Note Title 9/5/2017

Algebra 2 — Disuguaglianze M

Scambret

- → Ripasso
- → Convessità
- → (Forte) ABC
- → Disuguaglianze tra frazioni (CS)
- → Disuguaglianze tra radici

— o — o —

Ripasso

Riarrangiamento

$$a_1 \le a_2 \le \dots \le a_n$$
$$b_1 \le b_2 \le \dots \le b_n$$

$$S = \sum a_i b_{\sigma(i)} \qquad \sigma : \{1, \dots, n\} \to \{1, \dots, n\}$$

max $\sigma(i) = i$

min $\sigma(i) = n+1-i$

Dim $\hat{\sigma}$ (Esiste?) e vogliamo trovare il max

$$j > i \quad \hat{\sigma}(j) < \hat{\sigma}(i)$$

$$a_i b_{\sigma(i)} + a_j b_{\sigma(j)}$$
$$\wedge$$
$$a_i b_{\sigma(j)} + a_j b_{\sigma(i)}$$

Ex $\quad x_1, \dots, x_n > 0$

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}} \geq n \qquad e \quad x_{n+1} = x_1$$

wlog $\quad x_1 \geq x_2 \geq \dots \geq x_n$ $\qquad$ NO!!!

$$\frac{1}{x_1} \leq \dots \leq \frac{1}{x_n}$$

$y_1 = \max\{x_i\}$

$y_2 = 2^\circ$ più grande $\{x_i\}$

$\vdots$

$y_n = \min\{x_i\}$

$$y_1 \geq \dots \geq y_n$$

$$\frac{1}{y_1} \leq \dots \leq \frac{1}{y_n} \qquad\qquad n \leq \sum y_i \cdot \frac{1}{y_{\sigma(i)}}$$

■

$a_1 \leq \dots \leq a_n, \qquad b_1 \leq \dots \leq b_n$

$$\left(\frac{1}{n} \sum a_i\right) \overset{!}{\cdot} \left(\frac{1}{n} \sum b_i\right) \leq \frac{1}{n} \sum a_i b_i$$

Dim $\quad a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a_1 b_1 + \dots + a_n b_n$

$a_1 b_1 + \dots + a_n b_n \underset{!}{\geq} a_1 b_2 + a_2 b_3 + \dots + a_n b_1$

$$\sum a_i b_i \geq \sum a_i b_{i+k}$$

$$n\sum a_i b_i \geq \left(\sum a_i\right)\left(\sum b_i\right)$$

<u>CS</u> a, b con componenti in $\mathbb{R}$

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$\left(\sum a_i b_i c_i\right)^3 \leq \ \_ \ \_ \ \_ \qquad \text{NO!}$$

$$3\cancel{abc} \leq a^3+b^3+c^3 \qquad b=0 \leftarrow$$

$$2ab \leq a^2+b^2 \qquad \leftarrow$$

<u>Dim</u>

i) $$P(x) = \sum_{i=1}^{n} (a_i + x b_i)^2 \overset{!}{\geq} 0$$

$$= x^2 \sum b_i^2 + 2x \sum a_i b_i + \sum a_i^2 \geq 0$$

$$\Delta = \left(\sum a_i b_i\right)^2 - \sum a_i^2 \sum b_i^2 \leq 0$$

ii) $$\left(\sum a_i^2\right)\left(\sum b_i^2\right) - \left(\sum a_i b_i\right)^2$$

$$= \boxed{\sum_{1\leq i<j\leq n} (a_i b_j - a_j b_i)^2} \geq 0$$

iii) $$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$\lambda_1 (a_1, \dots, a_n), \mu (b_1, \dots, b_n) \qquad \cancel{\lambda^2 \mu^2}\, LHS \leq \cancel{\lambda^2 \mu^2}\, RHS$$

wlog $$\sum a_i^2 = \sum b_i^2 = 1 \rightsquigarrow \sum a_i b_i \leq 1$$

$$ab \le \frac{a^2+b^2}{2}$$

$$\sum a_i b_i \le \frac{\sum a_i^2 + \sum b_i^2}{2} = 1$$

AGGIUSTARE I DETTAGLI

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$$ Young $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

$p, q > 1$

Parenti a, b, c positivi

$$\left(\sum a_i b_i c_i\right)^3 \le \left(\sum a_i^3\right)\left(\sum b_i^3\right)\left(\sum c_i^3\right)$$

$$abc \le \frac{a^3+b^3+c^3}{3}$$

$$\sum a_i b_i \le \left(\sum a_i^p\right)^{\frac{1}{p}} \left(\sum b_i^q\right)^{\frac{1}{q}}$$ $\frac{1}{p}+\frac{1}{q}=1$

$p, q > 1$

$$\frac{a+b}{2} \ge \sqrt{ab}$$

$a_i = \frac{c_i}{\sqrt{d_i}}$ $\quad b_i = \sqrt{d_i}$ e ottenete il Lemma di Titu !!!

— o — o —

Medie $a_1, \dots, a_n > 0$, $p \in \mathbb{R}$

$$f(p) = \sqrt[p]{\frac{a_1^p + \dots + a_n^p}{n}} = M_p$$

"Definiamo" $p = +\infty$ $\quad f(p) = \max\{a_i\}$

$p = -\infty$ $\quad f(p) = \min\{a_i\}$

$p = 0$ $\quad$ GM

$p > q \Rightarrow f(p) > f(q)$

Dim $+$ $AM \geq GM$ (e questo qui sistema il confronto p contro 0)

$*$ $AM \leq M_p$ $\quad p \geq 1$ (questo sistema p contro q, $p, q \neq 0$)

$AM \geq GM$ $\quad$ 1) Induzione $\quad n=2$ ok!

$n = 2$ $\quad n \Rightarrow 2n$!!! $\quad 2 \to 4 \to 8 \to 16$

$n \Rightarrow n-1$ $\quad \downarrow 15 \downarrow 14 \downarrow 13$

Tizio fittizio

$$\left(a_1, \dots, a_{n-1}, \frac{a_1 + \dots + a_{n-1}}{n-1} \right)$$

$$b_1, \dots, b_{n-1}, b_n$$

$$\frac{b_1 + \dots + b_n}{n} \geq \sqrt{b_1 \cdots b_n} \Longrightarrow \quad b_n = AM_{n-1}, \quad AM_{n-1} \geq GM_{n-1}$$

II) Jensen $\quad f\left(\sum \lambda_i x_i\right) \leq \sum \lambda_i f(x_i)$ $\quad f$ convessa, $\sum \lambda_i = 1$

$\lambda_i = \frac{1}{n}$ $\quad f(x) = \log x$

$$f\left(\sum \frac{x_i}{n}\right) \geq \frac{1}{n}\sum f(x_i)$$

$$\log\left(\frac{x_1+\dots+x_n}{n}\right) \geq \frac{\log x_1 + \dots + \log x_n}{n}$$

$$\overset{!}{=} \frac{\log(x_1 \cdots x_n)}{n}$$

$$\overset{!}{=} \log(x_1 \cdots x_n)^{\frac{1}{n}}$$

$$\overset{!}{\Longrightarrow} \frac{x_1+\dots+x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

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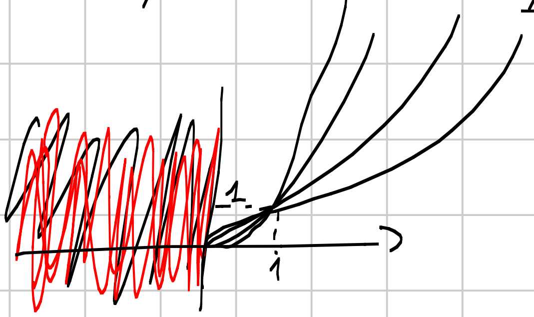
$AM \leq M_p \qquad p \geq 1$

Jensen $\quad f(x) = x^p \quad$ f convessa



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Bunching / Schur

$K \succ N \qquad k_1 \geq \dots \geq k_m$
$\qquad\qquad\quad n_1 \geq \dots \geq n_m$

$k_1 \geq n_1$

$k_2 + k_1 \geq n_2 + n_1$

$\vdots$

$k_{m-1} + k_{m-2} + \dots + k_1 \geq n_{m-1} + \dots + n_1$

$\sum k_i = \sum n_i$

$x_1, \dots, x_n \in \mathbb{R}^+$

$$\sum_{sym} x_i^{k_i} \geq \sum x_i^{n_i}$$

$$2(a^3+b^3+c^3) = \sum_{sym} a^3 b^0 c^0$$

$$\vee\!/$$

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b = \sum_{sym} a^2 b^1 c^0$$

$$\vee\!/$$

$$6abc = \sum_{sym} a^1 b^1 c^1$$

0, 1, 2

0, 0, 3

Dim

$$[3,0,0] \succ [2,1,0]$$

$$\frac{a^3+a^3+b^3}{3} \geq \sqrt[3]{a^6 b^3} = a^2 b$$

$$\frac{b^3+b^3+c^3}{3} \geq b^2 c$$

$$\frac{c^3+c^3+a^3}{3} \geq c^2 a$$

$$a^3+b^3+c^3 \geq a^2b + b^2c + c^2a$$

Schur

Forte + Debole $\geq$ Med(f)

$$\sum_{cyc} a(a-b)(a-c) \geq 0$$

vera

wlog $a \geq b \geq c$ No!

wlog $\max\{a,b,c\} = a$ Sì!

$$a^3+b^3+c^3+3abc \geq \sum_{sym} a^2b$$

$$\sum_{sym}(a^3+abc-2a^2b) \geq 0$$

$$\boxed{[3,0,0]+[1,1,1] \geq 2[2,1,0]}$$

$a=xy$ $b=yz$ $c=zx$

$$\sum_{cyc} a^m(a^n-b^n)(a^n-c^n) \geq 0$$

MICLE VARIANTI

BMO 12/2

$x,y,z \geq 0$

$$\sum_{cyc}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx)$$

Dm i) $(x+y)\sqrt{(z+x)(z+y)} \geq 2(xy)+yz+zx$!!!

ii) $x+y=a^2$
$y+z=b^2$
$z+x=c^2$

$$x=\frac{a^2+c^2-b^2}{2}$$

$$\sum_{cyc} a^2bc \geq \_\ \_\ \_\ \_$$

Schur

TST 09/6

$a_1,\dots,a_n>0$
$b_1,\dots,b_n>0$

$$a_1 \cdot \ldots a_n + b_1 \ldots b_n \leq (a_1^n + b_1^n)^{\frac{1}{n}} \ldots (a_n^n + b_n^n)^{\frac{1}{n}}$$

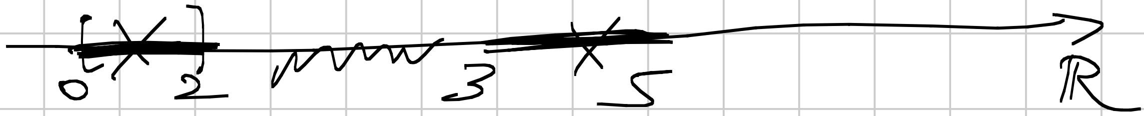
Dim

$$= \sqrt[n]{(a_1^n + b_1^n) \ldots (a_n^n + b_n^n)}$$

$\doteq$ Teji di CS a n vettori di $\mathbb{R}_{+}^{2}$

Jensen / Convessità

$I \subseteq \mathbb{R}$ se $\forall a,b \in I$ tutto il segmento $AB \in I$
I si dice convesso

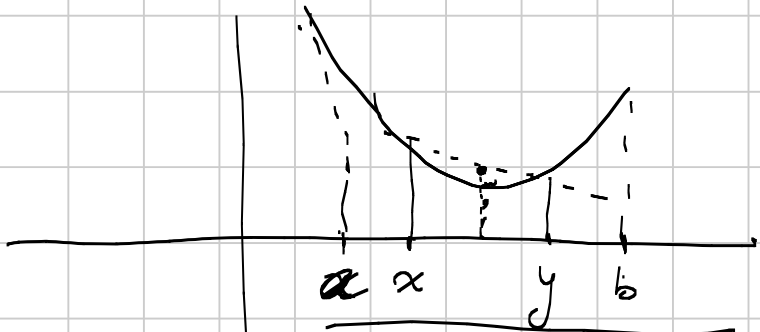


$f: I \to \mathbb{R}$ f convessa se $\forall x,y \in I$ e $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

$[\lambda = \frac{1}{2}]$

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}[f(x) + f(y)]$$



$f''(x) \geq 0 \quad \forall x \in I \Rightarrow \boxed{f \text{ è convessa}}$

- x^r $\to$ $r \geq 1$ f è convessa $[0, +\infty)$
  $\to$ $r < 1, r > 0$ f è concava $[0, +\infty)$

$r \geq 1$ $\quad f''(x) = \underbrace{r(r-1)}\underbrace{x^{r-2}}_{>0} \Rightarrow f$ è convessa

$0 < r \leq 1$ $\quad f$ è concava

$\log(x)$ è concavo $(0, +\infty)$

$-\frac{1}{x^2} < 0$

e^x è convessa in $\mathbb{R}$

$e^x > 0$

f, g convesse $\quad c > 0 \implies \left.\begin{array}{l} c \cdot f \\ f(ax+b) \\ f+g \end{array}\right\}$ convesse

Problema

$$a, b, c > 0$$

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}}$$

$$a \ln a + b \ln b + c \ln c \geq \frac{a+b+c}{3} \ln\left(\frac{a+b+c}{3}\right)$$

$f(x) = x \ln x$? si $\quad \dots \quad f''(x) = \frac{1}{x} > 0$

per Jensen
si finisce

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \qquad a, b, c > 0$$

Dim $\quad a+b+c = 1 \quad$ wlog

$$\frac{9}{2} \le \sum_{cyc} \frac{1}{1-a}$$

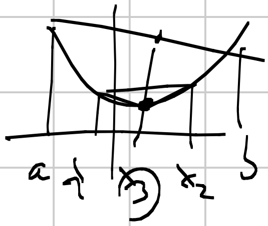
$$f(x) = \frac{1}{1-x}$$ convessa $(0,1)$

$$\frac{1}{x}$$

$f(ax+b)$ OK

$f(x) =$

END POINT CONVEX



f convessa $[a,b]$

$$\boxed{\max\{f(x) : x \in [a,b]\} = \max\{f(a), f(b)\}}$$

Step I Il massimo esiste!

Step II

$$f(\lambda x + \mu y) \overset{!}{\le} \lambda f(x) + \mu f(y) \quad \forall x,y \in I$$

$$\mu + \lambda = 1.$$

$x = a,\ y = b$

AGGIUSTARE I DETTAGLI...

i) Bulgaria 1995

$$x_1 + \dots + x_n - (x_1x_2 + \dots + x_nx_1) \le \left\lfloor \frac{n}{2} \right\rfloor$$

$$0 \le x_1, \dots, x_n \le 1$$

$x_1 - x_1x_2 - x_1x_n + \text{roba}$ $\qquad x_i = \{0,1\}$

$$\underbrace{x_1(1-x_2)}_{} + x_2(1-x_3) + \dots + x_n(1-x_1)$$

$x_1 = 0$ o $x_2 = 1$ $\qquad 1 \iff x_i = 1$ e $x_{i+1} = 0$

ii) USAMO 80/5

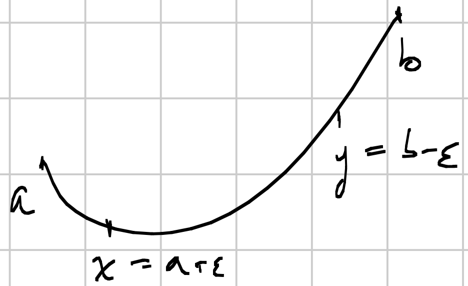
$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \le 1$$

$0 \le a,b,c \le 1.$

$$\frac{\text{roba}_2}{\text{roba}+a}$$

— o — o —

Smoothing



$a+b = x+y$

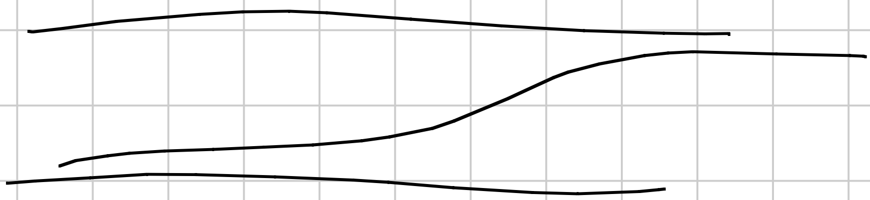
$x \le y$

$$f(a)+f(b) \ge f(a+\varepsilon)+f(b-\varepsilon)$$

Teorema Non esiste nessun numero reale >1.

Dim: Sia M $\qquad M^2 > M > 1$ ■

NO!



$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$(a, b, c) \to \left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$$ NO!

Wlog $a \leq b \leq c$

$$(a, b, c) \to (a+\varepsilon, b, c-\varepsilon)$$

$$\varepsilon_1 = \frac{a+b+c}{3} - a$$

$$\left(\frac{a+b+c}{3}\right)^2 \geq \sum_{cyc} a\sqrt{bc}$$

$$c\sqrt{ab} + \sqrt{c}\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)$$

$$(a, b, c) \to \left(\frac{a+b+c}{3}, b, c-\varepsilon_1\right)$$!

!

<u>Karamata</u> $\Rightarrow$ <u>Jensen</u>

$X \succ Y$ $\qquad$ f convessa

$$\sum f(x_i) \geq \sum f(y_i)$$

Disuguaglianze tra frazioni (CS)

TT 2017.1

$x, y, z > 0$

$$\frac{x}{2x+y} + \frac{y}{3y+z} + \frac{z}{4z+x} \geq k$$

$a, b, c > 0$

$$\sum_{\cancel{cyc}} \frac{a}{b+c} \geq \frac{3}{2} \qquad \left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$\sqrt{\frac{a}{b+c}}, \quad \sqrt{a(b+c)} \qquad a_i b_i = \sqrt{\frac{a}{b+c}}\sqrt{b+c}\sqrt{a}$$

$$(\text{TESTO})\underbrace{(ab+ac+bc+ba+ca+cb)}_{\text{Den}} \geq (a+b+c)^2$$

$$\text{TESTO} \overset{!}{\geq} \frac{(a+b+c)^2}{2(ab+bc+ca)} \overset{?}{\geq} \frac{3}{2} \qquad \text{ok}$$

Brillante

$$\frac{z}{4z+x} \geq \frac{z}{4(x+y+z)}$$

Bovino

$$\sqrt{\frac{x}{2x+y}} \qquad \sqrt{x(2x+y)}$$

$$\text{TESTO} \geq \frac{(x+y+z)^2}{(2x^2+xy+3y^2+yz+4z^2+zx)} \geq k$$

$1 \quad 1 \quad 1 \qquad\qquad k\cdot 2 \quad k\cdot 3 \quad k\cdot 4$

$$x^2+y^2+z^2+2(xy+yz+zx)$$

$$\geq \underline{k(2x^2)} + \underline{k(3y^2)} + k(4z^2)$$
$$+ k(xy+yz+zx)$$

$1 \geq 2k$ ✓

$1 \geq 3k$ ✓ $\qquad k \leq \frac{1}{4}$

$1 \geq 4k$ ✓

$\frac{1}{4} + a$

IMO 95/2

$abc = 1 \qquad \sum_{cyc} \frac{1}{a^3(b+c)} \geq \frac{3}{2} \qquad a = \frac{1}{bc}$

$\frac{1}{a} = bc$

$$\sqrt{\frac{1}{a^3(b+c)}}, \quad \sqrt{a(b+c)}$$

$$\text{TESTO} \geq \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{2(ab+bc+ca)} \geq \frac{3}{2}$$

$ab+bc+ca \geq 3 \qquad abc = 1 \qquad$ ok!

$$\boxed{\frac{a+b+c}{3} \geq \sqrt{\frac{ab+bc+ca}{3}} \geq \sqrt[3]{abc}}$$

IMO 05/3

$xyz \geq 1$ $$\sum_{cyc} \frac{x^5 - x^2}{x^5+y^2+z^2} \geq 0$$

$$\sum_{cyc} \frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq 3$$

$x^5+y^2+z^2 - x^2 - y^2 - z^2$

$$\sum_{cyc} \frac{1}{x^5+y^2+z^2} \overset{!}{\leq} \frac{3}{x^2+y^2+z^2}$$

$(\sqrt{x^5}, y, z)$ e $(\sqrt{\frac{1}{x}}, y, z)$ $\qquad$ $xyz \geq 1$, $yz \geq \frac{1}{x}$

$$(x^5+y^2+z^2)\left(\frac{1}{x}+y^2+z^2\right) \geq (x^2+y^2+z^2)^2$$

$$\Rightarrow \frac{1}{x^5+y^2+z^2} \leq \frac{\frac{1}{x}+y^2+z^2}{(x^2+y^2+z^2)^2} \leq \frac{yz+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\sum_{cyc} \frac{x^2+y^2+z^2}{x^5+y^2+z^2} \leq \sum \frac{yz+y^2+z^2}{x^2+y^2+z^2} = 2 + \frac{xy+yz+zx}{x^2+y^2+z^2} \leq 3 \quad \checkmark$$

IMO 01/2

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

Brillante

$\lambda = \frac{4}{3}$

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq \sum_{cyc} \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}} = 1$$

$$\frac{a^2}{a^2+b^2+c^2} \qquad \sum_{cyc}^{\text{CASA}} \frac{a}{b+2c+d} \geq 1$$

i) $\left(\frac{\sqrt{a}}{\sqrt[4]{a^2+8bc}}\right), \quad \left(\sqrt[4]{a^2+8bc}\,\sqrt{a}\right)$

$$\left(\sum \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum a\sqrt{a^2+8bc}\right) \overset{!}{\geq} \left(\sum a\right)^2$$

$$\left(\sum a\right)^2 \geq \underbrace{\sum a\sqrt{a^2+8bc}}\ \Rightarrow \boxed{\text{Tesi}}$$

1) Jensen $\quad \sum a\sqrt{a^2+8bc} \overset{!}{\leq} \sum a\sqrt{a^2+4(\underbrace{b^2+c^2}_{1-4a^2})} \overset{!}{\leq} \left(\sum a\right)^2$

$\wedge\!\!\mid \quad \leq$

$3 \leq \left(\sum a\right)^2 \qquad \sum a^2 = 1$

$\geq$

$x\sqrt{1-3x^2}$

2) Sostituzione $\quad \left(\sum a\right)^2 \geq \sum a\sqrt{a^2+8bc}$

$a^2+8bc = z^2 \qquad a$

$b^2+8ca = x^2$

$c^2+8ab = y^2$

3) Termine a termine $\quad a(a+b+c)+b(b+c+a)+\dots$

$$\cancel{a}(a+b+c) \geq \cancel{a}\sqrt{a^2+8bc}$$

$$\cancel{a^2}+\cancel{b^2}+\cancel{c^2}+\underline{2ab}+2bc+\underline{2ca} \geq \cancel{a^2}+\underline{8bc}$$

$$ab + ca \geq 2bc$$

$$a\sqrt{a^2+8bc} \overset{!}{\leq} \frac{a^2+a^2+8bc}{2} = a^2 + 4bc$$

Point of Incidence

1) AM-GM

$$3a\sqrt{a^2+8bc} \leq \frac{9a^2+a^2+8bc}{2} = 5a^2+4bc$$

$$a\sqrt{a^2+8bc} \leq \frac{5}{3}a^2 + \frac{4}{3}bc$$

$$\frac{5}{3}(a^2+b^2+c^2) + \frac{4}{3}(ab+bc+ca)$$

$$\leq (a^2+b^2+c^2) + 2(ab+bc+ca)$$ NO!

$$\left(\sum a\right)^2 \geq \sum a\sqrt{a^2+8bc}$$

$$2ab\sqrt{a^2+8bc}\sqrt{b^2+8ca}$$

~~3) (Power)~~

CS

5) "Idea" Jensen

omogeneizzato

$$abc = 1$$

$$\overset{!}{} \quad \frac{a^2+8}{2} \qquad \sqrt{a^4+8a}$$

$$\sum a^2 + \frac{2}{a} \overset{!}{\geq} \sum\sqrt{a^4+8a}$$

$$\cancel{a^4} + \cancel{4a} + \frac{4}{a^2} \geq \cancel{a^4} + 8/a$$

$$a^3 \leq 1 \rightarrow a \leq 1$$

$$a^4 + a + a + \dots + a$$

$$a^4 + 2a$$

$x = a+b$
$y = b+c$
$z = c+a$

$a^2 + 8bc = \cancel{z}w^2$

$\left(\frac{x+z-y}{2}\right)^2 + 8\left(\frac{y+z-x}{2}\right)\left(\frac{x+y-z}{2}\right)$

$= \boxed{}$ boh...

$$\left(\sum a\right)^2 \geq \sum a\sqrt{a^2+8bc}$$

>1

6) $\left(\sum a\right)^2 \geq \sum a\sqrt{a^2+8bc}$

CS

$a \geq b \geq c$

$a\sqrt{a^2+\frac{8}{a}} = \sqrt{a}$

$2 - \frac{4}{a^2}$ $\rightsquigarrow$

$\left(a \quad , \sqrt{a^2+8bc}\right)$

$$\sum a\sqrt{a^2+8bc} \leq \sqrt{\left(\sum a^2\right)}\sqrt{\left(\sum a^2+8bc\right)} \leq \left(\sum a\right)^2$$

$$\underbrace{\left(\sum a^2\right)}\underbrace{\left(\sum a^2+8bc\right)} \leq \underbrace{\left(\sum a\right)}^4$$ no!

$$\left(\sqrt{x}, \sqrt{a^3+8abc}\right) = \left(\frac{a}{\sqrt{x}}, \sqrt{x}\sqrt{a^2+8bc}\right)$$

$$\sum a\sqrt{a^2+8bc} \leq \sqrt{\left(\sum a\right)}\sqrt{\sum\left(a^3+8abc\right)} \leq \left(\sum a\right)^2$$

$$\sum (a^3 + 8abc) \leq \left(\sum a\right)^3$$

$$\sum a^3 + \underline{\underline{24abc}} \leq \sum a^3 +$$

Disuguaglianze tra radici

$LHS \le RHS$

i) $LHS \le c \le RHS \qquad c \in \mathbb{R}$

ii) Fondere le radici " $a\sqrt{a^2+8bc}$ "

iii) Maggiorare termine a termine

$$\sum_{cyc} \frac{a}{\sqrt{(a+b)(a+c)}} \le \frac{3}{2}$$

$$\frac{a}{\sqrt{(a+b)(a+c)}} \le \frac{1}{2} \quad !!!$$

$$\cancel{4}^{\,3} a^2 \le \cancel{a^2} + ab + ac + bc$$

$$\cancel{a}(b+c+a) \ge 3a^{\cancel{2}}$$

$$b+c \ge 2a$$

$$\frac{\cancel{a}}{\sqrt{(a+b)(a+c)}} \le \frac{3}{2} \frac{\cancel{a}}{a+b+c}$$

$$2(a+b+c) \le 3\sqrt{(a+b)(a+c)}$$

$$8ca + 8bc + 8ab + 4a^2 + \underline{4b^2+4c^2} \le 9a^2 + 9ab + 9ac + 9bc$$

$$\frac{3}{4} \frac{b+c}{a+b+c}$$

$$a+b = x^2$$
$$b+c = y^2$$
$$c+a = z^2$$

$$\sum_{cyc} \frac{x^2+z^2-y^2}{xz} \le 3$$

$$\sum_{cyc} y(x^2+z^2-y^2) \le 3xyz$$

$\boxed{\text{Schur}}$

$$\underbrace{\sum a\sqrt{b+c}} \le \frac{3}{2}\sqrt{a+b}\sqrt{b+c}\sqrt{c+a}$$

CS $\quad (a, \sqrt{b+c}) \leftarrow$ CONTI $\geq$

$(\sqrt{a}, \sqrt{ab+ac}) \leftarrow \quad \left(\frac{a}{\sqrt{a}}, \sqrt{a}\sqrt{b+c}\right)$

$$\sum a\sqrt{b+c} \overset{?}{\le} \sqrt{\left(\sum a\right)}\sqrt{\sum ab+ac} \overset{?}{\le} \frac{3}{2}\sqrt{a+b}\sqrt{b+c}\sqrt{c+a}$$

$$4\sum a \cdot 2\sum ab \le 3(a+b)(b+c)(c+a) \checkmark$$

Vera!

ii) $a+b+c=1 \qquad a,b,c \geq 0$

$$\underbrace{\sum \sqrt{1-a}} \le \sqrt{2}\left(\sqrt{\sum ab} + 2\sqrt{\sum a^2}\right)$$

Come si fa? $\qquad f(x) = \sqrt{1-x}$

RENDERSENE CONTO!

$$\sum \sqrt{b+c} = LHS$$

$$\boxed{\sqrt{A}+\sqrt{B}+\sqrt{C} \le \sqrt{3}\sqrt{A+B+C}} = \sqrt{3}\sqrt{3-a-b-c} = \sqrt{6}$$

$$\left(1,1,1\right) \quad \left(\sqrt{A}, \sqrt{B}, \sqrt{C}\right)$$

$$LHS \le \sqrt{6}$$

$$\cancel{\sqrt{6}}^{\sqrt{3}} \le \cancel{\sqrt{2}}\left(\sqrt{\sum ab} + 2\sqrt{\sum a^2}\right) \qquad \boxed{a+b+c=1}$$

$$\sum ab + 4\sum a^2 + 4\sqrt{\sum ab}\sqrt{\sum a^2} \ge 3$$

$$a^2+b^2+c^2 = 1-2(ab+bc+ca)$$

$$a+b+c=1 \qquad \sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3} \qquad ab+bc+ca \le \frac{1}{3}$$

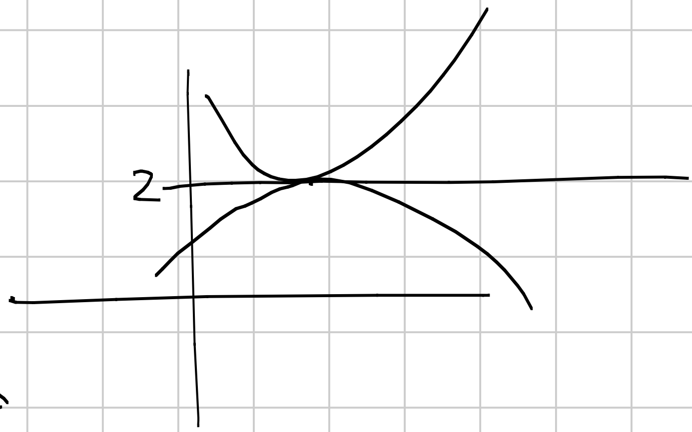
$$4 - 8\sum ab + \sum ab + 4\sqrt{\sum ab}\sqrt{\sum a^2} \ge 3$$

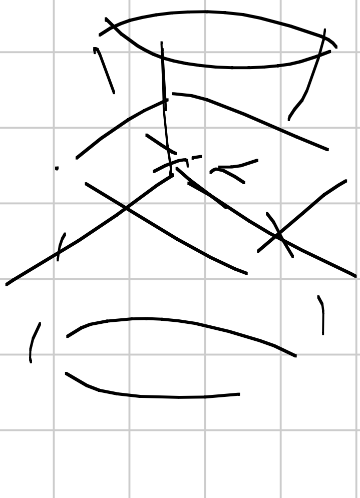
$$1 + 4\sqrt{\sum ab}\sqrt{\sum a^2} \ge 7\sum ab$$

$$a^2+b^2+c^2+2(ab+bc+ca)$$

$$\sum a^2 + \boxed{4\sqrt{\sum ab}\sqrt{\sum a^2}} \ge 5\sum ab$$

$$\sum a^2 \ge \sum ab$$





IMO SL 09/A4

$ab + bc + ca \le 3abc$ $\qquad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le 3$

$$\Longrightarrow \sum \sqrt{\frac{a^2+b^2}{a+b}} + 3 \le \sqrt{2} \sum \sqrt{a+b}$$

$$\boxed{\sqrt{2}\sqrt{a+b} \overset{?}{\ge} \sqrt{\frac{a^2+b^2}{a+b}} + 1}$$

$$\sqrt{2}(a+b) \ge \sqrt{a^2+b^2} + \sqrt{a+b}$$

$$2a^2 + 4ab + 2b^2$$

$$\sqrt{\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}} \ge \sqrt{3}$$

$$a^2 > a$$

$$\sqrt{2}\sqrt{a+b} \ge \sqrt{\frac{a^2+b^2}{a+b}} \; + \; \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \le \ldots \quad \text{no!}$$

$$\sqrt{2}\sqrt{a+b} \qquad \sqrt{\frac{a^2+b^2}{a+b}}$$

$$\sqrt{\frac{a^2+b^2}{2}} \cdot \frac{1}{\sqrt{a+b}}$$

CS $\qquad \sqrt{2}\sqrt{a+b} \qquad \sqrt{\frac{a^2+b^2}{a+b}}$

$$= \sqrt{2}\sqrt{\frac{a^2+b^2+2ab}{a+b}} \underset{\boxed{CS}}{\ge} \sqrt{\frac{a^2+b^2}{a+b}} + \sqrt{\frac{2ab}{a^2+b^2}}$$

$$\sum \sqrt{\frac{2ab}{a+b}} \ge 3 \qquad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le 3$$

Brillante

$$M_{\frac{1}{2}} \ge M_{-\frac{1}{2}}$$

Bovina $\sum \sqrt{\frac{2}{x+y}} \geq 3$

PER CASA

$\frac{1}{a} = x$

$x+y+z \leq 3$