G2 Basic — Luca Mac

Titolo nota — 05/09/2018

- CARTESIANE
- COMPLESSI
- VETTORI

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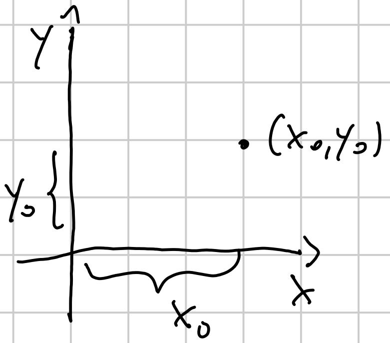
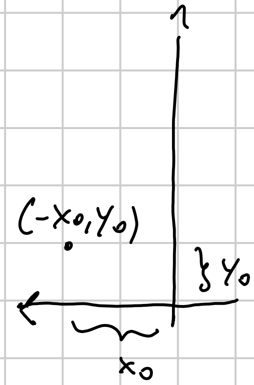
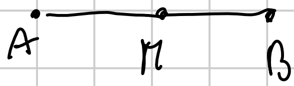
CARTESIANE

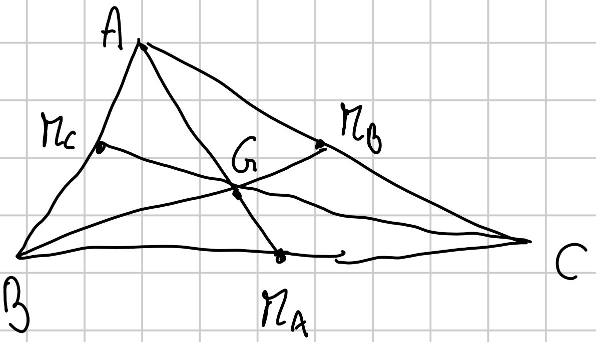




$A=(x_A, y_A)$ $B=(x_B, y_B)$ M pt medio di A e B?

$$M=\left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$$





$AG = 2GM_A$

$$G=\left(\frac{x_A+x_B+x_C}{3}, \frac{y_A+y_B+y_C}{3}\right)$$

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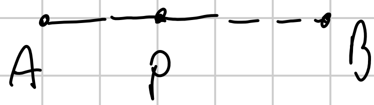
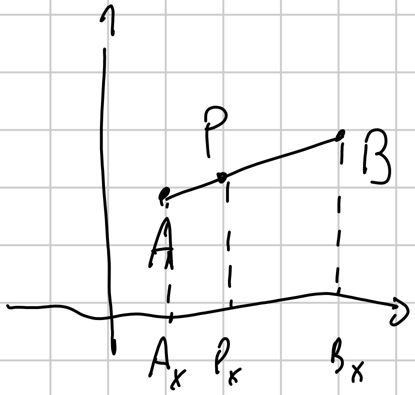
Supponiamo di avere A e B. Prendiamo $P \in \overline{AB}$ con

$AP = \lambda AB$ ($\Rightarrow PB = (1-\lambda)AB$)



$$P=\left(\lambda x_B+(1-\lambda)x_A, \lambda y_B+(1-\lambda)y_A\right)$$

Controllare



$$\underbrace{A_xP_x}_{x_P - x_A} = \lambda \underbrace{A_xB_x}_{x_B - x_A} \Rightarrow x_P = \lambda x_B + (1-\lambda)x_A$$

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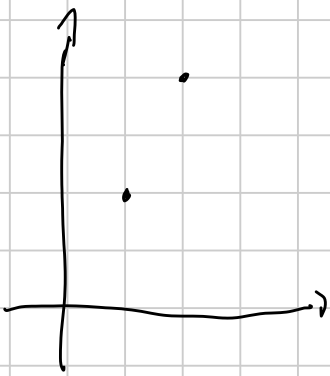
Retta per due punti.

$A, B \qquad \{ P \mid P, A, B \text{ allineati} \}$

P, A, B allineati $\Leftrightarrow$ $\underset{(x_P - x_A, y_P - y_A)}{P - A}, O, \underset{(x_B - x_A, y_B - y_A)}{B - A}$



$$\Leftrightarrow \frac{y_B - y_A}{x_B - x_A} = \frac{y_P - y_A}{x_P - x_A}$$

La retta per A, B è $(x - x_A)(y_B - y_A) = (y - y_A)(x_B - x_A)$

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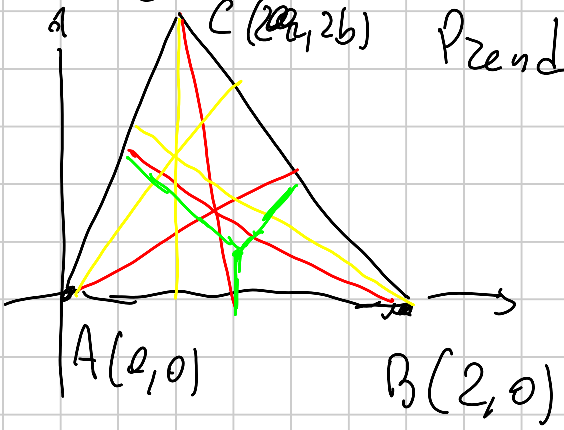


Prendo ABC triangolo

Sappiamo già $G = \left(\frac{2}{3}(a+1), \frac{2}{3}b\right)$

$CH \perp AB$ e $AH \perp BC$

$x_H = x_C = 2a$

$$m_{BC} = \frac{b}{a-1} \left(= \frac{2b-0}{2a-2} \right)$$

$$\Rightarrow m_{AH} = -\frac{a-1}{b} \qquad AH: \ y = -\frac{a-1}{b}x$$

$$y_H = -\frac{(a-1)}{b}(2a)$$

$$H = \left(2a, \frac{2a(1-a)}{b}\right)$$

Calcoliamo O:

$M_C O \perp AB$ e $M_B O \perp AC$

$$x_O = x_{M_C} = \frac{x_A + x_B}{2} = 1$$

$$m_{AC} = \frac{2b}{2a} = \frac{b}{a} \qquad m_{M_B O} = -\frac{a}{b} \qquad M_B = (a, b)$$

$$M_B O: \ y - y_{M_B} = -\frac{a}{b}(x - x_{M_B})$$

$$M_B O: \ y - b = -\frac{a}{b}(x - a)$$

$$y_O = b - \frac{a}{b}(1-a)$$

$$O = \left(1, \ b - \frac{a}{b}(1-a)\right)$$

O, G, H ALLINEATI. Basta verificare

$$\frac{x_H - x_G}{y_H - y_G} \stackrel{?}{=} \frac{x_O - x_G}{y_O - y_G}$$

$$\frac{\cancel{2}a - \frac{\cancel{2}}{3}(a+1)}{\frac{\cancel{2}a(1-a)}{b} - \frac{\cancel{2}}{3}b} \stackrel{?}{=} \frac{1 - \frac{2}{3}(a+1)}{b - \frac{a}{b}(1-a) - \frac{2}{3}b}$$

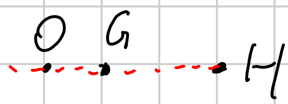
$$\frac{\frac{2}{3}a - \frac{1}{3}}{\frac{3a(1-a) - b^2}{3b}} \overset{?}{=} \frac{\frac{1}{3} - \frac{2}{3}a}{\frac{\cancel{3}b^2 - 3a(1-a) - \cancel{2b^2}}{3b}}$$

quindi è vero!

Quanto vale $\frac{GH}{GO}$? Beh $\frac{GH}{GO} = \frac{x_H - x_G}{x_O - x_G}$

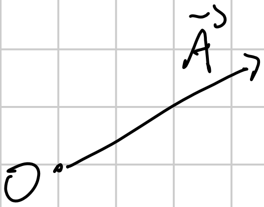
SAPENDO CHE SONO ALLINEATI

$$\frac{x_H - x_G}{x_O - x_G} = \frac{2a - \frac{2}{3}(a+1)}{1 - \frac{2}{3}(a+1)} = \frac{\frac{4}{3}a - \frac{2}{3}}{\frac{1}{3} - \frac{2}{3}a} = -2 < 0$$ perché non sono nell'ordine giusto.

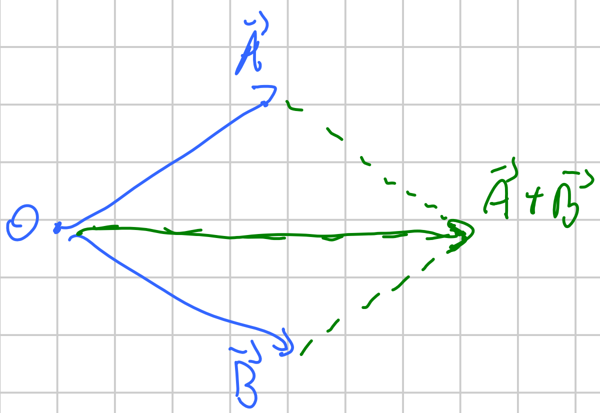


$(HG = 2GO)$

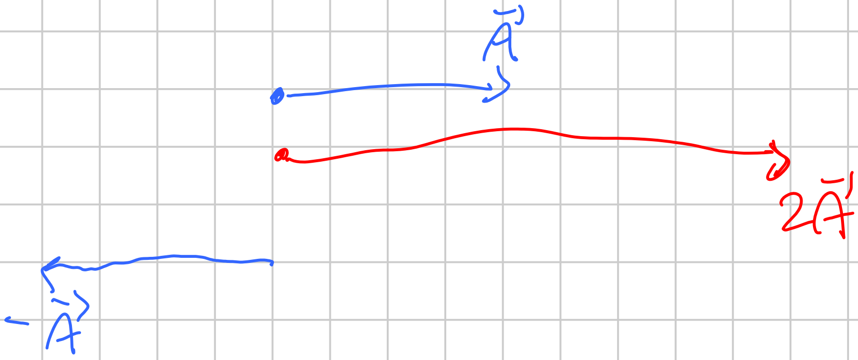
VETTORI



SOMMA DI DUE VETTORI? REGOLA DEL PARALLELOGRAMMA



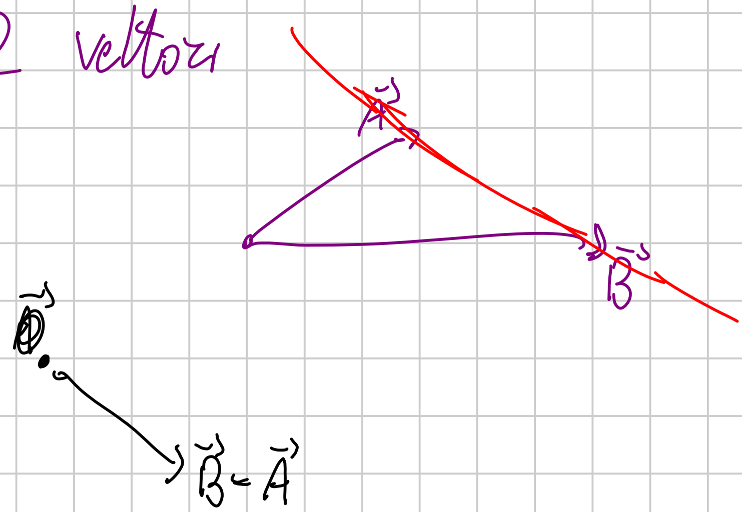
MOLTIPLICAZIONE PER UNO SCALARE



Retta di un certo vettore è l'insieme dei $\vec B$ tali che

$$\vec B = \lambda \vec A$$

Retta per 2 vettori

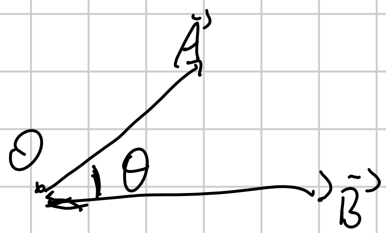


È l'insieme dei $\vec P$ tali che $\vec P - \vec A = \lambda(\vec B - \vec A)$

$\iff \vec P = \lambda \vec B + (1-\lambda)\vec A \quad \lambda \in \mathbb{R}$

Stare invece sul segmento vuol dire $\lambda \in [0,1]$.

NORMA E PRODOTTO SCALARE



$$\vec A \cdot \vec B = \|\vec A\| \cdot \|\vec B\| \cos\theta$$

Osserviamo che $\vec A \cdot \vec A = \|\vec A\|^2$

$\|\vec{A}\| = 0 \iff \vec{A} = \vec{0}$ possiamo dire che $\vec{A} \perp \vec{B} \iff \vec{A}\cdot\vec{B} = 0$

Dim: ($\Rightarrow$) $\vec{A}\perp\vec{B} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0 \Rightarrow \vec{A}\cdot\vec{B} = 0$

($\Leftarrow$) Se uno tra $\vec{A}$ e $\vec{B}$ è $\vec{0}$, allora è banale.

Supponiamo $\vec{A}, \vec{B} \neq \vec{0}$

$0 = \vec{A}\cdot\vec{B} = \|\vec{A}\|\cdot\|\vec{B}\|\cdot\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

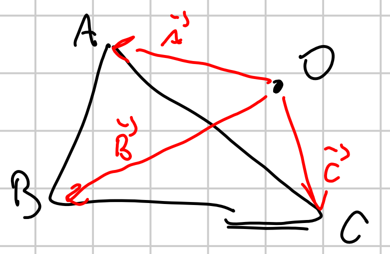
$\Rightarrow \vec{A}\perp\vec{B}$.

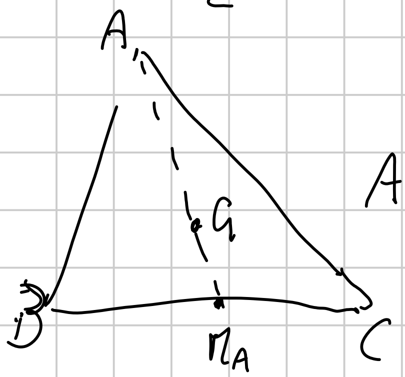
CALCOLIAMO UN PO' DI PUNTI.

$\vec{A}, \vec{B}, \vec{C}$. Quanto vale $\vec{M}_A$?

$\vec{M}_A = \frac{\vec{B}+\vec{C}}{2}$ (AUDIO) $\qquad \vec{M}_B = \frac{\vec{A}+\vec{C}}{2} \qquad \vec{M}_C = \frac{\vec{A}+\vec{B}}{2}$.

$\vec{G} = \frac{\vec{A}+\vec{B}+\vec{C}}{3}$





$AG = 2GM_A$

$\vec{G} - \vec{A} = 2(\vec{M}_A - \vec{G})$

$3\vec{G} = \vec{A}+\vec{B}+\vec{C}$

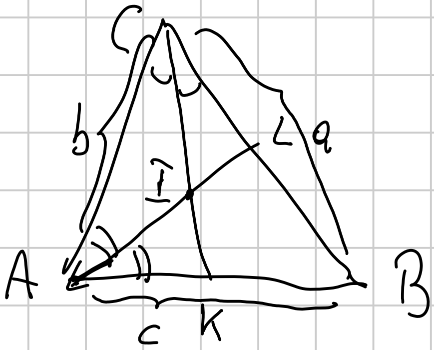
$2\vec{OG} = \vec{GH}$ $\qquad 2\vec{G} - 2\vec{O} = \vec{H} - \vec{G}$

circocentro

$\vec{H} + 2\vec{O} = 3\vec{G} = \vec{A}+\vec{B}+\vec{C}$

Se metto l'origine nel circocentro O, trovo $\vec{H} = \vec{A}+\vec{B}+\vec{C}$

E l'incentro I?



$$\vec{K} = \frac{b}{a+b}\vec{B} + \frac{a}{a+b}\vec{A}$$

$$\frac{AK}{KB} = \frac{AC}{BC} = \frac{b}{a}$$

$$\frac{AK}{AB} = \frac{b}{a+b} \Rightarrow AK = \frac{bc}{a+b}$$

$$\frac{CI}{IK} = \frac{AC}{AK} = \frac{b}{\frac{bc}{a+b}} = \frac{a+b}{c} \Rightarrow \frac{CI}{CK} = \frac{a+b}{a+b+c}$$

$$\vec{I} = \frac{a+b}{a+b+c}\vec{K} + \frac{c}{a+b+c}\vec{C} = \frac{a\vec{A}+b\vec{B}+c\vec{C}}{a+b+c}$$

(È VERA anche se l'origine NON è il circocentro)

Verifichiamo che $\vec{AH} \perp \vec{BC}$

$$(\vec{H}-\vec{A})\cdot(\vec{C}-\vec{B}) \overset{?}{=} 0$$

$$(\vec{B}+\vec{C})(\vec{C}-\vec{B}) \overset{?}{=} 0$$

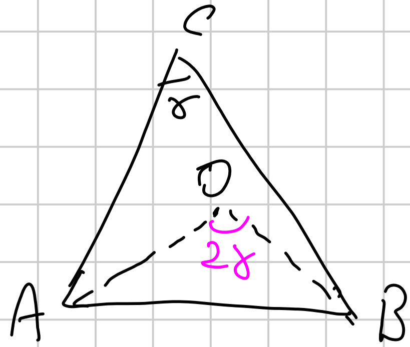
$$\vec{B}\cdot\vec{B} \overset{?}{=} \vec{C}\cdot\vec{C}$$

$$R = \|\vec{B}\| \overset{?}{=} \|\vec{C}\| = R$$

Quanto vale OH?

$$OH^2 = \|\vec{OH}\|^2 = \vec{OH}\cdot\vec{OH} = \vec{H}\cdot\vec{H} = (\vec{A}+\vec{B}+\vec{C})(\vec{A}+\vec{B}+\vec{C}) = 3R^2 +$$

$$+ 2\vec{A}\cdot\vec{B} + 2\vec{A}\cdot\vec{C} + 2\vec{B}\cdot\vec{C}$$



$$\vec{A}\cdot\vec{B} = R^2\cos(2\gamma) = R^2\left(1 - \frac{4\sin^2\gamma}{2}\right) = R^2 - \frac{1}{2}(2R\sin\gamma)^2 = R^2 - \frac{c^2}{2}$$

$$OH^2 = 3R^2 + 2\left(3R^2 - \frac{a^2+b^2+c^2}{2}\right) = 9R^2 - (a^2+b^2+c^2)$$

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Calcoliamo OI

$$OI^2 = \vec{OI}\cdot\vec{OI} = \vec{I}\cdot\vec{I} = \frac{1}{(a+b+c)^2}(a\vec{A}+b\vec{B}+c\vec{C})(a\vec{A}+b\vec{B}+c\vec{C}) =$$

$$= \frac{1}{(a+b+c)^2}\left(R^2(a^2+b^2+c^2) + 2ab\left(R^2-\frac{c^2}{2}\right) + 2ac\left(R^2-\frac{b^2}{2}\right) + 2bc\left(R^2-\frac{a^2}{2}\right)\right) =$$

$$= \frac{1}{(a+b+c)^2}\left(R^2(a+b+c)^2 - \overbrace{abc(a+b+c)}^{-abc^2-ab^2c-a^2bc}\right) = R^2 - \frac{abc}{a+b+c}$$

$$R = \frac{abc}{4S} \qquad r = \frac{2S}{a+b+c} \qquad 2Rr = \frac{abc}{a+b+c}$$

$$OI^2 = R^2 - 2Rr = R(R-2r)$$

oss: Ho appena dimostrato che $R \geq 2r$

Provate a calcolare GH. $\left(GH^2 = 4R^2 - \frac{4}{9}(a^2+b^2+c^2)\right)$

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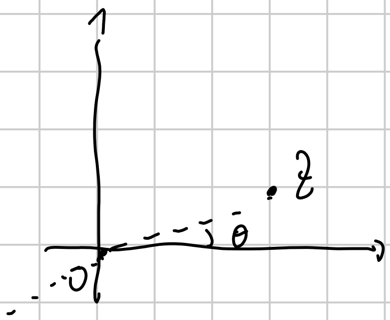
COMPLESSI

$i^2 = -1$

Iniziamo da quando 3 numeri complessi sono allineati:

Caso 1 Uno dei 3 è l'origine



$$\frac{z}{\bar{z}} = \frac{e^{i\theta}}{e^{-i\theta}} = e^{2i\theta}$$

osserviamo che se al posto di θ ci metto $\theta+\pi$

ho $e^{2i(\theta+\pi)} = e^{2i\theta}\cdot e^{2i\pi} = e^{2i\theta}$

$$e^{2i\theta} = e^{2i\varphi} \iff \theta = \varphi \lor \theta = \varphi \pm \pi$$

Dunque $0, z, w$ allineati se e solo se $\frac{z}{\bar z} = \frac{w}{\bar w}$

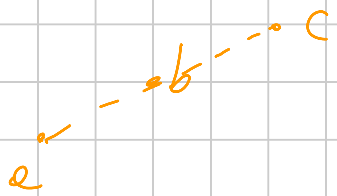
Ora x, y, z allineati $\iff$ $0, y-x, z-x$ allineati

$$\iff \frac{y-x}{\bar y - \bar x} = \frac{z-x}{\bar z - \bar x}$$

Al solito $m = \frac{a+b}{2}$

Supponiamo di avere a e b

$b = \frac{a+c}{2} \leadsto c = 2b - a$



Lemma: il simmetrico di H rispetto a M_A è il diametralmente opposto ad A sulla circoscritta. E il simmetrico di H rispetto a BC è sulla circoscritta.

Dim: Possiamo supporre che $a\bar a = 1$ $b\bar b = 1$ $c\bar c = 1$

Il circocentro è l'origine. Quindi $h = a+b+c$

$m_a = \frac{b+c}{2}$.

Dunque il simmetrico di H rispetto ad M_A è

$$2m_a - h = \cancel{b}+\cancel{c} - a - \cancel{b} - \cancel{c} = -a.$$

La retta BC è $\frac{z-b}{\bar z-\bar b}=\frac{b-c}{\bar b-\bar c}=\frac{b-c}{\frac{1}{b}-\frac{1}{c}}=\frac{b-c}{\frac{c-b}{bc}}=-bc$

$BC: z=b+c-\bar z bc$

$AH: \frac{z-a}{\bar z-\bar a}=\frac{b+c}{\bar b+\bar c}=\frac{b+c}{\frac{b+c}{bc}}=bc$

$AH: z=a+bc\left(\bar z-\frac{1}{a}\right)$

$\{H_A\}=BC\cap AH \Rightarrow \begin{cases} h_a=b+c-\bar h_a bc \\ h_a=a+bc\left(\bar h_a-\frac{1}{a}\right) \end{cases}$

Quindi $2h_a=a+b+c-\frac{bc}{a}$

Ora il simmetrico è $2h_a-h=\left(a+b+c-\frac{bc}{a}\right)-(a+b+c)=-\frac{bc}{a}$

$\left(-\frac{bc}{a}\right)\overline{\left(-\frac{bc}{a}\right)}\stackrel{?}{=}1$

$\left(-\frac{bc}{a}\right)\left(-\frac{\bar b\bar c}{\bar a}\right)\stackrel{?}{=}1$

$\left(-\frac{bc}{a}\right)\left(-\frac{a}{bc}\right)\stackrel{?}{=}1$

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<u>Es:</u> <u>PI</u> 12, 22, 24

<u>PII</u> 6, 10, 16

<u>PIII</u> 2014 B1

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Problema 6

Mettiamo l'origine nel circocentro di ABC.

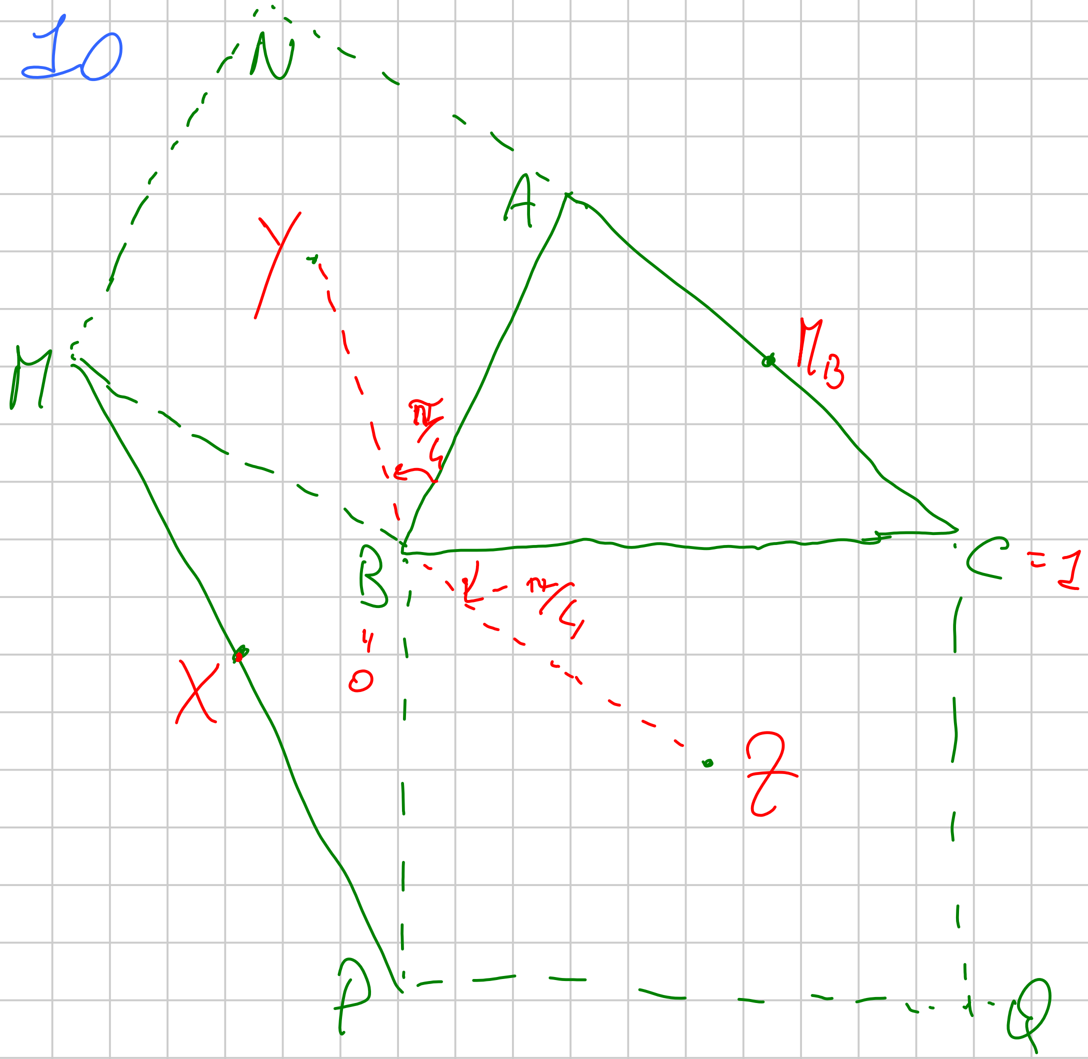
$\ell_c^2=\|\overrightarrow{CM_c}\|^2=(\vec{M_c}-\vec C)\cdot(\vec{M_c}-\vec C)=\frac{1}{4}(\vec A+\vec B-2\vec C)(\vec A+\vec B-2\vec C)=$

$$= \frac{1}{4}\left[2R^2 + 4R^2 + 2\left(R^2 - \frac{c^2}{2}\right) - 4\left(R^2 - \frac{b^2}{2}\right) - 4\left(R^2 - \frac{a^2}{2}\right)\right] =$$

$$= \frac{1}{4}\left(-c^2 + 2b^2 + 2a^2\right) = \frac{2a^2 + 2b^2 - c^2}{4}$$

$$l_a^2 + l_b^2 + l_c^2 = \frac{1}{4}\left(3a^2 + 3b^2 + 3c^2\right) = \frac{3}{4}\left(a^2 + b^2 + c^2\right).$$

Probleme 10

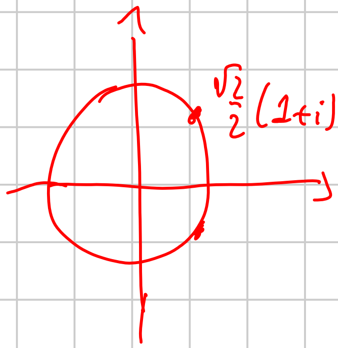


Complessi: Mettiamo $b = 0$ wlog
e anche $q = 1 - i$ wlog
(cioè $c = 1$)

$$z = \boxed{\frac{1-i}{2}}$$

$$m_B = \boxed{\frac{a+1}{2}}$$

$$p = -i$$



$y = a \cdot \frac{\sqrt{2}}{2}(1+i) \cdot \frac{1}{\sqrt{2}}$ (perché $BY = \frac{AB}{\sqrt{2}}$)

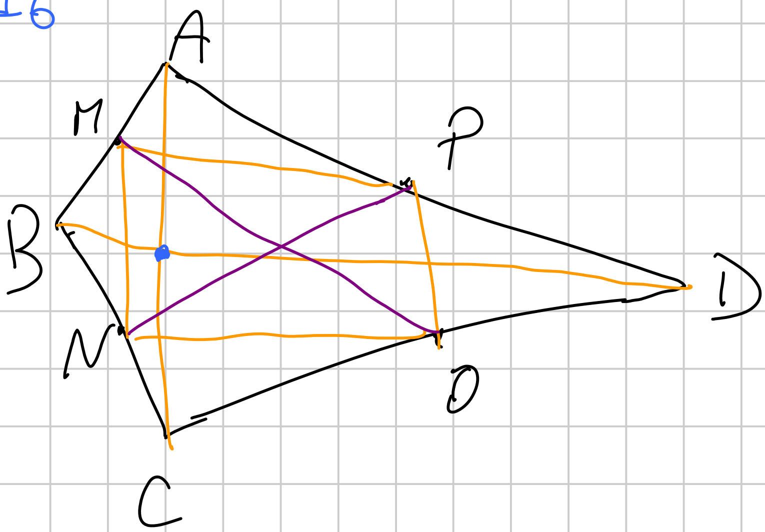
$$y = \boxed{\frac{a(1+i)}{2}}$$

$m = ai$

$$x = \frac{p+m}{2} = \boxed{i\,\frac{a-1}{2}}$$

e qui diventa una verifica.

Problema 16



$$\vec{C} = -\alpha\vec{A} \qquad \alpha,\beta \in \mathbb{R}^+$$
$$\vec{D} = -\beta\vec{B}$$

$$\vec{M} = \frac{\vec{A}+\vec{B}}{2} \qquad \vec{N} = \frac{\vec{B}+\vec{C}}{2} = \frac{\vec{B}-\alpha\vec{A}}{2} \qquad \vec{O} = \frac{\vec{C}+\vec{D}}{2} = \frac{-\alpha\vec{A}-\beta\vec{B}}{2}$$

$$\vec{P} = \frac{\vec{A}+\vec{D}}{2} = \frac{\vec{A}-\beta\vec{B}}{2}.$$

$$\vec{AC} \perp \vec{BD} \iff (\vec{C}-\vec{A})\cdot(\vec{D}-\vec{B}) = 0 \iff (\alpha+1)(\beta+1)\vec{A}\cdot\vec{B} = 0$$
$$\iff \vec{A}\cdot\vec{B} = 0$$

$$MO = NP \iff MO^2 = NP^2 \iff \left(\frac{\vec{A}(\alpha+1)+\vec{B}(\beta+1)}{2}\right)\left(\ ---\ \right) =$$
$$= \left(\frac{\vec{A}(\alpha+1)-\vec{B}(\beta+1)}{2}\right)\left(\ ----\ \right)$$

$\iff \frac{4}{4}(\alpha+1)(\beta+1)\vec{A}'\cdot\vec{B}'=0 \iff \vec{A}'\cdot\vec{B}'=0$

Quindi $AC \perp BD \iff \vec{A}'\cdot\vec{B}'=0 \iff MQ = NP$.

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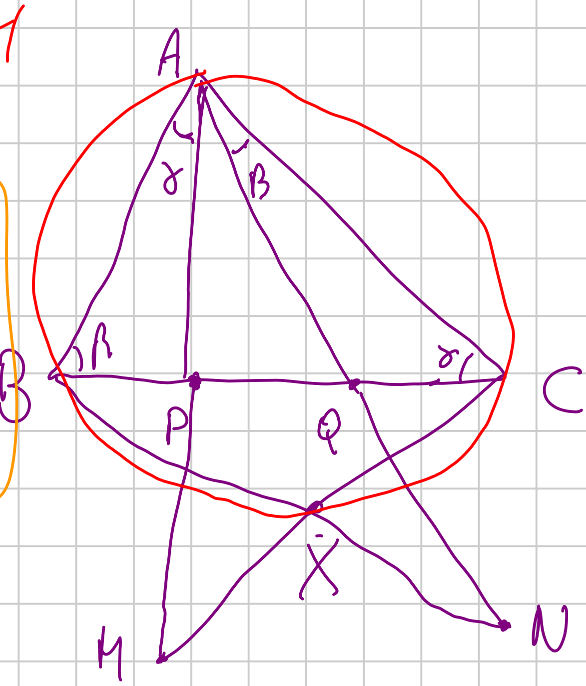
IMO 2014.4

$\boxed{\text{Disegno sbagliato}}$



$BPA \sim BAC$

$CQA \sim CAB$

$$\frac{BP}{AB} = \frac{AB}{BC} \leadsto BP = \frac{c^2}{a}$$

$$\frac{BP}{BC} = \frac{c^2}{a^2} \qquad \vec{P} = \frac{c^2}{a^2}\vec{C} + \left(1-\frac{c^2}{a^2}\right)\vec{B}$$

$$\vec{Q} = \frac{b^2}{a^2}\vec{B} + \left(1-\frac{b^2}{a^2}\right)\vec{C}$$

$$\vec{M} = 2\vec{P} - \vec{A} = 2\frac{c^2}{a^2}\vec{C} + \frac{2(a^2-c^2)}{a^2}\vec{B} - \vec{A}$$

$$\vec{N} = 2\vec{Q} - \vec{A} = \frac{2b^2}{a^2}\vec{B} + \frac{2(a^2-b^2)}{a^2}\vec{C} - \vec{A}$$

$$BM: \ \lambda\vec{M} + (1-\lambda)\vec{B} = \vec{B} + \lambda\left[2\frac{c^2}{a^2}\vec{C} + \frac{a^2-2c^2}{a^2}\vec{B} - \vec{A}\right] \quad \lambda \in \mathbb{R}$$

$$CN: \ \mu \vec{N} + (1-\mu)\vec{C} = \vec{C} + \mu\left[2\frac{b^2}{a^2}\vec{B} + \frac{a^2-2b^2}{a^2}\vec{C} - \vec{A}\right] \quad \mu \in \mathbb{R}$$

$(\vec{A})$ $\lambda = \mu$

$(\vec{B})$ $1 + \lambda\frac{a^2-2c^2}{a^2} = \lambda\, 2\frac{b^2}{a^2} \rightsquigarrow \lambda = \frac{a^2}{2b^2+2c^2-a^2}$

$(\vec{C})$ è uguale!

$$\vec{X} = \vec{B} + \frac{1}{2b^2+2c^2-a^2}\left(2c^2\vec{C} + (a^2-2c^2)\vec{B} - a^2\vec{A}\right)$$

$$\vec{X} = \frac{1}{2b^2+2c^2-a^2}\left(2c^2\vec{C} + 2b^2\vec{B} - a^2\vec{A}\right)$$

$\vec{X}\cdot\vec{X} \overset{?}{=} R^2 \iff$ Tesi

$$\frac{1}{(2b^2+2c^2-a^2)^2}\left(2c^2\vec{C}+2b^2\vec{B}-a^2\vec{A}\right)\left(\_\ \_\ \_\right) \overset{?}{=} R^2$$

$$\left(2c^2\vec{C}+2b^2\vec{B}-a^2\vec{A}\right)\left(\_\ \_\ \_\right) \overset{?}{=} R^2(2b^2+2c^2-a^2)^2$$

$$4c^4R^2 + 4b^4R^2 + a^4R^2 + 8b^2c^2\left(R^2-\frac{a^2}{2}\right) - 4a^2b^2\left(R^2-\frac{c^2}{2}\right)$$
$$-4a^2c^2\left(R^2-\frac{b^2}{2}\right) \overset{?}{=} R^2(2b^2+2c^2-a^2)^2$$

$$R^2(2b^2+2c^2-a^2)^2 - \cancel{4a^2b^2c^2} + \cancel{2a^2b^2c^2} + \cancel{2a^2b^2c^2} \overset{?}{=} R^2(2b^2+2c^2-a^2)^2$$

Quindi è vera.