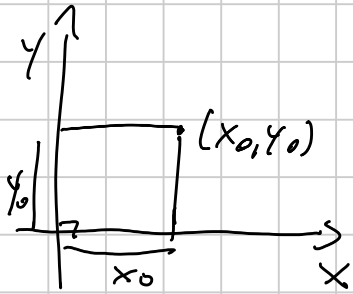
G2 Basic LucaMac

Note Title 9/5/2017

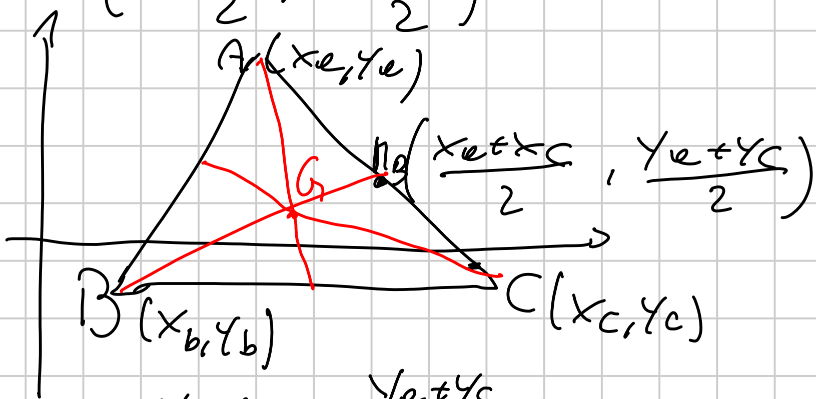
Cartesiane, Complessi e Vettori.

CARTESIANE



$A=(x_0,y_0)$ $B=(x_1,y_1)$ M il punto medio di AB

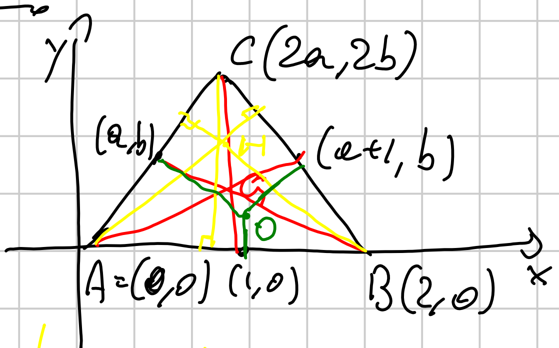
$$M=\left(\frac{x_0+x_1}{2}, \frac{y_0+y_1}{2}\right)$$



BG: $\frac{y-y_b}{x-x_b}=\frac{\frac{y_a+y_c}{2}-y_b}{\frac{x_a+x_c}{2}-x_b}$ AG: $\frac{y-y_a}{x-x_a}=\frac{\frac{y_b+y_c}{2}-y_a}{\frac{x_b+x_c}{2}-x_a}$

(generale: la retta per due punti è $\frac{y-y_0}{x-x_0}=\frac{y_1-y_0}{x_1-x_0}$)

$$G=\left(\frac{x_a+x_b+x_c}{3}, \frac{y_a+y_b+y_c}{3}\right)$$



$$G=\left(\frac{2}{3}(a+1), \frac{2}{3}b\right)$$

Calcoliamo H: $CH \perp AB$ $x_H=x_c=2a$

$AH \perp BC$

$$m_{BC}=\frac{y_b-y_c}{x_b-x_c}=\frac{-2b}{2-2a}=\frac{b}{a-1}$$

$m_{AH} = \frac{1-a}{b}$ $\qquad$ $AH: \ y = \frac{1-a}{b}x$

$H = \left(2a, \frac{2a-2a^2}{b}\right)$

Calcoliamo O:

Sta sull'asse di $AB \Rightarrow x_O = 1$

sta sull'asse di AC $\qquad y = -\frac{a}{b}x + \frac{a^2}{b} + b$

$O = \left(1, -\frac{a}{b} + \frac{a^2}{b} + b\right)$

$$\frac{x_H - x_G}{y_H - y_G} \stackrel{?}{=} \frac{x_O - x_G}{y_O - y_G}$$

$$\frac{2a - \frac{2}{3}(a+1)}{\frac{2a-2a^2}{b} - \frac{2}{3}b} \stackrel{?}{=} \frac{1 - \frac{2}{3}(a+1)}{-\frac{a}{b} + \frac{a^2}{b} + b - \frac{2}{3}b}$$

$$\frac{b(6a - 2(a+1))}{6a - 6a^2 - 2b^2} \stackrel{?}{=} \frac{b(3 - 2(a+1))}{-3a + 3a^2 + 3b^2 - 2b^2}$$

$$\frac{4a-2}{6a-6a^2-2b^2} \stackrel{?}{=} \frac{1-2a}{-3a+3a^2+b^2}$$

$$\frac{2a-1}{3a-3a^2-b^2} \stackrel{?}{=} \frac{1-2a}{-3a+3a^2+b^2} \qquad \text{Vero!}$$

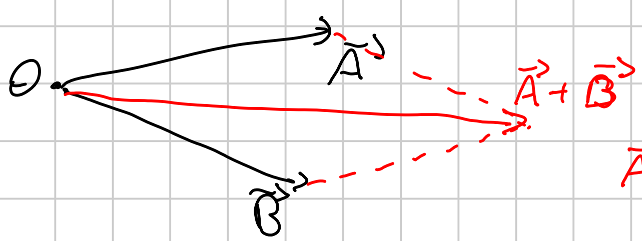
H, G, O allineati.

C'è di più: $\qquad y_H - y_G = -2(y_O - y_G)$

$\qquad x_H - x_G = -2(x_O - x_G) \Rightarrow HG = 2GO$

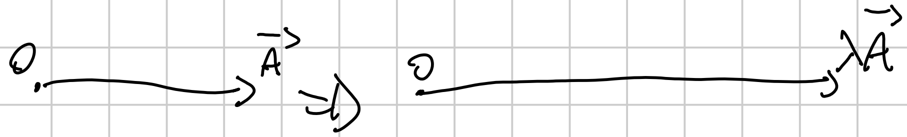
Vettori:

Somma:



$\vec{A} + \vec{B} = \vec{B} + \vec{A}$

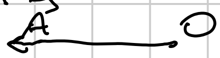
Moltiplicazione per scalare:



$\lambda = 0$

$\lambda = 1$ resta $\vec{A}$

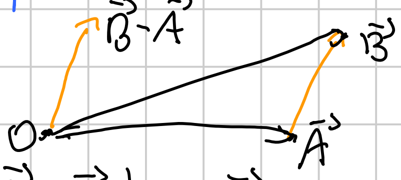
$\lambda = -1$ $-\vec{A}$



- retta del vettore $\vec{A}$

$\vec{B} = \lambda \vec{A}$

- retta per 2 due vettori, $\vec{A}, \vec{B}$

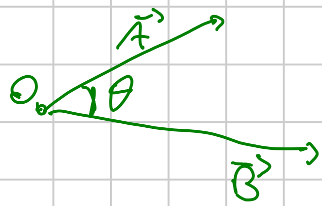


$r: \lambda(\vec{B} - \vec{A}) + \vec{A}$

$r: \lambda \vec{B} + (1-\lambda)\vec{A}$

SEGMENTO si ha $0 \leq \lambda \leq 1$

PRODOTTO SCALARE:



$\vec{A} \cdot \vec{B} = \|\vec{A}\| \cdot \|\vec{B}\| \cdot \cos\theta$

"quanto è lungo $\vec{A}$"

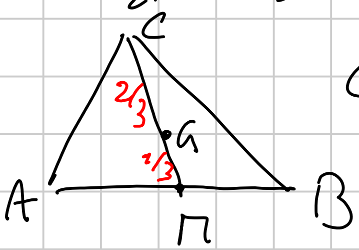
OSS: $\vec{A} \cdot \vec{A} = \|\vec{A}\|^2$

$\vec{A} \cdot (-\vec{A}) = -\|\vec{A}\|^2$

CALCOLIAMO UN PO' DI PT NOTI:

dati $\vec{A}$ e $\vec{B}$ il punto medio $\vec{M}$ è

$\vec{M} = \frac{1}{2}\vec{B} + (1-\frac{1}{2})\vec{A} = \frac{1}{2}\vec{A} + \frac{1}{2}\vec{B}$

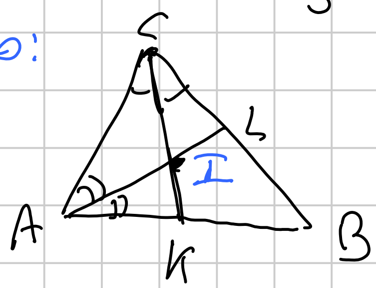


$CG = 2GM$

$\frac{CG}{CM} = 2/3$

$\vec{G} = \lambda \vec{C} + (1-\lambda)\vec{M} = \frac{1}{3}\vec{C} + \frac{2}{3}\vec{M} = \frac{\vec{A} + \vec{B} + \vec{C}}{3}$

Incentro:



$\frac{AK}{KB} = \frac{AC}{BC} = \frac{b}{a}$

$\frac{AK}{AB} = \frac{b}{a+b}$ $AK = \frac{bc}{a+b}$

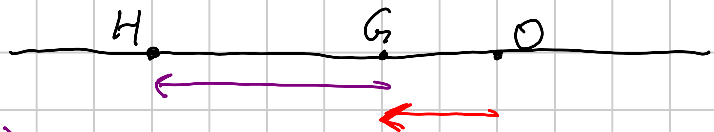
$\vec{K} = \frac{b}{a+b}\vec{B} + \frac{a}{a+b}\vec{A}$

$\frac{CI}{IK} = \frac{AC}{AK} = \frac{b}{\frac{bc}{a+b}} = \frac{a+b}{c}$

$$\frac{CI}{CK} = \frac{CI}{CI+IK} = \frac{1}{\frac{CI+IK}{CI}} = \frac{1}{1+\frac{IK}{CI}} = \frac{1}{1+\frac{c}{a+b}} = \frac{a+b}{a+b+c}$$

$$\vec{I} = \frac{a+b}{a+b+c}\vec{K} + \frac{c}{a+b+c}\vec{C} = \frac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c}$$

Ortocentro:



$$\vec{H}-\vec{G} = 2(\vec{G}-\vec{O})$$

$$\vec{H} = 3\vec{G} - 2\vec{O} = \vec{A}+\vec{B}+\vec{C} - 2\vec{O}$$

E' molto molto comodo prendere come origine del sistema vettoriale il circocentro O.

In questo caso si ha

$$\vec{H} = \vec{A}+\vec{B}+\vec{C}$$

OSS: $\vec{A}$ e $\vec{B}$ sono $\perp \iff \vec{A}\cdot\vec{B} = 0$

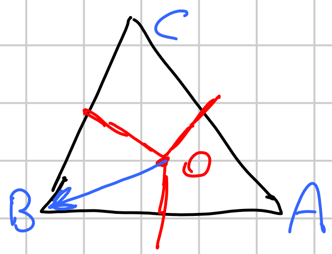
E' vero che $\vec{H} = \vec{A}+\vec{B}+\vec{C}$?

$$(\vec{H}-\vec{C})\cdot(\vec{B}-\vec{A}) \overset{?}{=} 0$$

$$(\vec{B}+\vec{A})\cdot(\vec{B}-\vec{A}) \overset{?}{=} 0$$

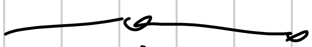
$$\vec{B}\cdot(\vec{B}-\vec{A}) + \vec{A}\cdot(\vec{B}-\vec{A}) = \vec{B}\cdot\vec{B} - \cancel{\vec{B}\cdot\vec{A}} + \cancel{\vec{A}\cdot\vec{B}} - \vec{A}\cdot\vec{A} \overset{?}{=} 0$$

$$\vec{B}\cdot\vec{B} = \|\vec{B}\|^2$$



$$\vec{B}\cdot\vec{B} = R^2 = \vec{A}\cdot\vec{A}$$

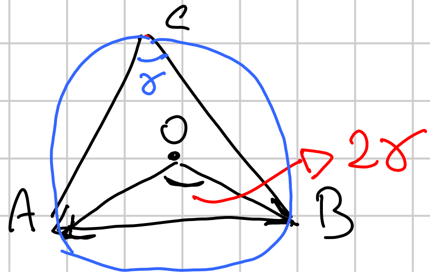
Quindi fa proprio 0.



$\overline{OH}$

$$OH^2 = \|\vec{OH}\|^2 = \vec{OH}\cdot\vec{OH} = (\vec{A}+\vec{B}+\vec{C})\cdot(\vec{A}+\vec{B}+\vec{C}) =$$

$$= \vec{A}\cdot\vec{A} + \vec{B}\cdot\vec{B} + \vec{C}\cdot\vec{C} + 2\vec{A}\cdot\vec{B} + 2\vec{A}\cdot\vec{C} + 2\vec{B}\cdot\vec{C}$$



$$\vec{A}\cdot\vec{B} = R^2\cos 2\gamma = R^2 - \frac{4R^2\sin^2\gamma}{2}$$

$$\cos 2\gamma = 1 - 2\sin^2\gamma$$

$$\vec{A}\cdot\vec{B} = R^2 - \frac{1}{2}\left[(2R\sin\gamma)^2\right] = R^2 - \frac{c^2}{2}$$

$$\vec{A}\cdot\vec{C} = R^2 - b^2/2 \qquad \vec{B}\cdot\vec{C} = R^2 - \frac{a^2}{2}$$

$$OH^2 = 3R^2 + 2\left(R^2 - \frac{c^2}{2}\right) + 2\left(R^2 - \frac{b^2}{2}\right) + 2\left(R^2 - \frac{a^2}{2}\right)$$

$$OH^2 = 9R^2 - (a^2+b^2+c^2).$$

CALCOLIAMO OI

$$OI^2 = \|\vec{OI}\|^2 = \vec{OI}\cdot\vec{OI} = \left(\frac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c}\right)\left(\frac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c}\right) =$$

$$= \frac{1}{(a+b+c)^2}\left[(a^2+b^2+c^2)R^2 + 2\left(ab\,\vec{A}\cdot\vec{B} + ac\,\vec{A}\cdot\vec{C} + bc\,\vec{B}\cdot\vec{C}\right)\right] =$$

$$= \frac{1}{(a+b+c)^2}\left[R^2(a^2+b^2+c^2) + 2R^2(ab+ac+bc) - abc(a+b+c)\right] =$$

$$= \frac{1}{(a+b+c)^2}\left[R^2(a+b+c)^2 - abc(a+b+c)\right] = R^2 - \frac{abc}{a+b+c}$$

$$R = \frac{abc}{4S} \qquad r = \frac{2S}{a+b+c} \implies 2Rr = \frac{abc}{a+b+c}$$

$$OI^2 = R^2 - 2Rr$$

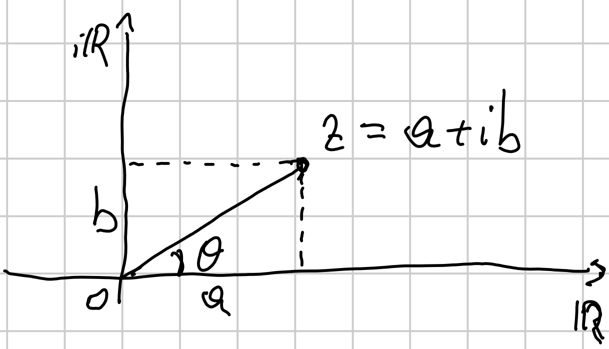
Abbiamo $R^2 - 2Rr \geq 0$

$$\boxed{R \geq 2r}$$

CALCOLATE $\overline{GH}$.

$$GH^2 = 4R^2 - \frac{4}{9}(a^2+b^2+c^2).$$

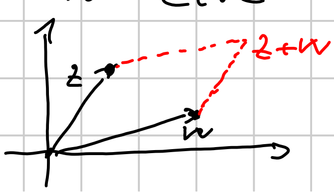
COMPLESSI



$a, b \in \mathbb{R}$

$$z = a+ib = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} = \left(\sqrt{a^2+b^2}\right)e^{i\theta}$$

$z = a+ib \qquad w = c+id \qquad z+w = (a+c)+i(b+d)$

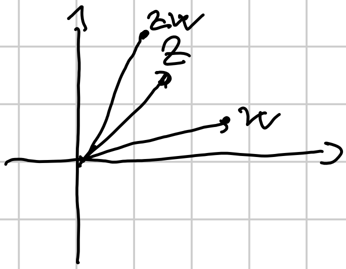


$$zw = ac + (ad+bc)i + i^2 bd \qquad i^2 = -1$$

$$zw = (ac - bd) + i(ad+bc)$$

$$z = |z| e^{i\theta} \qquad w = |w| e^{i\varphi}$$

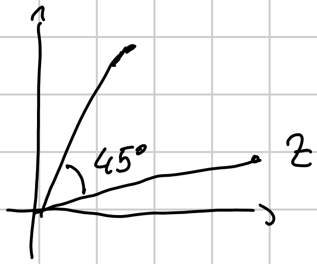
$$zw = |z| \cdot |w| \cdot e^{i\theta + i\varphi} = |z| \cdot |w| \cdot e^{i(\theta+\varphi)}$$

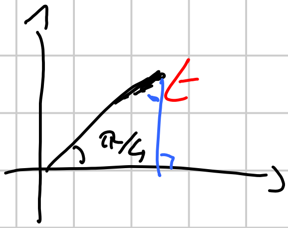


$$|zw| = |z| \cdot |w|$$

$$i = e^{i\theta} = e^{i\pi/2}$$

$$-1 = e^{i\pi}$$



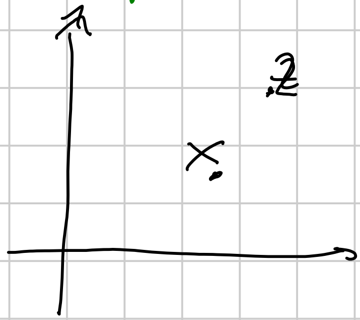


$t = a + ib \qquad |t| = 1 \Rightarrow a^2 + b^2 = 1$

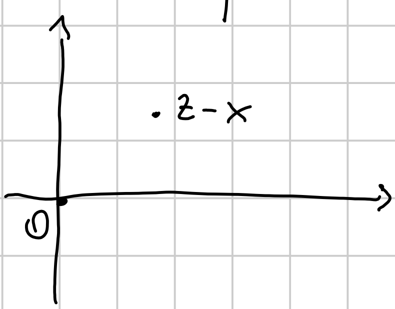
$a = b > 0 \qquad a = b = \frac{\sqrt{2}}{2} \qquad t = \frac{\sqrt{2}}{2}(1+i)$

Quindi per ruotare z di $45°$ intorno all'origine dobbiamo moltiplicarlo per $\frac{\sqrt{2}}{2}(1+i) = e^{i\pi/4}$.

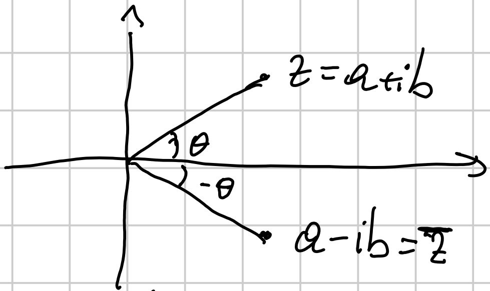
Dato x complesso come ruoto intorno ad x?



fissiamo θ



$$z' = (z-x)e^{i\theta} + x$$

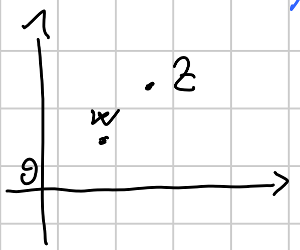


$\bar{z}$ si dice coniugato di z.

$$z\bar{z} = (a+ib)(a-ib) = a^2 - (ib)^2 = a^2 - i^2 b^2 = a^2 + b^2 = |z|^2 = |\bar{z}|^2$$

$$z = |z| e^{i\theta} \qquad \bar{z} = |z| e^{-i\theta}$$

Cosa vuol dire $z, w, 0$ allineati?

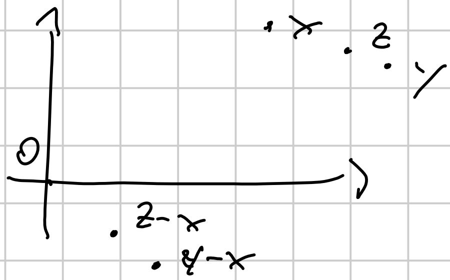


$$z = |z| e^{i\theta} \qquad w = |w| e^{i\varphi}$$

$$\frac{z}{\bar{z}} = e^{2i\theta} \qquad e^{2i(\pi+\theta)} = \underbrace{e^{2i\pi}}_{1} e^{2i\theta} = e^{2i\theta}$$

Sono allineati se e solo se $\dfrac{z}{\bar{z}} = \dfrac{w}{\bar{w}}$

Cosa vuol dire x, y, z allineati?



x, z, y allineati

$\Updownarrow$

$0, z-x, y-x$ allineati

$$\frac{z-x}{\overline{z-x}} = \frac{y-x}{\overline{y-x}}$$

$$\frac{z-x}{\bar{z}-\bar{x}} = \frac{y-x}{\bar{y}-\bar{x}}$$

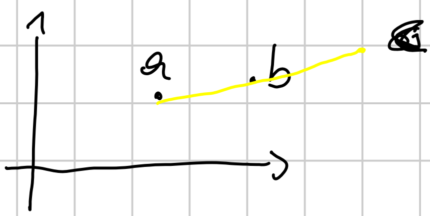
OSS: $\overline{w+t} = \bar{w} + \bar{t}$

$\overline{wt} = \bar{w}\bar{t}$

Come si trova il punto medio tra a e b?

$$m = \frac{a+b}{2}.$$

Se vogliamo riflettere un punto risp ad un altro, ad esempio a rispetto a b



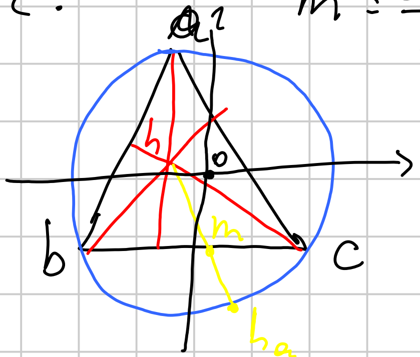
$$\frac{a+c}{2} = b \qquad c = 2b - a$$

Vogliamo riflettere $\underbrace{1+i}_{a}$ rispetto a $\underbrace{3+7i}_{b}$

$$c = 2b - a = 6+14i - 1 - i = 5+13i$$

Vogliamo dimostrare che il simmetrico dell'ortocentro H rispetto al punto medio di BC è sulla circoscritta ed è il diametralmente opposto ad A.

Sol Prendiamo come circoscritta la circonferenza unitaria. Allora $a\bar{a}=1$ $b\bar{b}=1$ $c\bar{c}=1$.
$o=0$ e $h=a+b+c$. $m=\frac{b+c}{2}$



$h_a = 2m - h$

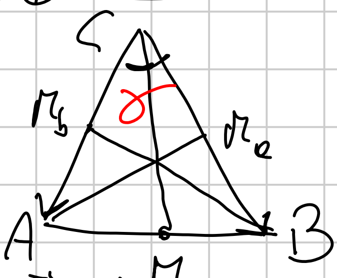
$h_a = b+c-a-b-c$

$h_a = -a$. $(-a)(\overline{-a}) = (-a)(-\bar{a}) = a\bar{a} = 1$

Inoltre $a+(-a)=0$ quindi $-a$ è il diametralmente opposto ad a.

Es: PI: 12, 22, 24
PII: 6, 10, 16
PIII: 2014 B1

P6



$$CM_c^2 + BM_B^2 + AM_A^2$$

$$\vec{M_c} = \frac{\vec{A}+\vec{B}}{2}$$

$$CM_c^2 = \vec{CM_c}\cdot\vec{CM_c} = (\vec{M_c}-\vec{C})(\vec{M_c}-\vec{C}) = \left(\frac{\vec{A}+\vec{B}}{2}-\vec{C}\right)\left(\frac{\vec{A}+\vec{B}}{2}-\vec{C}\right)$$

Origine vettoriale in $\vec{C}$

$$CM_c^2 = \frac{1}{4}(\vec{A}+\vec{B})(\vec{A}+\vec{B}) = \frac{1}{4}\vec{A}\cdot\vec{A} + \frac{1}{4}\vec{B}\cdot\vec{B} + \frac{1}{2}\vec{A}\cdot\vec{B} = \frac{1}{4}b^2 + \frac{1}{4}a^2 + \frac{1}{2}\vec{A}\cdot\vec{B}$$

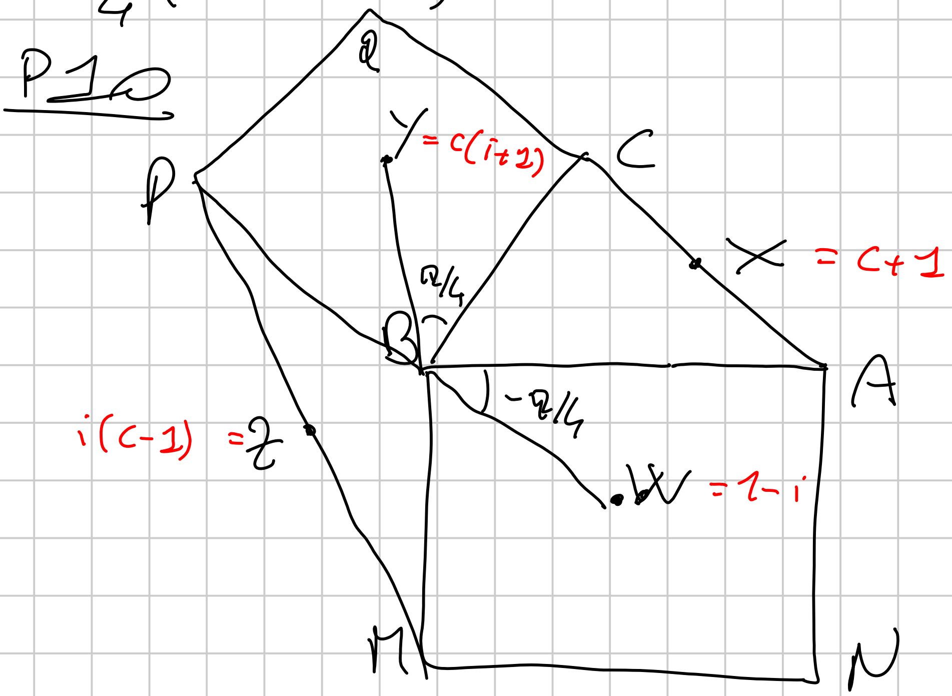
$$CM_c^2 = \frac{1}{4}a^2 + \frac{1}{4}b^2 + \frac{2}{4}ab\cos\gamma$$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad 2ab\cos\gamma = a^2 + b^2 - c^2$

$CM_c^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2)$

$CM_c^2 + AM_A^2 + BM_B^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2 + 2b^2 + 2c^2 - a^2 + 2a^2 + 2c^2 - b^2) =$
$= \frac{3}{4}(a^2 + b^2 + c^2)$

P10



$B = 0 \qquad A = 2 \qquad C = 2c$

$X = c + 1$

$BW = \frac{\sqrt{2}}{2} BA$

$w = 2 \cdot \frac{\sqrt{2}}{2} \cdot \left(\frac{\sqrt{2}}{2}(1-i)\right) = 1 - i$

$Y = 2c \cdot \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}(1+i)\right) = c(1+i)$

$m = 2(-i) = -2i$

$p = 2ci = 2ic$

$z = i(c-1)$

$w \overset{?}{=} i(y-x) + x$	$w \overset{?}{=} -i(y-z) + z$
$1 - i \overset{?}{=} i(c + ci - c - 1) + c + 1$	$1 - i = -i(c + ci - ci + i) + ci - i$
$1 - i \overset{?}{=} \cancel{ic} - \cancel{c} - \cancel{ic} - i + \cancel{c} + 1$	$1 - i = -\cancel{ci} + \cancel{c} - \cancel{c} + 1 + \cancel{ci} - i$

Allora YXW e YZW sono isosceli e rettangoli su base YW $\Rightarrow$ XYZW è quadrato.

P16

$\vec{M} = \frac{\vec{A}+\vec{B}}{2}$ $\quad \vec{N} = \frac{\vec{B}+\vec{C}}{2}$ $\quad \vec{O} = \frac{\vec{C}+\vec{D}}{2}$ $\quad \vec{P} = \frac{\vec{A}+\vec{D}}{2}$

$MO = NP \iff \|\vec{MO}\| = \|\vec{NP}\| \iff \|\vec{C}+\vec{D}-\vec{A}-\vec{B}\| = \|\vec{A}+\vec{D}-\vec{B}-\vec{C}\|$

$\vec{x} = \vec{C}-\vec{A} \quad \vec{y} = \vec{D}-\vec{B}$

$\Updownarrow$

$\|\vec{y}+\vec{x}\| = \|\vec{y}-\vec{x}\|$

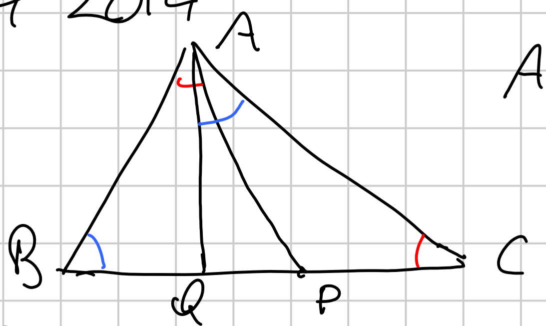
$\iff (\vec{y}+\vec{x})(\vec{y}+\vec{x}) = (\vec{y}-\vec{x})(\vec{y}-\vec{x}) \iff 4\vec{x}\cdot\vec{y} = 0 \iff \vec{x}\cdot\vec{y} = 0$

$\vec{x} = \vec{C}-\vec{A} = \vec{AC} \quad \vec{y} = \vec{BD}$

$\Updownarrow$

$AC \perp BD \iff \vec{AC}\cdot\vec{BD} = 0$

IMO 4 2014



$ABP \sim CBA$

$\frac{BP}{AB} = \frac{BA}{CB} \quad BP = \frac{c^2}{a}$

$\frac{BP}{BC} = \frac{c^2}{a^2}$

$\vec{P} = \frac{c^2}{a^2}\vec{C} + \frac{a^2-c^2}{a^2}\vec{B}$

$\vec{Q} = \frac{b^2}{a^2}\vec{B} + \frac{a^2-b^2}{a^2}\vec{C}$

$ACQ \sim BCA$

$\frac{CQ}{AC} = \frac{CA}{BC} \quad CQ = \frac{b^2}{a}$

$\vec{M} = 2\vec{P} - \vec{A} = \frac{2c^2}{a^2}\vec{C} + \frac{2a^2-2c^2}{a^2}\vec{B} - \vec{A}$

$\vec{N} = \frac{2b^2}{a^2}\vec{B} + \frac{2a^2-2b^2}{a^2}\vec{C} - \vec{A}$

$BM: \lambda\left(\frac{2c^2}{a^2}\vec{C} + \frac{2a^2-2c^2}{a^2}\vec{B} - \vec{A}\right) + (1-\lambda)\vec{B}$

$CN: \mu\left(\frac{2b^2}{a^2}\vec{B} + \frac{2a^2-2b^2}{a^2}\vec{C} - \vec{A}\right) + (1-\mu)\vec{C}$

$\vec{X} = \frac{1}{2b^2+2c^2-a^2}\left(-a^2\vec{A} + 2b^2\vec{B} + 2c^2\vec{C}\right)$

$\lambda = \frac{a^2}{2b^2+2c^2-a^2} = \mu$

Resta da verificare $\vec{X}\cdot\vec{X} \overset{?}{=} R^2$ (origine in O)

$R^2(2b^2+2c^2-a^2)^2 \overset{?}{=} \left(-a^2\vec{A}+2b^2\vec{B}+2c^2\vec{C}\right)\left(-a^2\vec{A}+2b^2\vec{B}+2c^2\vec{C}\right)$

$$RHS = (a^4+4b^4+4c^4)R^2 - 4a^2b^2\underbrace{\left(R^2-\frac{c^2}{2}\right)}_{\vec{A}\cdot\vec{B}} - 4a^2c^2\underbrace{\left(R^2-\frac{b^2}{2}\right)}_{\vec{A}\cdot\vec{C}} +$$

$$+\, 8b^2c^2\underbrace{\left(R^2-\frac{a^2}{2}\right)}_{\vec{B}\cdot\vec{C}}$$

$$RHS = R^2(2b^2+2c^2-a^2)^2 + \cancel{2a^2b^2c^2} + \cancel{2a^2b^2c^2} - \cancel{4a^2b^2c^2} = LHS$$

Abbiamo dimostrato che $ABXC$ è ciclico.