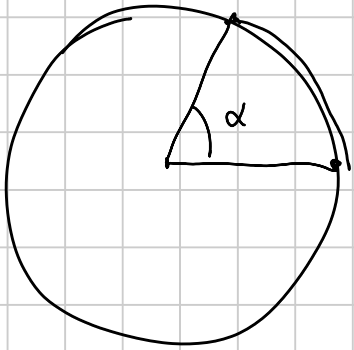
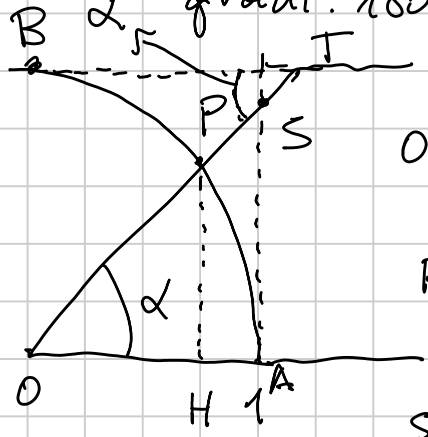
G1 - Basic (Trigonometria)

Titolo nota 03/09/2018



gradi : 180 = radianti : π



OH = coseno di α
$\cos\alpha$

PH = seno di α
$\operatorname{sen}\alpha$

SA = tangente di α
$\operatorname{tg}\alpha$ $\tan\alpha$

BT = cotangente di α
$\operatorname{cotg}\alpha$ $\cot\alpha$

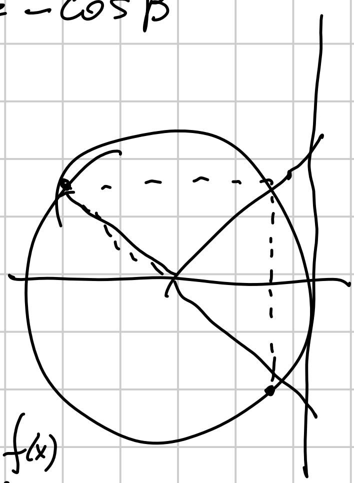
$\operatorname{tg}\alpha = \dfrac{\operatorname{sen}\alpha}{\cos\alpha}$ $\qquad \operatorname{cotg} = \dfrac{\cos\alpha}{\operatorname{sen}\alpha} = \dfrac{1}{\operatorname{tg}\alpha}$

Tutte queste sono periodiche di periodo 2π (tg e cotg π)

Angoli complementari $\alpha+\beta=\frac{\pi}{2}$ $\qquad \operatorname{sen}\alpha = \cos\beta$ $\quad \cos\alpha = \operatorname{sen}\beta$ $\quad \operatorname{tg}\alpha = \operatorname{cotg}\beta$

supplementari $\alpha+\beta=\pi$ $\qquad \operatorname{sen}\alpha = \operatorname{sen}\beta$ $\quad \cos\alpha = -\cos\beta$ $\quad \operatorname{tg}\alpha = -\operatorname{tg}\beta$

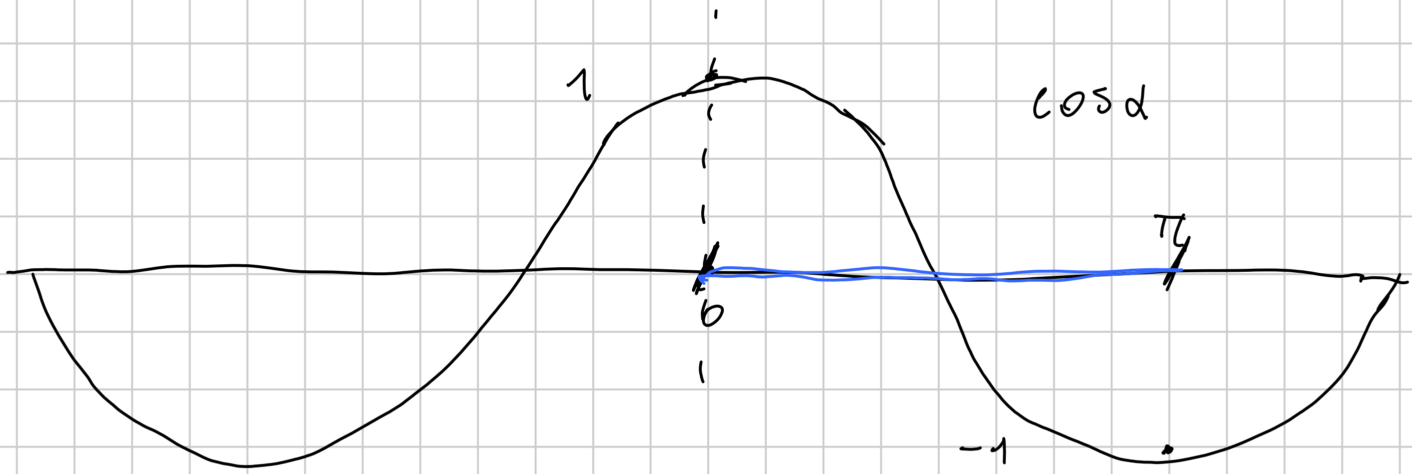
esplementari (opposti) $\alpha+\beta = 2\pi$
$\alpha = -\beta$
$\operatorname{sen}\alpha = -\operatorname{sen}\beta$
$\cos\alpha = \cos\beta$



$\operatorname{sen}\alpha$ è una funzione dispari $f(-x) = -f(x)$
$\cos\alpha$ " pari $f(-x) = f(x)$
$\operatorname{tg}\alpha$ dispari
$\operatorname{cotg}\alpha$ "

$\text{sen } 60° = \frac{\sqrt{3}}{2}$ $\text{sen } 30° = \frac{1}{2}$ $\text{sen } 45° = \frac{\sqrt{2}}{2}$

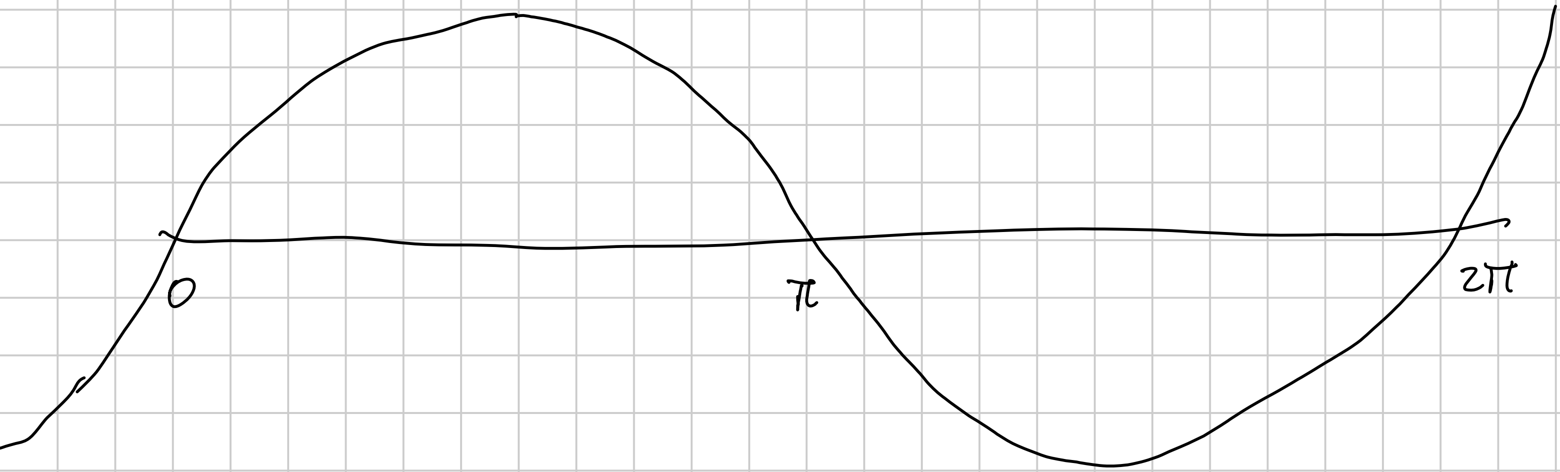
$\text{sen}^2\alpha + \cos^2\alpha = 1$



decrescente tra 0 e π ⇒

biettiva tra [0, π] e [−1, 1]

arccos x = angolo tra 0 e π che ha x come coseno
$x \in [-1, 1]$



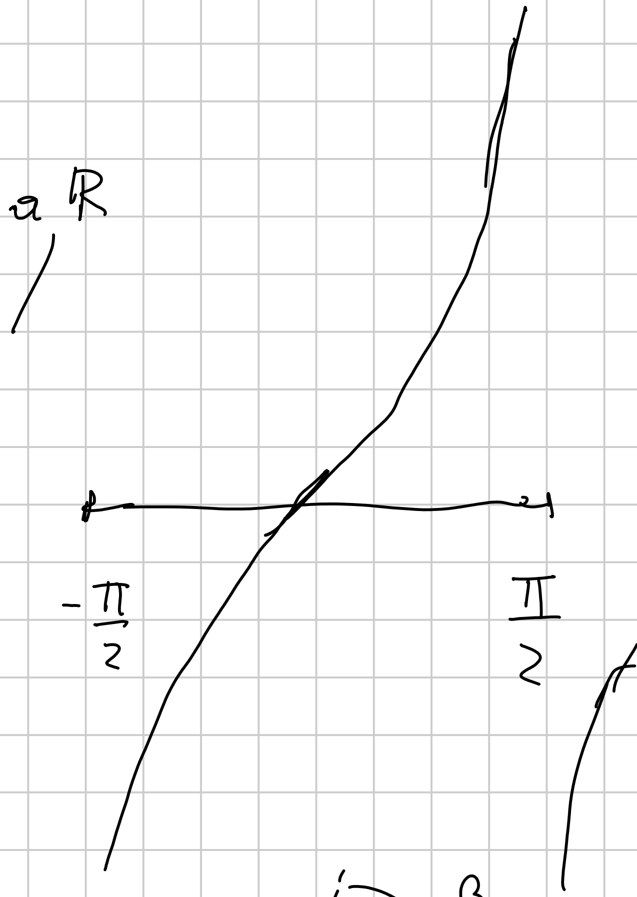
$\text{sen}\left(\alpha - \frac{\pi}{2}\right) = \cos\alpha$

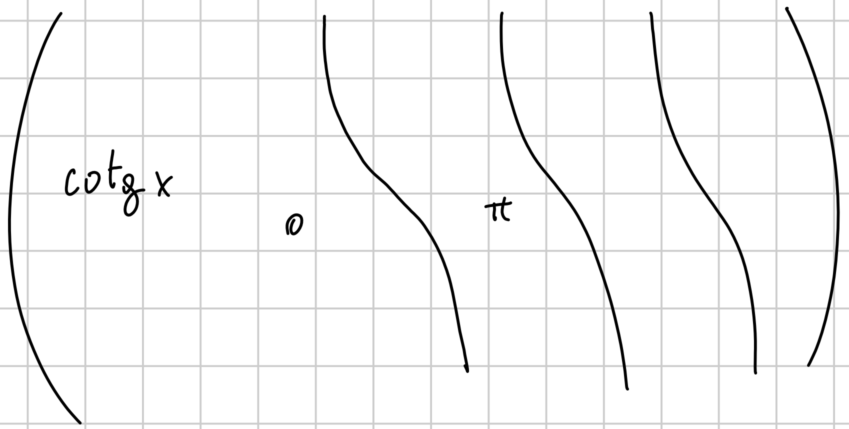
sen α è biettivo tra $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ e $[-1, 1]$

arc sen x : $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

tg α è biettiva da $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ a $\mathbb{R}$

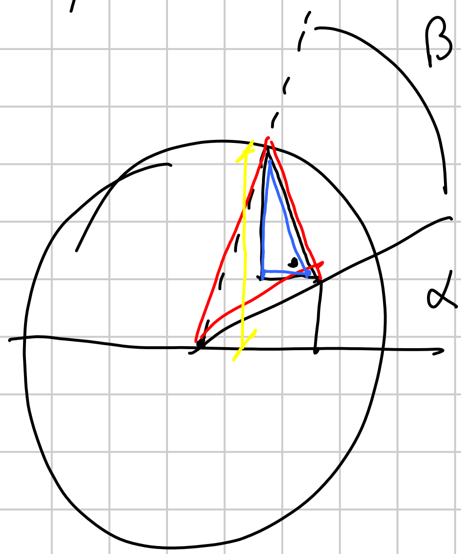
$\Rightarrow \operatorname{arctg} x : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$





Formule

Addizione $\quad \operatorname{sen}(\alpha+\beta)$?



ha angoli $90°, \beta, 90°-\beta$

ha angoli $90°, \alpha, 90°-\alpha$

$\operatorname{sen}(\alpha+\beta) = \; + \; = \; \cos\alpha + \; \operatorname{sen}\alpha =$

$= 1 \cdot \operatorname{sen}\beta \cdot \cos\alpha + 1 \cdot \cos\beta \cdot \operatorname{sen}\alpha =$

$= \operatorname{sen}\alpha \cos\beta + \cos\alpha \operatorname{sen}\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta$

$\operatorname{sen}(\alpha-\beta) = \operatorname{sen}\alpha\cos\beta + \cos\alpha(-\operatorname{sen}\beta)$

$\operatorname{tg}(\alpha+\beta) = \frac{\operatorname{sen}(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha \operatorname{tg}\beta}$

Derivo le formule per sen 2β $\quad \alpha = \beta$

$$\cos 2\beta = \cos^2\beta - \operatorname{sen}^2\beta = 2\cos^2\beta - 1 = 1 - 2\operatorname{sen}^2\beta$$

$2\beta = \alpha$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\operatorname{sen}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \operatorname{tg}\frac{\alpha}{2} = \frac{\operatorname{sen}\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$t = \operatorname{tg}\frac{\alpha}{2} \qquad \operatorname{tg}\alpha = \frac{2t}{1-t^2} \qquad \cos\alpha = \frac{1-t^2}{1+t^2} \qquad \operatorname{sen}\alpha = \frac{2t}{1+t^2}$$

Triplicazione? n-plicazione?

$\mathbb{C}$ numeri complessi $\quad \{a+ib \mid a,b \in \mathbb{R} \quad i^2=-1\}$

$$(a_1+ib_1)+(a_2+ib_2) = a_1+a_2+i(b_1+b_2)$$

$$(a_1+ib_1)(a_2+ib_2) = a_1a_2 + i(b_1a_2+a_1b_2) + i^2 b_1b_2 =$$

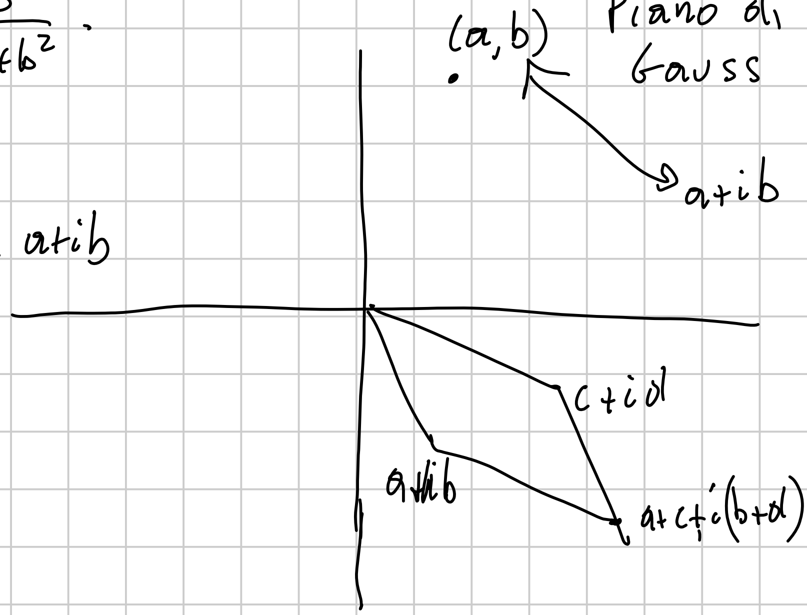
$$= a_1a_2 - b_1b_2 + i(b_1a_2+a_1b_2)$$

$$\frac{1}{i} = -i \qquad \frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2-(-b^2)} = \frac{a-ib}{a^2+b^2} =$$

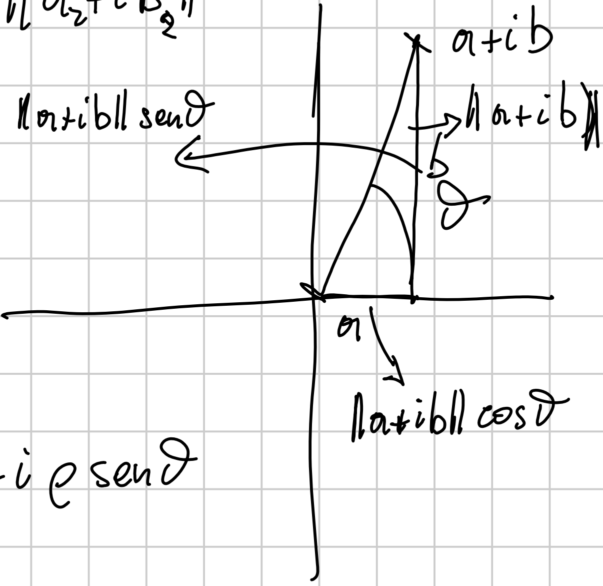
$$= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

$\|a+ib\| = \sqrt{a^2+b^2}$ norma di $a+ib$

Piano di Gauss



$$\|(a_1+ib_1)(a_2+ib_2)\| = \|a_1+ib_1\|\,\|a_2+ib_2\|$$



ϑ? $\operatorname{tg}\vartheta = \frac{b}{a}$

$$\vartheta = \operatorname{arctg}\left(\frac{b}{a}\right)$$

$\rho = \|a+ib\|$ $a+ib = \rho\cos\vartheta + i\rho\,\operatorname{sen}\vartheta$

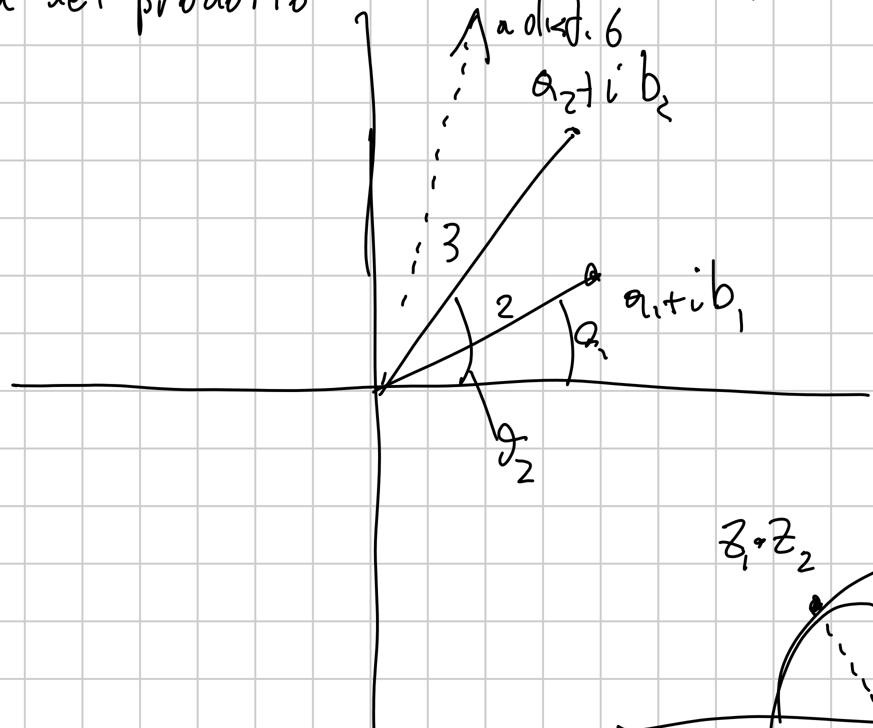
$$(a_1+ib_1)(a_2+ib_2) = \rho_1(\cos\vartheta_1 + i\operatorname{sen}\vartheta_1)\,\rho_2(\cos\vartheta_2 + i\operatorname{sen}\vartheta_2) =$$

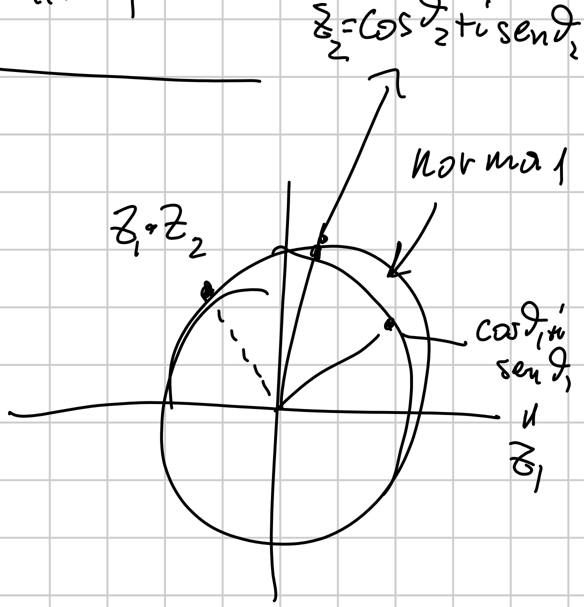
$$= \rho_1\rho_2\,(\underline{\cos\vartheta_1\cos\vartheta_2 - \operatorname{sen}\vartheta_1\operatorname{sen}\vartheta_2} + i\operatorname{sen}\vartheta_1\cos\vartheta_2 + i\cos\vartheta_1\operatorname{sen}\vartheta_2)$$

$$= \rho_1\rho_2\,(\cos(\vartheta_1+\vartheta_2) + i\operatorname{sen}(\vartheta_1+\vartheta_2))$$

norma del prodotto

angolo del prodotto è $\vartheta_1+\vartheta_2$

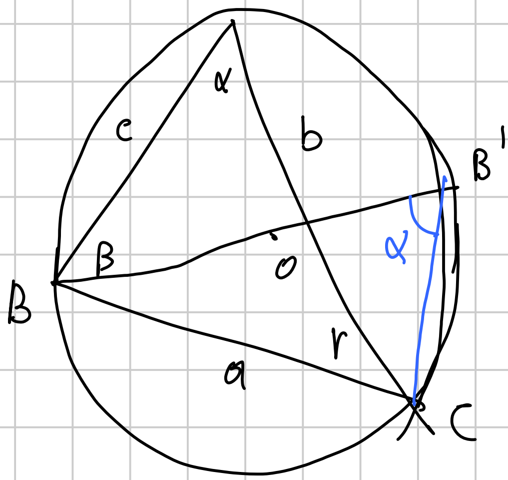




$$(\cos\vartheta + i\,\mathrm{sen}\,\vartheta)^n = \cos n\vartheta + i\,\mathrm{sen}\,n\vartheta$$

ha come angolo $n\vartheta \pmod{2\pi}$

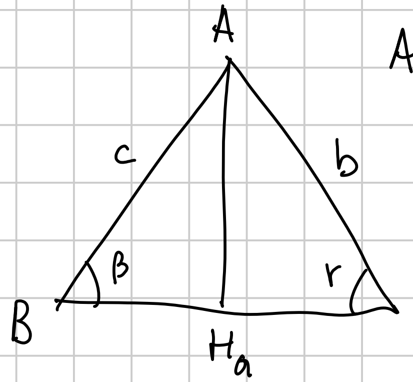
Triangoli



$$2R = BB' = \frac{a}{\mathrm{sen}\,\alpha} = \frac{b}{\mathrm{sen}\,\beta} = \frac{c}{\mathrm{sen}\,\gamma}$$

$B\hat{C}B' = 90°$

Teorema dei seni

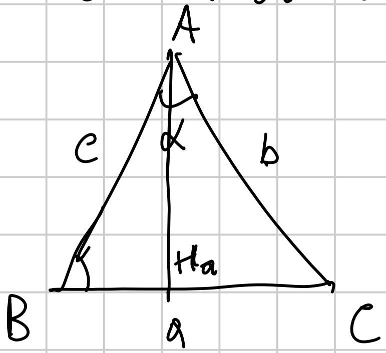


$$\text{Area} = \frac{BC \cdot AH_a}{2} = \frac{ac\sin\beta}{2} = \frac{ab\sin\gamma}{2} = \frac{bc\sin\alpha}{2}$$

$$\mathrm{sen}\,\alpha = \frac{a}{2R}$$

$$\text{Area} = \frac{bc\,\mathrm{sen}\,\alpha}{2} = \frac{abc}{4R}$$

Teorema di Carnot (Pitagora per triangoli non rettangoli)



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$a = BH_a + H_aC$$

$$b^2 = AH_a^2 + H_aC^2 \qquad c^2 = AH_a^2 + H_aB^2$$

Svolgo i conti $\quad a^2 = BH_a^2 + H_aC^2 + 2\,BH_a \cdot H_aC$

$$AH_a = c\,\mathrm{sen}\,\beta = b\,\mathrm{sen}\,\gamma \qquad BH_a = c\cos\beta \quad H_aC = b\cos\gamma$$

$$2\,BH_a \cdot H_aC = 2\,AH_a^2 - 2bc\cos\alpha$$

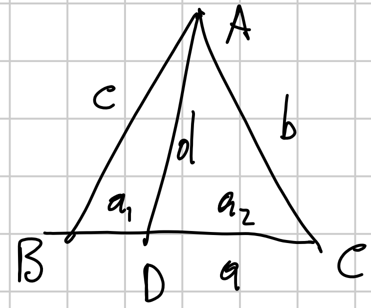
$$2bc\cos\beta\cos\gamma = 2 \cdot c\,\mathrm{sen}\,\beta \cdot b\,\mathrm{sen}\,\gamma - 2bc\cos\alpha$$

$$\cos\alpha = \operatorname{sen}\beta \operatorname{sen}\gamma - \cos\beta\cos\gamma = -\cos(\beta+\gamma)$$

$$\| $$

$$\cos(\pi-(\beta+\gamma))$$

Stewart:



$$b^2 a_1 + c^2 a_2 = a \cdot a_1 \cdot a_2 + d_a^2$$

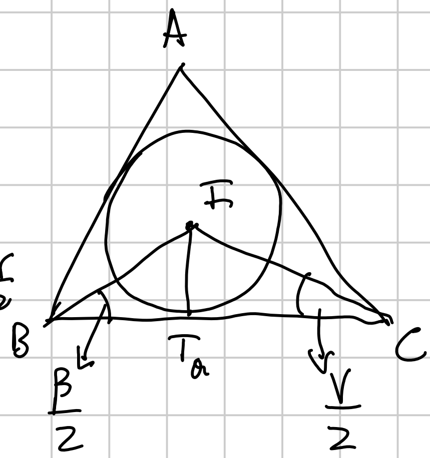
Prostaferesi: $\operatorname{sen}\alpha + \operatorname{sen}\beta = 2\operatorname{sen}\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Werner: $\operatorname{sen}\alpha \cdot \operatorname{sen}\beta = \frac{1}{2}\left[\operatorname{sen}(\alpha+\beta) - \cos(\alpha-\beta)\right]$

$$\operatorname{sen}\alpha = \operatorname{sen}\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) \qquad t = \frac{\alpha+\beta}{2} \quad s = \frac{\alpha-\beta}{2}$$

$$\operatorname{sen}\beta = \operatorname{sen}\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right) \qquad \alpha = t+s \quad \beta = t-s$$

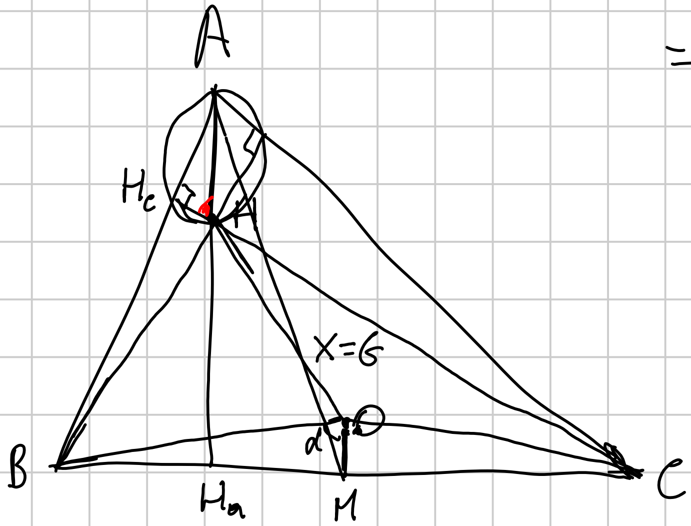
Raggio circ. inscritta:



$$IT_a = BT_a \operatorname{tg}\frac{\beta}{2} \qquad IT_a = CT_a \operatorname{tg}\frac{\gamma}{2}$$

$$a = BT_a + CT_a = \frac{r}{\operatorname{tg}\frac{\beta}{2}} + \frac{r}{\operatorname{tg}\frac{\gamma}{2}} \longrightarrow r = a\,\frac{1}{\frac{1}{\operatorname{tg}\frac{\beta}{2}} + \frac{1}{\operatorname{tg}\frac{\gamma}{2}}} =$$

$$= a\,\frac{\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}}{\operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2}}$$



OM? AH?

$$\widehat{BOC} = 2\alpha \qquad \widehat{BOM} = \alpha$$

$$OM = R\cos\alpha$$

AH_c? è in ACH_c rettangolo $b\cos\alpha$

e $H_c\hat{H}A$? $90° - A_c\hat{A}H = \beta$ Allora $AH = \dfrac{b\cos\alpha}{\operatorname{sen}\beta} =$

$= 2R\cos\alpha = 2\,OM$.

AM è mediana $AH \parallel OM$ $AHX \sim OMX$

Ah! Ma allora poiché $AH = 2\cdot OM$, anche

$AX = 2\cdot XM$ ma quindi $X = G$ baricentro!

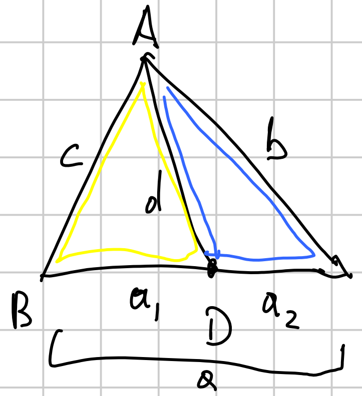
e $HG = 2\,GO$

1) ABC triangolo allora $a^2+b^2+c^2 \geq 4\sqrt{3}\,S$ S=area.

Esercizi : 2,3 pag. 3 Prostaferesi e Werner Stewart

7, 8, 9 11 (una di tante)

Es. 8, 9 pag. 36



(2, 3, 7, 8) Stewart:

$$a_1 b^2 + a_2 c^2 = a_1 a_2 a + a d^2$$

$$a_2\cdot \quad c^2 = a_1^2 + d^2 - 2a_1 d\cos B\hat{D}A$$

$$a_1\cdot \quad b^2 = a_2^2 + d^2 - 2a_2 d\cos(\pi - B\hat{D}A) = a_2^2 + d^2 + 2a_2 d\cos B\hat{D}A$$

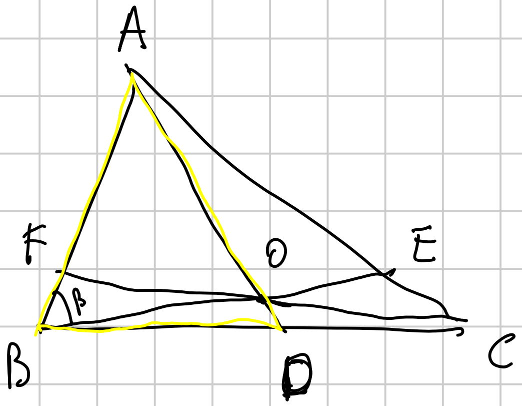
$$a_2\cdot c^2 + a_1 b^2 = a_2\cdot a_1^2 + a_1\cdot a_2^2 + a_2\cdot d^2 + a_1\cdot d^2 = a_1 a_2 \underbrace{(a_1+a_2)}_{a} + d^2 \underbrace{(a_1+a_2)}_{a}$$

$\operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha \operatorname{tg}\beta \operatorname{tg}\gamma.$ $\qquad \alpha = \pi - (\beta+\gamma)$

$$-\operatorname{tg}(\beta+\gamma) + \operatorname{tg}\beta + \operatorname{tg}\gamma = -\operatorname{tg}(\beta+\gamma)\operatorname{tg}\beta\operatorname{tg}\gamma$$

$$-\frac{\operatorname{tg}\beta+\operatorname{tg}\gamma}{1-\operatorname{tg}\beta\operatorname{tg}\gamma} + \operatorname{tg}\beta + \operatorname{tg}\gamma = -\frac{\operatorname{tg}\beta+\operatorname{tg}\gamma}{1-\operatorname{tg}\beta\operatorname{tg}\gamma}\operatorname{tg}\beta\operatorname{tg}\gamma$$

$$-\cancel{\operatorname{tg}\beta} - \cancel{\operatorname{tg}\gamma} + \cancel{\operatorname{tg}\beta} + \cancel{\operatorname{tg}\gamma} - \operatorname{tg}\beta\operatorname{tg}\gamma(\operatorname{tg}\beta+\operatorname{tg}\gamma) = -(\operatorname{tg}\beta+\operatorname{tg}\gamma)\operatorname{tg}\beta\operatorname{tg}\gamma$$



$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{AO} \qquad AO = R$$

$$A\hat{O}B = 2\gamma \Rightarrow B\hat{A}O = B\hat{A}D = 90° - \gamma$$

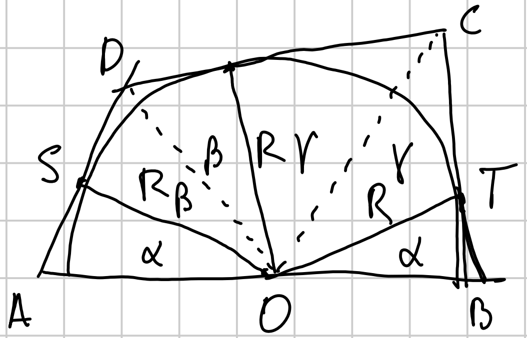
$$A\hat{D}B = 90° + \gamma - \beta$$

$$AD \overset{?}{=} \frac{AB}{\operatorname{sen} A\hat{D}B} = \frac{AD}{\operatorname{sen}\beta} \qquad AD = \frac{c\operatorname{sen}\beta}{\cos(\gamma-\beta)} = \frac{2R\operatorname{sen}\gamma\operatorname{sen}\beta}{\cos(\gamma-\beta)}$$

$$\frac{\cos(\gamma-\beta)}{2\cancel{R}\operatorname{sen}\gamma\operatorname{sen}\beta} + \frac{\cos(\beta-\alpha)}{2\cancel{R}\operatorname{sen}\beta\operatorname{sen}\alpha} + \frac{\cos(\alpha-\gamma)}{2\cancel{R}\operatorname{sen}\alpha\operatorname{sen}\gamma} = \frac{2}{\cancel{R}}$$

$$(\cos\gamma\cos\beta + \operatorname{sen}\gamma\operatorname{sen}\beta)\operatorname{sen}\alpha + (\qquad\qquad)\operatorname{sen}\gamma + (\qquad)\operatorname{sen}\beta =$$

$$= 4\operatorname{sen}\alpha\operatorname{sen}\beta\operatorname{sen}\gamma \quad \rightsquigarrow \operatorname{tg}\alpha + \operatorname{tg}\beta + \operatorname{tg}\gamma = \operatorname{tg}\alpha\operatorname{tg}\beta\operatorname{tg}\gamma$$



$$AB^2 = 4AD \cdot BC$$

$$2\alpha + 2\beta + 2\gamma = 180°$$

$$AO = \frac{R}{\cos\alpha} \qquad AD = AS + SD =$$

$$= R\operatorname{tg}\alpha + R\operatorname{tg}\beta \qquad BC = BT + TC = R(\operatorname{tg}\alpha + \operatorname{tg}\gamma)$$

$$\frac{\cancel{4R^2}}{\cos^2\alpha} = \cancel{4R^2}(\operatorname{tg}\alpha+\operatorname{tg}\beta)(\operatorname{tg}\alpha+\operatorname{tg}\gamma)$$

$$\frac{1}{\cos^2\alpha} = \frac{\operatorname{sen}^2\alpha+\cos^2\alpha}{\cos^2\alpha} = \cancel{\operatorname{tg}^2\alpha} + 1 = \cancel{\operatorname{tg}^2\alpha} + \operatorname{tg}\alpha\operatorname{tg}\gamma + \operatorname{tg}\beta\operatorname{tg}\alpha + \operatorname{tg}\beta\operatorname{tg}\gamma$$

Identità nota!

$$a^2+b^2+c^2 \geq 4\sqrt{3}\,S$$

$$\cancel{2}(b^2+c^2) - \cancel{2}bc\cos\alpha \geq \cancel{4}^{2}\sqrt{3}\,\frac{1}{\cancel{2}}\,bc\operatorname{sen}\alpha$$

$$b^2+c^2 \geq bc\left(\sqrt{3}\operatorname{sen}\alpha+\cos\alpha\right) = 2bc\left(\frac{\sqrt{3}}{2}\operatorname{sen}\alpha+\frac{1}{2}\cos\alpha\right) =$$

$$= 2bc\left(\operatorname{sen}60^\circ\operatorname{sen}\alpha+\cos60^\circ\cos\alpha\right) = 2bc\cos(\alpha-60^\circ)$$

$$b^2+c^2 \geq 2bc \geq 2bc\cos(\alpha-60^\circ)$$

IMO 1961/2