G1 ADVANCED LucaMac

Titolo nota 03/09/2018

- Parallele

$x+y+z=0$

BMO 2015.2

ABC scaleno, I incentro e ω circoscritta. $AI \cap \omega = \{A, D\}$, $BI \cap \omega = \{B, E\}$; $CI \cap \omega = \{C, F\}$. La parallela a BC per I interseca EF in K. Analogamente si definiscono L, M.

<u>Tesi:</u> K, L, M allineati.

Soluzione

$AI:\ yc = zb \qquad \omega: \sum_{cyc} a^2yz = 0$

$D = (-a^2, b(b+c), c(b+c))$

$E = (a(a+c), -b^2, c(a+c))$

$F = (a(a+b), b(a+b), -c^2)$

$EF: -xbc\cancel{a}\cancel{\left(\sum_{cyc} a\right)} + y\cancel{a}c(a+c)\cancel{\left(\sum_{cyc} a\right)} + z\cancel{a}b(a+b)\cancel{\left(\sum_{cyc} a\right)} = 0$

$EF:\ -xbc + yc(a+c) + zb(a+b) = 0$

$\infty_{BC} = (0, 1, -1)$

$I = (a, b, c)$

$I\infty_{BC}:\ x(b+c) - ya - za = 0$

$K = (a(b-c), b^2, -c^2)$

$L = (a^2, -b^2, c(a-b))$

$M = (-a^2, b(c-a), c^2)$

Tesi diventa $\det\begin{pmatrix} \cancel{a}(b-c) & b^{\cancel{2}} & -c^{\cancel{2}} \\ a^{\cancel{2}} & -b^{\cancel{2}} & \cancel{c}(a-b) \\ -a^{\cancel{2}} & \cancel{b}(c-a) & c^{\cancel{2}} \end{pmatrix} \stackrel{?}{=} 0$

o si fa il determinante

$$-bc(b-c) - ab(a-b) - ac(c-a) \stackrel{?}{=} (b-c)(a-b)(c-a)$$

$$-\sum_{cyc} a^2 b + \sum_{cyc} ab^2 \stackrel{?}{=} -\sum_{cyc} a^2 b + \sum_{cyc} ab^2 \quad \underline{\text{vera!}}$$

CONIUGATI (in ABC)

ISOGONALI $(\alpha, \beta, \gamma) \longleftrightarrow \left(\frac{a^2}{\alpha}, \frac{b^2}{\beta}, \frac{c^2}{\gamma}\right)$

ISOTOMICI $(\alpha, \beta, \gamma) \longleftrightarrow \left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$

DISTANZE

$P_1 ; P_2$ normalizzati $\sum_{cyc} x = 1$

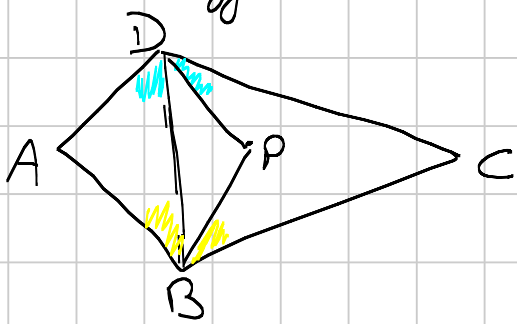
$\vec{P_1P_2} = (x_1, y_1, z_1) \quad \sum_{cyc} x_1 = 0$

$P_1P_2^2 = -\sum_{cyc} a^2 y_1 z_1$

IMO 2004.5

ABCD quadrilatero convesso tale che BD NON è bisettrice né di $A\hat{B}C$ né di $A\hat{D}C$. P interno ad ABCD con $P\hat{B}C = D\hat{B}A$ e $P\hat{D}C = B\hat{D}A$.

Tesi: ABCD ciclico iff $AP = CP$



Soluzione

1° oss A e C coniugati isogonali in PBD,

E PBD non degenere perché BD non è bisettrice.

Facciamo baricentriche su PBD.

$BD = a \quad PB = c \quad PD = b$.

$A = (\alpha, \beta, \gamma)$ $\qquad$ $C = \left(\frac{a^2}{\alpha}, \frac{b^2}{\beta}, \frac{c^2}{\gamma}\right) = (a^2\beta\gamma, b^2\alpha\gamma, c^2\alpha\beta)$

oss: $\alpha\beta\gamma \neq 0$

$\boxed{1}$ ABCD ciclico

$$\Gamma: \sum_{cyc} a^2yz = \left(\sum_{cyc} x\right) ux$$

$$A \to \sum_{cyc} a^2\beta\gamma = \left(\sum_{cyc} \alpha\right) \alpha u$$

$$C \to \cancel{a^2} b^2c^2\alpha\cancel{\beta\gamma} \sum_{cyc} \alpha = \left(\sum_{cyc} a^2\beta\gamma\right) \cancel{a^2\beta}\gamma u$$

ABCD ciclico iff $\boxed{\alpha^2b^2c^2\left(\sum_{cyc}\alpha\right)^2 = \left(\sum_{cyc} a^2\beta\gamma\right)^2}$

$\boxed{2}$ $AP = CP$ ($\iff AP^2 = CP^2$)

$P = (1, 0, 0)$

$A = \frac{1}{\sum\alpha}(\alpha, \beta, \gamma)$ $\qquad$ $\vec{AP} = \left(\frac{\beta+\gamma}{\sum\alpha}, \frac{-\beta}{\sum\alpha}, \frac{-\gamma}{\sum\alpha}\right)$

$$AP^2 = \frac{1}{(\sum\alpha)^2}\left(-a^2\beta\gamma + (\beta+\gamma)(\beta c^2 + \gamma b^2)\right)$$

$$CP^2 = \frac{1}{(\sum a^2\beta\gamma)^2}\left(-a^2b^2c^2\alpha^2\beta\gamma + (b^2\alpha\gamma + c^2\alpha\beta)(b^2c^2)(\alpha\gamma + \alpha\beta)\right)$$

$$= \frac{b^2c^2\alpha^2}{(\sum a^2\beta\gamma)^2}\left(\underbrace{-a^2\beta\gamma + (b^2\gamma + c^2\beta)(\gamma+\beta)}_{\text{schifo}}\right)$$

Quindi diventa la cosa di prima dopo aver dimostrato che schifo $\neq 0$

schifo $\neq 0$ perché $A \neq P$. ☺

POTENZE

Sempre prendiamo le coordinate normalizzate, diciamo $P = (x, y, z)$

$\Gamma: \sum_{cyc} a^2yz = (\sum x)(\sum ux)$ per opportuni $u, v, w \in \mathbb{R}$

$$Pow_\Gamma(P) = -\sum_{cyc} a^2yz + \left(\sum_{cyc} x\right)\left(\sum_{cyc} ux\right)$$

IMO SL 2011 G2

A_1, A_2, A_3, A_4 non conciclici, O_i ed r_i centro e raggio di $\odot\{A_j\}_{j\neq i}$

<u>Tesi:</u> $\displaystyle\sum_{cyc}' \frac{1}{O_iA_i^2 - r_i^2} = 0$

Soluzione

$$O_iA_i^2 - r_i^2 = Pow_{\Gamma_i}(A_i)$$

Tesi equivale a $\displaystyle\sum_{cyc}' \frac{1}{Pow_{\Gamma_i}(A_i)} \stackrel{?}{=} 0$

Riferimento $A_1A_2A_3$ $\quad A_1A_2 = c \quad A_1A_3 = b \quad A_2A_3 = a$

$A_4 = (\alpha, \beta, \gamma) \quad \sum_{cyc} \alpha \neq 0$

$\Gamma_4 : -\sum_{cyc} a^2 yz = 0 \qquad \Gamma_3 : -\sum_{cyc}' a^2 yz + \left(\sum_{cyc}' x\right) z \cdot \dfrac{\sum_{cyc}' a^2\beta\gamma}{\gamma \sum_{cyc} \alpha} = 0$

$$Pow_{\Gamma_4}(A_4) = -\left(\sum_{cyc}' a^2\beta\gamma\right) \cdot \frac{1}{\left(\sum_{cyc}' \alpha\right)^2}$$

$$Pow_{\Gamma_3}(A_3) = \frac{\sum_{cyc}' a^2\beta\gamma}{\gamma \sum_{cyc} \alpha}$$

Tesi equivale a $\displaystyle -\frac{\left(\sum_{cyc}' \alpha\right)^2}{\sum_{cyc}' a^2\beta\gamma} + \sum_{cyc} \left(\frac{\gamma \sum_{cyc} \alpha}{\sum_{cyc}' a^2\beta\gamma} \right) \stackrel{?}{=} 0$ <u>vero!</u>

Perpendicolari

$\vec{v_1} = (x_1, y_1, z_1) \qquad \vec{v_2} = (x_2, y_2, z_2) \qquad \sum_{cyc}' x_1 = 0$

$\vec{v_1} \perp \vec{v_2}$ iff $-\sum_{cyc}' a^2(y_1z_2 + y_2z_1) = 0$

<u>Dim:</u> $\vec{v_1} = x_1\vec{A} + y_1\vec{B} + z_1\vec{C}$

$\vec{v_1} \cdot \vec{v_2} = \underbrace{\cancel{(x_1+y_1+z_1)}}_{0}(x_2+y_2+z_2)R^2 - \frac{1}{2}\sum_{cyc} a^2(y_1z_2 + y_2z_1)$

<u>OSS:</u> Funziona assumendo $\sum \vec{v_i} \neq 0$ solo se ho messo il centro vettoriale nel circocentro di ABC.

IMO SL 2012 G4

ABC con $AB \neq AC$ ed O circocentro. Bisettrice di $B\hat{A}C$ interseca BC in D. Riflettendo D rispetto al pt medio di BC si ottiene E. Le $\perp$ a BC per D ed E intersecano risp AO ed AD in X e Y.

Tesi: $BXCY$ ciclico

Soluzione

$$O = (a^2S_A, b^2S_B, c^2S_C)$$
$$D = (0, b, c) \qquad E = (0, c, b)$$
$$\infty_{AH} = (-a^2, S_C, S_B)$$

$$DX: \ x(bS_B - cS_C) - ya^2c + za^2b = 0$$
$$AO: \ yc^2S_C = zb^2S_B$$
$$x(bS_B - cS_C)bS_B + y\big(\underbrace{-a^2cbS_B + a^2c^2S_C}_{a^2c(cS_C - bS_B)}\big) = 0$$
$$X = (a^2bc, b^2S_B, c^2S_C)$$

$$EY: \ x(cS_B - bS_C) - ya^2b + za^2c = 0$$
$$AD: \ yc = zb$$
$$xb(cS_B - bS_C) + y(-a^2b^2 + a^2c^2) = 0$$
$$cS_B - bS_C = \frac{c(a^2+c^2-b^2)}{2} - \ldots = \tfrac{1}{2}\left[a^2(c-b) + (c-b)(c^2+cb+b^2) + bc(c-b)\right] = \tfrac{1}{2}(c-b)(a^2+(b+c)^2)$$

$$xb(a^2+(b+c)^2) + 2ya^2(b+c) = 0$$

$$Y = (-2a^2(b+c), b(a^2+(b+c)^2), c(a^2+(b+c)^2))$$

$$\Gamma: \ \sum_{cyc} a^2yz = \Big(\sum_{cyc} x\Big)ux$$

$$X: \quad \cancel{a^2}b^2c^2(S_BS_C + a^2bc) = (a^2bc + b^2S_B + c^2S_C)u \cdot \cancel{a^2}bc$$
$$u = \frac{bc(S_BS_C + a^2bc)}{a^2bc + b^2S_B + c^2S_C}$$

(Y) $\cancel{a^2}bc\left(a^2+(b+c)^2\right)\left[\cancel{a^2+(b+c)^2}-2(b+c)^2\right]=$

$=(b+c)\left(\cancel{(b+c)^2-a^2}\right)u\left(\cancel{+}2\cancel{a^2}(b+c)\right)$

$$u=\frac{bc\left(a^2+(b+c)^2\right)}{2(b+c)^2}$$

Tesi iff $2(S_BS_C+a^2bc)(b+c)^2=\left(a^2+(b+c)^2\right)\left(a^2bc+b^2S_B+c^2S_C\right)$

$(b+c)^2\left(2S_BS_C+a^2bc-b^2S_B-c^2S_C\right)\overset{?}{=}a^2\left(a^2bc+b^2S_B+c^2S_C\right)$

$(b+c)^{\cancel{2}}\left(\cancel{a^2}bc-S_A\cancel{a^2}\right)\overset{?}{=}\cancel{a^2}(bS_B+cS_C)\cancel{(b+c)}$

$(b+c)(bc-S_A)\overset{?}{=}(bS_B+cS_C)$

$bc(b+c)\overset{?}{=}c^2b+b^2c$ vero!

IMO SL 2005 G5

ABC acutangolo, $AB\neq AC$. H ortocentro, $\vec{M}=\frac{\vec{B}+\vec{C}}{2}$. $D\in AB$, $E\in AC$.
$AE=AD$; D, H, E allineati.
Tesi: $HM\perp$ asse radicale tra $\odot(ABC)$ e $\odot(ADE)$.

Soluzione

$D=(c-\ell,\ell,0)$ $\quad E=(b-\ell,0,\ell)$ $\quad H=(S_BS_C,\ -\ -\ -)$

$$\det\begin{pmatrix} c-\ell & \ell & 0\\ b-\ell & 0 & \ell\\ S_BS_C & S_AS_C & S_AS_B\end{pmatrix}=0$$

$$-\cancel{\ell}(c-\ell)S_AS_C+\ell^{\cancel{2}}S_BS_C-\cancel{\ell}(b-\ell)S_AS_B=0$$

$$\boxed{\ell(S_AS_C+S_BS_C+S_AS_B)=S_A(cS_C+bS_B)}$$

$\odot ADE$: $\sum_{cyc}a^2yz=\left(\sum_{cyc}x\right)(vy+wz)$

(D) $c^2\ell(c-\ell)=c(\ell v)\leadsto v=c(c-\ell)$

(E) $w=b(b-\ell)$

l'asse radicale è $c(c-\ell)y + b(b-\ell)z = 0$

Due punti e caso ① A

② $\cap BC = P\,(0, b(b-\ell), -c(c-\ell))$

A meno di scalare $\vec{AP} = (c(c-\ell)-b(b-\ell), b(b-\ell), -c(c-\ell))$

$\vec{HM} = \vec{M} - \vec{H} = \frac{\vec{B}+\vec{C}}{2} - (\vec{A}+\vec{B}+\vec{C}) = (-1, -\frac{1}{2}, -\frac{1}{2}) \sim (2,1,1)$

Tesi diventa:

$$a^2(b(b-\ell)-c(c-\ell)) + b^2(-b(b-\ell)-c(c-\ell)) + c^2(c(c-\ell)+b(b-\ell)) \stackrel{?}{=} 0$$

$$a^2(b(b-\ell)-c(c-\ell)) + (c^2-b^2)(c(c-\ell)+b(b-\ell)) \stackrel{?}{=} 0$$

$$a^2(b+c-\ell) \stackrel{?}{=} (b+c)(c(c-\ell)+b(b-\ell))$$

$$\ell \stackrel{?}{=} \frac{(b+c)(b^2+c^2-a^2)}{(b+c)^2-a^2}$$

Ora sappiamo $\sum_{cyc} S_A S_B = \frac{1}{4}(a+b+c)\prod_{cyc}(a+b-c)$

$$S_A = \frac{b^2+c^2-a^2}{2}$$

$$cS_C + bS_B = \frac{1}{2}\left[a^2(b+c) + bc(b+c) - (b+c)(b^2-bc+c^2)\right] =$$

$$= \frac{1}{2}(b+c)(a^2-(b-c)^2) = \frac{1}{2}(b+c)(a+b-c)(a+c-b)$$

e ora è ovvio!

POLI e POLARI

Come si fanno? Sdoppiamento?

$x^2 \mapsto x x_0$

$xy \mapsto \frac{x y_0 + y x_0}{2}$

La polare di (α, β, γ) risp a $\sum_{cyc} a^2 yz = 0$

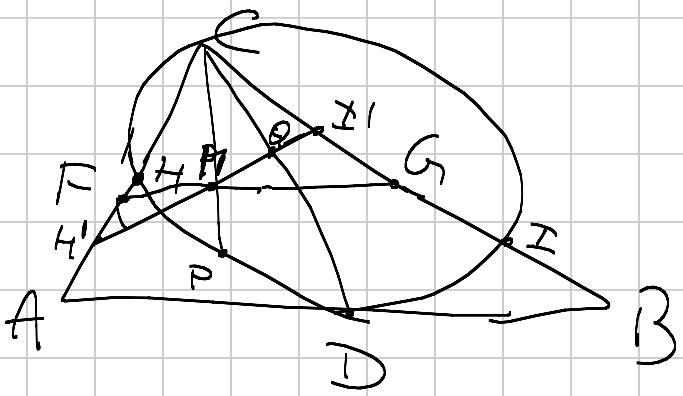
$$\sum_{cyc} a^2(y\gamma + z\beta) = 0$$

La polare di $(1,0,0)$ risp a $\sum_{cyc} a^2yz = (\sum x)(\sum ux)$

$b^2z + c^2y = 2ux + vy + wz$

TST 2016.6 (IMO SL 2015 G5)

ABC triangolo con $AC \neq BC$. D, F, G punti medi di AB, AC, BC risp. Γ passa per C e tange AB in D. $\Gamma \cap \overline{AF}$ è H e $\Gamma \cap \overline{BG}$ è I. H' e I' i simmetrici di H ed I rispetto a F e G, rispettivamente. $H'I'$ interseca CD in Q e FG in M. $CM \cap \Gamma = \{C, P\}$. Tesi: $CQ = QP$.



Soluzione

$D = (1,1,0)$

$$\Gamma : \sum_{cyc} a^2yz = (\sum_{cyc} x)(ux + vy)$$

$$c^2 = 2(u+v)$$

$c^2xy = (x+y)(ux+vy)$ ha $\Delta = 0$

ovvero $4uv = (u+v-c^2)^2 = \frac{c^4}{4}$

$uv = \frac{c^4}{16} \quad u+v = \frac{c^2}{2}$

$u = v = \frac{c^2}{4}$

$$\Gamma : \sum_{cyc} a^2yz = \frac{c^2}{4}(\sum_{cyc} x)(x+y)$$

$\Gamma \cap \{x=0\} \Rightarrow a^2\cancel{y}z = \frac{c^2}{4}\cancel{y}(y+z) \qquad yc^2 = z(4a^2-c^2)$

$I = (0, 4a^2-c^2, c^2) \qquad H = (4b^2-c^2, 0, c^2)$

$I' = (0, c^2, 4a^2-c^2) \qquad H' = (c^2, 0, 4b^2-c^2)$

$CD:\ x=y$

$$H'I':\ \det\begin{pmatrix} x & y & z \\ 0 & c^2 & 4a^2-c^2 \\ c^2 & 0 & 4b^2-c^2 \end{pmatrix}=0$$

$$H'I':\ x\cancel{c^2}(4b^2-c^2)+y\cancel{c^2}(4a^2-c^2)-zc^2\cdot\cancel{c^2}=0$$

$$(4b^2+4a^2-2c^2)x=zc^2$$

$$Q=(c^2,\ c^2,\ 2(2a^2+2b^2-c^2))$$

$FG:\ z=x+y$

$$AM:\ x(2b^2-c^2)+y(2a^2-c^2)=0$$

$$\begin{array}{ccc} x & y & z \\ c^2 & c^2 & 2(2a^2+2b^2-c^2) \end{array}$$

$$\infty_A=(-a^2,\ S_C,\ S_B)$$

$$Q\infty_A:\ x\left(c^2S_B-2S_C(2a^2+2b^2-c^2)\right)+$$
$$+y\left(-2a^2(2a^2+2b^2-c^2)-c^2S_B\right)+zc^2(a^2+S_C)=0$$

$$Q\infty_A\cap\{x=0\}\leadsto Q_A=\left(0,\ c^2(a^2+S_C),\ c^2S_B+2a^2(2a^2+2b^2-c^2)\right)$$

ω di centro Q che passa per C $\left(\Rightarrow \text{per } 2Q_A-C \text{ e } 2Q_B-C\right)$

$$\sum Q_A=4a^2(a^2+b^2)\leadsto C=(0,\ 0,\ 4a^2(a^2+b^2))$$

$$2Q_A-C=\left(0,\ c^2(a^2+S_C),\ c^2S_B+2a^2(2a^2+2b^2-c^2)-2a^2(a^2+b^2)\right)$$

$$2Q_A-C=\left(0,\ c^2(a^2+S_C),\ c^2S_B+2a^2(a^2+b^2-c^2)\right)$$

$$\omega:\ \sum_{cyc}a^2yz=\left(\sum_{cyc}x\right)(ux+vy)$$

$2Q_A-C$

$$\cancel{a^2}\cancel{c^2}\cancel{(a^2+S_C)}\left(c^2S_B+2a^2(a^2+b^2-c^2)\right)=2\cancel{a^2}(a^2+b^2)v\cancel{c^2}\cancel{(a^2+S_C)}$$

$$v = \frac{c^2 S_B + 2a^2(a^2+b^2-c^2)}{2(a^2+b^2)} = \frac{c^2 S_B + 4a^2 S_C}{2(a^2+b^2)}$$

$$U = \frac{c^2 S_A + 4b^2 S_C}{2(a^2+b^2)}$$

Tesi equivale ad CM : coincide con $\left(CM: x(2b^2-c^2)+y(2a^2-c^2)=0 \right)$

$$\left(\frac{c^2 S_B + 4a^2 S_C}{2(a^2+b^2)} - \frac{c^2}{4} \right) y + (\text{sym}) x = 0$$

$$\frac{2c^2 S_B + 8a^2 S_C - a^2c^2 - b^2c^2}{4\cancel{(a^2+b^2)}} y + (\text{sym}) x = 0$$

Tesi $\iff$ $\dfrac{2c^2 S_B + 8a^2 S_C - a^2c^2 - b^2c^2}{2a^2-c^2}$ e' sym in a e b

$$c^2(\cancel{a^2}+c^2-b^2) + 4a^2(a^2+b^2-c^2) - \cancel{a^2}c^2 - b^2c^2$$

$$\underline{4a^4} + \underline{4a^2b^2} - \underline{4a^2c^2} - \underline{2b^2c^2} + \underline{c^4} \quad \Big| \; \frac{2a^2-c^2}{2a^2+2b^2-c^2}$$

$$0 \qquad 0 \qquad \underline{-2a^2c^2} \qquad 0 \qquad 0$$

$$0$$

Tesi $\iff$ $2a^2+2b^2-c^2$ sym in a e b ☺