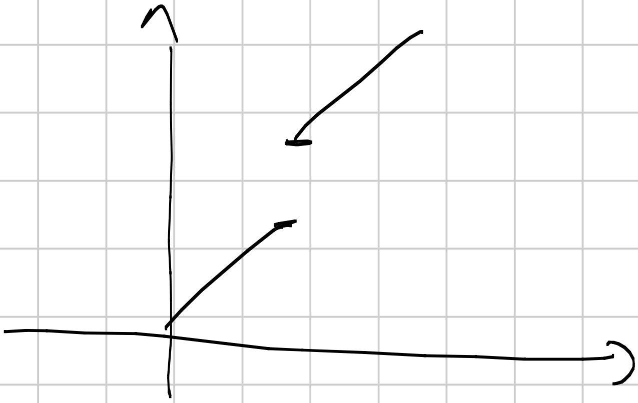
$$\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$$

Fatto: f continua + iniettiva $\implies$ monotona





Fatto: monotona + surgettiva $\implies$ continua

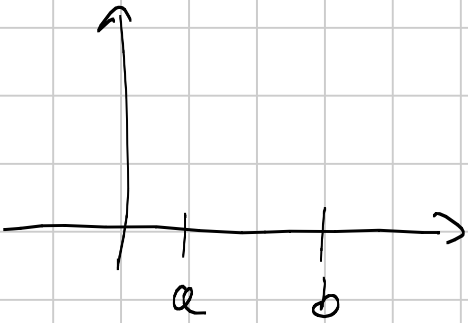


·) continuità, max & min

Teo di WEIERSTRASS

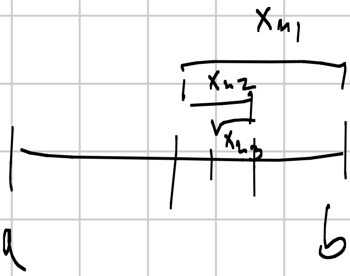
$f: [a,b] \to \mathbb{R}$ continua

$\Rightarrow f$ ha max e min in $[a,b]$



1) f limitata: per assurdo $\forall M>0\ \exists\, x_M \in [a,b]$ t.c. $f(x_M) > M$ (wlog)

$[a,b] \supseteq \{x_n\}$ t.c. $f(x_n) > n \quad \forall n \in \mathbb{N}$



$x_{n_j} \xrightarrow[j\to+\infty]{} c \in [a,b]$

$\Rightarrow \exists\, x_{n_j} \to c \in [a,b] \quad \Rightarrow \lim_{j\to+\infty} f(x_{n_j}) \overset{\text{continuità}}{=} f(c)$

$\geq \lim_{j\to+\infty} n_j = +\infty$

Varianti di W

1) $f: \mathbb{R} \to \mathbb{R}$ continua + periodica

2) $\exists\, K = [a,b]$ t.c. max e min devono stare in K.

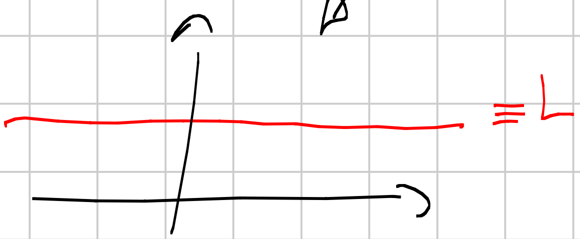
$\Rightarrow f$ ha max e min

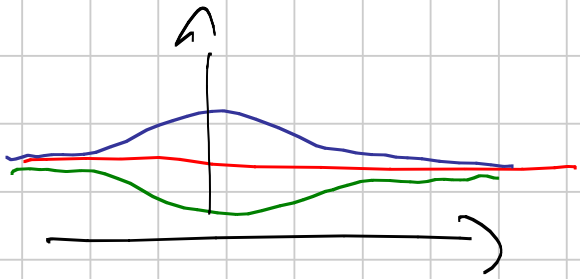
f cont + limitata <u>no</u> <u>Es</u>: $f(x) = \arctan x$

$$f(x) = \frac{1}{1+x^2}$$

<u>Es di 2</u>: $\lim_{x\to\pm\infty} f(x) = 0$ + $f(x) > 0 \ \forall x \in \mathbb{R}$ $\Rightarrow$ $\exists$ max

$\lim_{x\to\pm\infty} f(x) = L \in \mathbb{R} \Rightarrow \exists$ max o min o entrambi





$L = \pm\infty$ $\quad \lim_{x\to\pm\infty} f(x) = +\infty \Rightarrow \exists$ min

$\lim_{x\to\pm\infty} f(x) = -\infty \Rightarrow \exists$ max

<u>Es</u>: $f: \mathbb{R} \to \mathbb{R}$ continua t.c. $f(x+y+f(xy)) = f(x+y) + xy \quad \forall x,y \in \mathbb{R}$

1) $y = 0 \to f(x + f(0)) = f(x) + 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ periodica se $f(0) \neq 0$

$\Rightarrow f$ ha max e min per Weierstrass

$x = -y \Rightarrow f(f(-x^2)) = -x^2 + f(0) \Rightarrow f(\mathbb{R}) \supset (-\infty, 0]$

$\Rightarrow$ assurdo $\Rightarrow f(0) = 0$

2) $f(f(-x^2)) = -x^2 \Rightarrow f(x) \neq f(y) \ \forall\, x \neq y \ \text{e}\ x, y < 0$

3) f continua + iniettiva in $(-\infty, 0) \Rightarrow$ monotona

se $\exists\, t<0$ t.c. $f(t) < 0 \Rightarrow f(t) = t \ \forall t < 0$

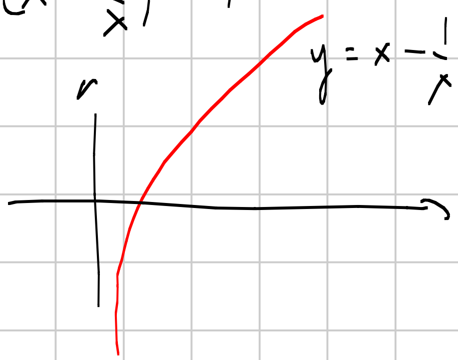
$$\forall n > 1 \quad f(n-1+f(-n)) = f(n-1) - n$$

$$\downarrow$$

$$f(\cancel{n}-1-\cancel{n}) = f(-1) = -1 \qquad \Rightarrow f(n-1) = n-1$$

se $\exists\, t<0$ t.c. $f(t) > 0 \Rightarrow f$ decrescente in $(-\infty, 0)$

$y = -\frac{1}{x} \Rightarrow f\left(x - \frac{1}{x} + f(-1)\right) = f\left(x - \frac{1}{x}\right) - 1$



$y = -\frac{k}{x}$ $\quad f\left(x - \frac{k}{x} + f(-k)\right) = f\left(x - \frac{k}{x}\right) - k$

$k > 0$

$x - \frac{k}{x} = z \qquad x - \frac{k}{x} + f(-k) = w$

$\Rightarrow f$ decrescente $+ f(f(-x^2)) = -x^2 \Rightarrow f$ inj + surj

$$f(a + \underset{\substack{\| \\ b>0}}{f(-k)}) = f(a) - k$$

$$f(b) = ff(-k) = -k$$

$$f(a+b) = f(a) + f(b) \qquad a, b > 0$$

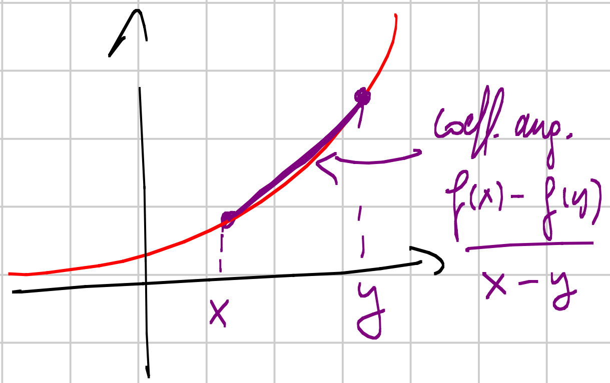
$$\Rightarrow \text{cauchy su } \mathbb{R} \qquad \lambda = \pm 1$$

<u>Es sbagliato</u> $f : [1, 2017] \to [1, 2017]$ t.c.

$$|f(x) - f(y)| \geq |x-y| \quad \forall\, x, y \in [1, 2017]$$

<u>sol</u> (NO!) :

$$\frac{|f(x) - f(y)|}{|x-y|} \geq 1$$



e se non esiste?

$$\Rightarrow |f'(x)| \geq 1 \quad \forall x \in [1, 2017]$$

wlog $f'(x) \geq 1 \quad \forall x \in [1, 2017]$!! ??

$$2017 \geq f(2017) = \int_1^{2017} f'(x)\,dx + f(1) \geq 2016 + f(1) \geq 2017$$

$$\Rightarrow f'(x) \equiv 1 \Rightarrow f(x) = x$$

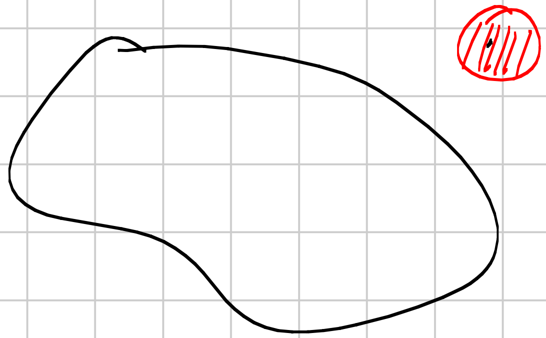
<u>NO!!</u>

Massimi e minimi

$K \subseteq \mathbb{R}^n$ - è chiuso se $\forall x \notin K \ \exists r>0$ t.c.

$$\|y-x\| < r \Rightarrow y \notin K$$

- è limitato se
$\exists R>0$
t.c. $\|x\|<R \ \forall x \in K$



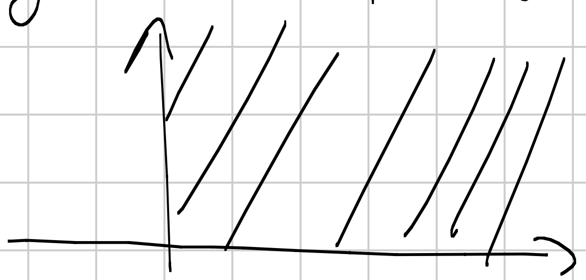
es: $x>0, y>0$ descrive un insieme non chiuso e non limitato.

Teo W: $f: K \to \mathbb{R}$ continua + K chiuso e limitato
$\Rightarrow f$ ha max e min su K.

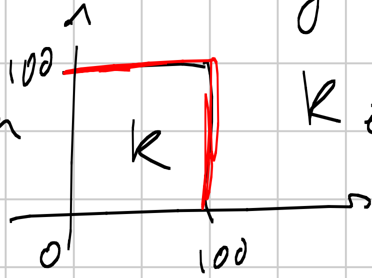
Teo: $f: K \to \mathbb{R}$ continua e con derivate continue
Max e min di f, se esistono si trovano

1) dove tutte le der di f si annullano
2) sul bordo di K

Es: $x^4 + y^5 - x^2 y$ minimo per $x, y \geq 0$ — non limitato.
$= f(x,y)$



$f(0,0)=0$. Se $x, y \geq 100 \Rightarrow x^4+y^5-x^2y>0$

$\Rightarrow$ mi restringo a  K è chiuso e limit.

Posso dire lo stesso con $K' = \{0 \leq x \leq 90,\ 0 \leq y \leq 90\}$

$\Rightarrow$ il min non sta nel bordo rosso.

<u>Derivate</u>:

$x^a \to a x^{a-1}$ $\qquad$ $\arctan x \to \dfrac{1}{1+x^2}$

$e^x \to e^x$

$\sin x \to \cos x$ $\qquad$ $f \cdot g \to f' \cdot g + g' f$

$\cos x \to -\sin x$

$\log \to \dfrac{1}{x}$ $\qquad$ $f(g) = f'(g) \cdot g'$

$$\frac{f}{g} = \frac{f'g - g'f}{g^2}$$

$$f(x,y) = x^4 + y^5 - x^2 y$$

$$f_x(x,y) = 4x^3 + 0 - y \cdot 2x$$

$$f_y(x,y) = 5y^4 - x^2$$

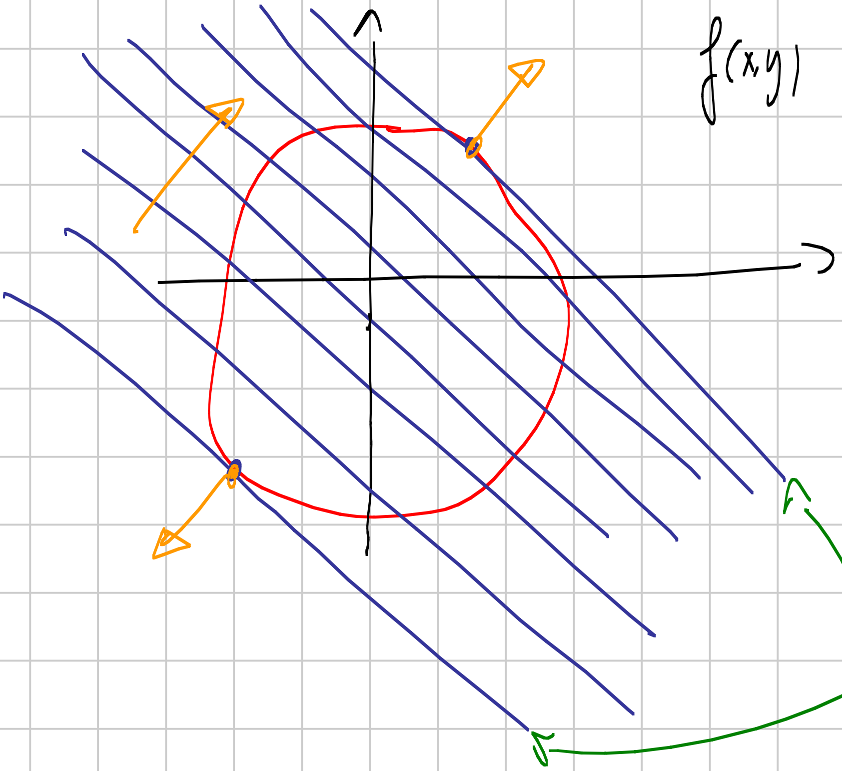
$$\begin{cases} 4x^3 - 2xy = 0 \\ 5y^4 - x^2 = 0 \end{cases}$$

$$\begin{cases} y = 2x^2 \\ 5 \cdot 16 x^8 - x^2 = 0 \end{cases} \qquad x^2(80x^6 - 1) = 0$$

$x = 0 \Rightarrow y = 0$

$x = \dfrac{1}{\sqrt[6]{80}} \Rightarrow y = \dots$

$\overset{0}{\overset{\parallel}{f(0,0)}}$, $f(\text{soluz}, \text{soluz}) = \dots$



$f(x,y)$ affine

$f \equiv$ cost retta

val min e max di f sulla curva rossa

Il vett. normale a $f(x,y)=c$ è dato da $\begin{pmatrix} f_x \\ f_y \end{pmatrix}$ calcolato nei pt. opportuni.

← vett. delle derivate

<u>Teo</u>: f, g continue con der. continue, $\nabla g \neq 0$ su $\{g=0\}$

$\Rightarrow$ max e min di f ristretta a $\{g=0\}$ sono pt. tali che

$$\nabla f = \lambda \cdot \nabla g \qquad \lambda \in \mathbb{R}$$

[oppure nei pt. di bordo di $\{g=0\}$, se esistono]

<u>Es</u>: dim che $a^3+b^3+c^3 \leq (a^2+b^2+c^2)^{\frac{3}{2}} + 3abc$ con $a+b+c>0$

<u>Sol</u>: è omogenea $\Rightarrow$ scelgo $a^2+b^2+c^2=1$ che è un vincolo chiuso e limitato

$$f(a,b,c) = a^3+b^3+c^3-3abc \qquad g(a,b,c) = a^2+b^2+c^2-1$$

$$\nabla g = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \qquad \nabla f = \begin{pmatrix} 3a^2 - 3bc \\ 3b^2 - 3ac \\ 3c^2 - 3ab \end{pmatrix}$$

$$\boxed{\nabla g = 0 \iff a = b = c = 0 \not\in \{g = 0\}}$$

$$\nabla f = \lambda \nabla g \quad + \; g = 0$$

$$\begin{cases} 3a^2 - 3bc = 2a \cdot \lambda \\ 3b^2 - 3ac = 2b \cdot \lambda \\ 3c^2 - 3ab = 2c \cdot \lambda \\ a^2 + b^2 + c^2 = 1 \end{cases}$$

$$\frac{3}{2}\left(\frac{a^2 - bc}{a}\right) = \frac{3}{2}\left(\frac{b^2 - ac}{b}\right) = \frac{3}{2}\left(\frac{c^2 - ab}{c}\right)$$

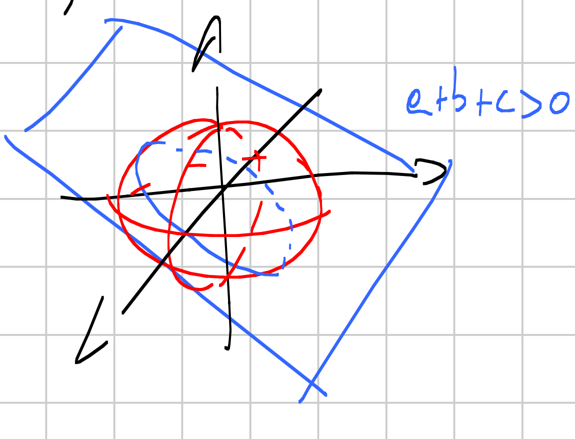
$$a^2 b - b^2 c = ab^2 - a^2 c \qquad (a-b)(ab+bc+ca) = 0$$

1) $c = a = b \Rightarrow a = b = c = \frac{1}{\sqrt{3}} \quad f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 0$

2) $ab + bc + ca = 0 \qquad (a+b+c)^2 = a^2 + b^2 + c^2 = 1$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ac - ba - cb) =$$

$$= 1 \cdot (1 - 0) = 1$$



$\underline{Es}$: Minimo M t.c

$$\left|\sum_{cyc} ab(a^2-b^2)\right| \le M(a^2+b^2+c^2)^2 \quad \forall a,b,c \in \mathbb{R}$$

$a^2+b^2+c^2=1$ $\qquad$ $g(a,b,c) = a^2+b^2+c^2-1$

$$f(a,b,c) = \sum_c ab(a^2-b^2)$$

$$\nabla g = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \qquad \nabla f = \begin{pmatrix} -b^3+ba^2+2ca^2b+c^3-ca^2-2a^2c \\ \vdots \\ \end{pmatrix}$$

$$3ba^2-b^3+c^3-3a^2c = 2\lambda a \quad \cdot b$$
$$3b^2c-c^3+a^2-3b^2a = 2\lambda b \quad \cdot a$$

$$3b^2a^2-b^4+c^3b-3a^2cb = 3ab^2c - ac^3+a^4-3b^2a^2$$

(1) $a^4+b^4-6a^2b^2 = -3bca^2-3acb^2+ac^3+bc^3 =$
$= (a+b)(c^3-3abc)$

(2) $a^4+c^4-6a^2c^2 = (a+c)(b^3-3abc)$

(3) $b^4+c^4-6b^2c^2 = (b+c)(a^3-3abc)$

(1) + (2) + (3)

$a^4+b^4+c^4 = a^4+b^4+c^4$

$$3(a+b+c)\overbrace{(a+b-c)(a-b+c)(-a+b+c)}^{A} =$$
$$= (a+b+c)\underbrace{(a^3+b^3+c^3-6abc)}_{B}$$

$3A - B = 0$

$(2a-b-c)(2b-a-c)(2c-a-b)=0$

$\frac{1}{k}\left(-11 \pm 6\sqrt{2},\ 7,\ -2 \pm 3\sqrt{2}\right)$ since

$\frac{9}{16\sqrt{2}}$