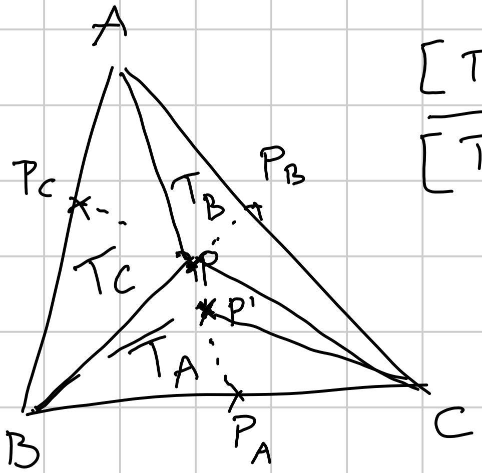
G1 - Medium

leck / elianto84

Titolo nota 03/09/2018

1) Ceva, Meneleo, Van Obel etc
2) Coord tril e bar
3) $\mathbb{C}$, config di Torricelli / Steiner / Fermat / Napoleone
4) Huygens-Steiner etc (parallel axis theorem)

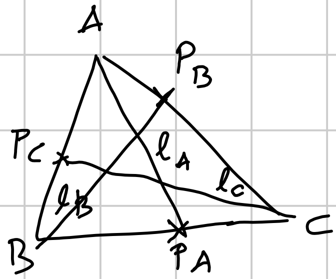


$$\frac{[T_C]}{[T_B]} = \frac{[ABP_A]}{[ACP_A]} = \frac{BP_A}{CP_A}$$

aree orientate

$$\frac{[T_A]}{[T_B]} \cdot \frac{[T_B]}{[T_C]} \cdot \frac{[T_C]}{[T_A]} = 1$$

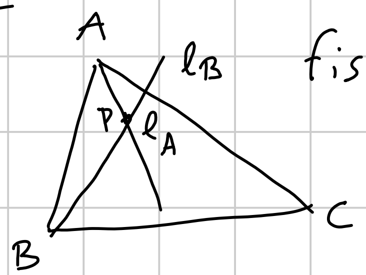
$$\frac{AP_C}{P_CB} \cdot \frac{BP_A}{P_AC} \cdot \frac{CP_B}{P_BA} = 1$$



Teorema di Ceva

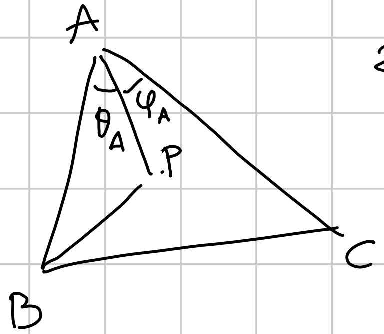
lunghezze con segno

ℓ_A, ℓ_B, ℓ_C concorrono $\Longleftrightarrow$ $\frac{AP_C}{P_CB} \cdot \frac{BP_A}{P_AC} \cdot \frac{CP_B}{P_BA} = 1$



fissati $\frac{[APB]}{[APC]}$, $\frac{[APB]}{[BPC]}$

è fissato anche $\frac{[APC]}{[BPC]}$ $\square$

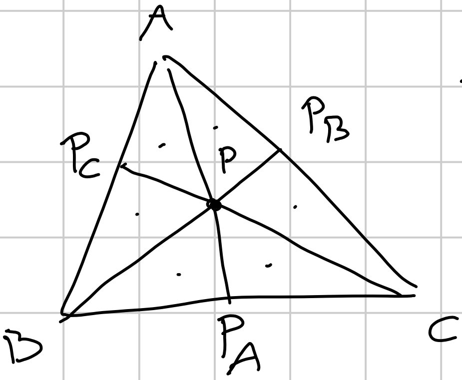


$$2\,[APB] = AP \cdot AB \cdot \sin(\theta_A)$$

Trig Ceva $\sin\theta_A \cdot \sin\theta_B \cdot \sin\theta_C = \sin\varphi_A \sin\varphi_B \sin\varphi_C$

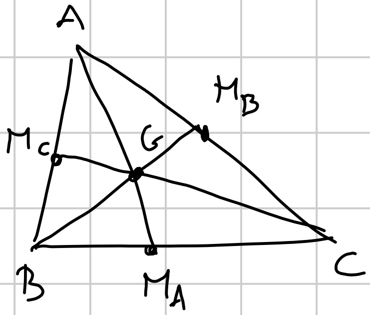
Ceva $\Longleftrightarrow$ Meneleo

Van Obel



$$\frac{AP}{PP_A} = \frac{AP_C}{P_CB} + \frac{AP_B}{P_BC}$$

$$\frac{[ABPC]}{[BPC]} = \frac{[APB]+[APC]}{[BPC]}$$



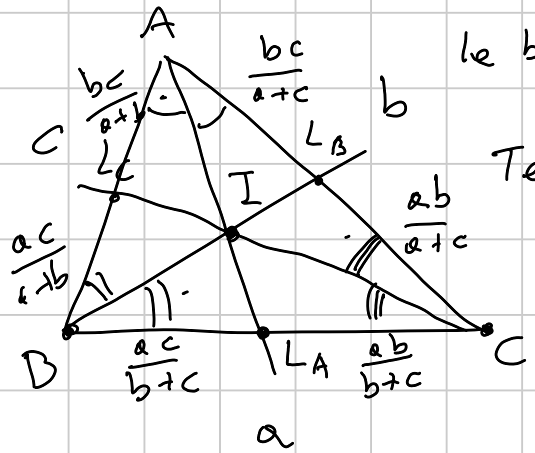
Il baricentro esiste <–> AM_A, BM_B, CM_C concorrono

↕

Ceva vale ovviamente

$$\frac{AG}{GM_A} = \frac{AM_C}{M_CB} + \frac{AM_B}{M_BC} = 1+1 = 2$$

G cade a 2/3 di ogni mediana



L'incentro esiste, ossia le bisettrici concorrono. <—> trig Ceva

Teorema della bisettrice

$$\frac{BL_A}{L_AC} = \frac{c}{b}$$

$$\frac{2[ABL_A]}{2[ACL_A]} = \frac{AB\cdot \cancel{AL_A}\cdot \cancel{\sin\frac{A}{2}}}{AC\cdot \cancel{AL_A}\cdot \cancel{\sin\frac{A}{2}}}$$

$$\| $$

$$\frac{BL_A}{CL_A}$$

$$BL_A = \frac{c}{b+c}\cdot a \qquad CL_A = \frac{b}{b+c}\cdot a$$

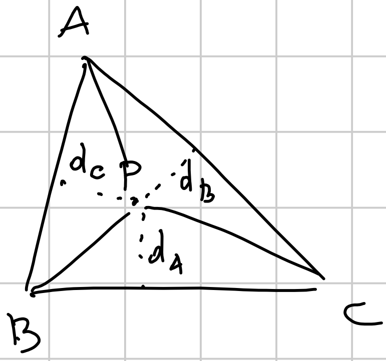
$$\frac{AI}{IL_A} = \frac{AL_C}{L_CB} + \frac{AL_B}{L_BC} = \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a}$$

$$AI = \frac{b+c}{b+c+a}\cdot AL_A$$

→ si trova con Stewart

↑

teorema del coseno



$$2[APB] = AB \cdot d_C$$

$$\text{Ceva} \iff \frac{d_A}{d_B} \cdot \frac{d_B}{d_C} \cdot \frac{d_C}{d_A} = 1$$

$[d_A ; d_B ; d_C]$ coordinate trilineari esatte di P

$[\gamma d_A ; \gamma d_B ; \gamma d_C] \quad \forall \gamma \neq 0$

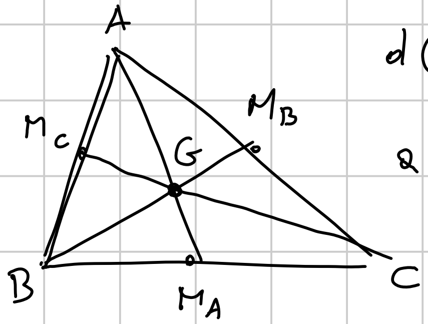
la terna $[d_A ; d_B ; d_C]$, definita a meno di moltiplicaz. per costanti non nulle, ci permette di identificare univocamente il punto P. coordinate trilin.

Incentro coordinate tril. esatte $[r ; r ; r]$

coord tril. $[1 ; 1 ; 1]$ $\cdot \lambda$

$$a d_A + b d_B + c d_C = 2[ABC]$$
$$a r + b r + c r = 2[ABC]$$

$$\lambda = \frac{2[ABC]}{a+b+c}$$



$$d(G, BC) = \frac{1}{3} \cdot \frac{2\Delta}{a} \qquad d(G, AB) = \frac{2\Delta}{3c}$$

$$a \cdot h_A = 2\Delta \qquad d(G, AC) = \frac{2\Delta}{3b}$$

$$h_A = \frac{2\Delta}{a}$$

$G\left[\frac{2\Delta}{3a} ; \frac{2\Delta}{3b} ; \frac{2\Delta}{3c}\right]$ tril. esatte

$G\left[\frac{1}{a} ; \frac{1}{b} ; \frac{1}{c}\right]$ Trilineari

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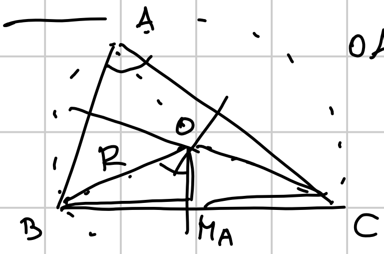


$$OA = OB = OC = R$$

$$\widehat{BOC} = 2\hat{A} \qquad \widehat{BOM_A} = \hat{A} \qquad OM_A = R\cos A$$

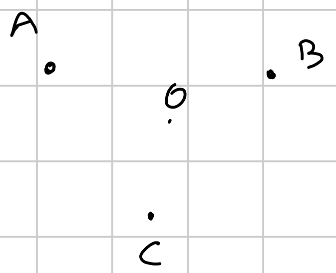
O $\quad [R\cos A, R\cos B, R\cos C]$ tril. esatte

$[\cos A, \cos B, \cos C]$ tril

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\| $$

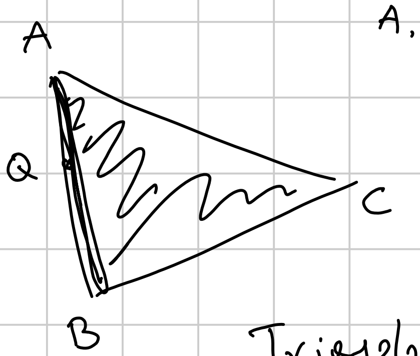
$$[\underbrace{a(b^2+c^2-a^2)}_{\text{triangle center function } (g)}; \dots ; \dots]$$



Coordinate di $A, B, C \in \mathbb{R}^3$

$\downarrow$

Coordinate di O



$A, B, C \in \mathbb{R}^n$

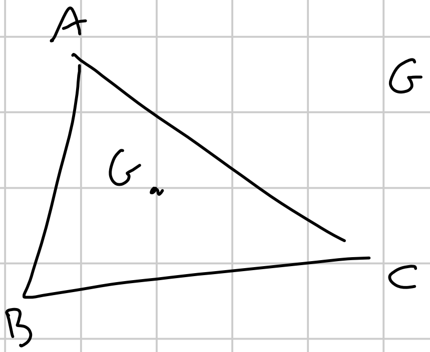
$$\forall Q \in AB \quad \exists \lambda \in [0,1] : \quad Q = \underbrace{\lambda A + (1-\lambda) B}_{\text{comb. convesse di } A \text{ e } B}$$

$$\text{Triangolo } ABC \equiv \left\{ xA + yB + zC : \begin{array}{l} x+y+z=1 \\ x,y,z \in [0,1] \end{array} \right\}$$

inviluppo convesso di A, B, C (convex hull) più piccolo (rispetto a $\subseteq$) insieme convesso che contiene A, B e C.

$[x; y; z]$ coordinate baricentriche esatte

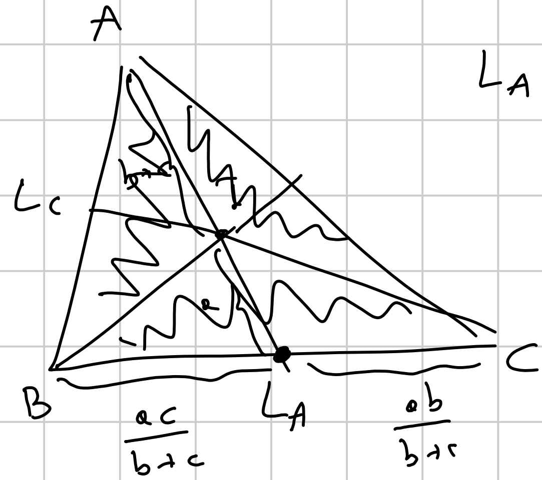
$[x; y; z]$ coordinate baricentriche (a meno di moltiplic. per scalari non nulli)



$$G = \frac{A+B+C}{3}$$

coord baricentriche di $G = [1;1;1]$

Coord baricentriche dell'incentro



$$L_A = B + (C-B)\cdot\frac{ac}{a}\cdot\frac{1}{b+c}$$

$$= B + (C-B)\frac{c}{b+c} = \frac{b}{b+c}\cdot B + \frac{c}{b+c}\, C$$

$$I = \frac{aA + bB + cC}{a+b+c}$$

$I = [a, b, c]$ baricentriche

Conversione tra trilineari e baricentriche

$[\alpha, \beta, \gamma]$ trilineari $\longrightarrow$ $[a\alpha, b\beta, c\gamma]$ baricentriche

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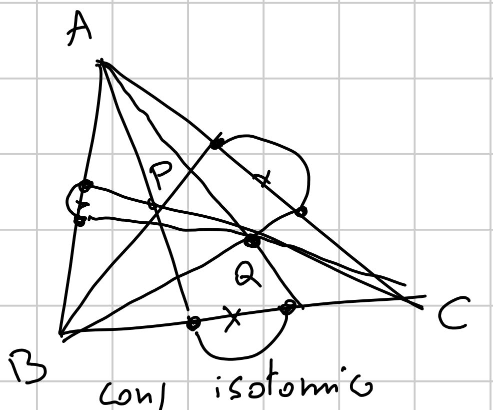
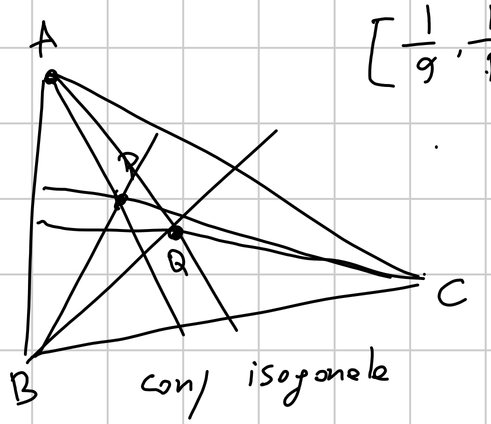
Ceva $\longrightarrow$ coniug. isotomico

Trig Ceva $\longrightarrow$ coniug. isogonale



coniug isotomico

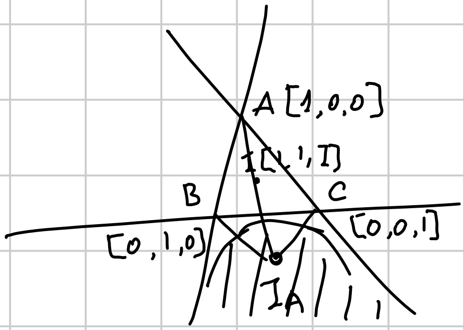
vengono reciprocate le coord. baricentriche



coniug isogonale

$\left[\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right]$ coord tril. del coniug. isogon. di P

$[\alpha, \beta, \gamma]$ coord. tril di P coord. tril. esatte $[\alpha, \beta, \gamma]$



$[-,+,+]$
$[+,-,-]$

baricentriche $[1;1;-2]$

$$\frac{A+B-2C}{1+1-2}$$

$\mathbb{R}^2 \subseteq \mathbb{P}^2(\mathbb{R})$

i punti con coord baricentriche $[\alpha, \beta, \gamma]$ con $\alpha+\beta+\gamma=0$ appartengono alla retta all'infinito di $\mathbb{P}^2(\mathbb{R})$

In coordin tril (o bar) le rette si scrivono come oggetti del tipo $u\alpha + v\beta + w\gamma = 0$

$\hookrightarrow$ $[u, v, w]$ coord tril (duali) delle rette

dualità proiettiva rette $\longleftrightarrow$ punti

Retta OH in trilineari.

$O = [\cos A, \cos B, \cos C]$ tril
coniug. iso
$H = \left[\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right]$ tril

$\downarrow$

$[u; v; w]$

$$\begin{cases} u\cos A + v\cos B + w\cos C = 0 \\ \dfrac{u}{\cos A} + \dfrac{v}{\cos B} + \dfrac{w}{\cos C} = 0 \end{cases}$$

$$\det \begin{vmatrix} \cos A & \cos B & \cos C \\ \frac{1}{\cos A} & \frac{1}{\cos B} & \frac{1}{\cos C} \\ u & v & w \end{vmatrix}$$

$$u = \frac{\cos B}{\cos C} - \frac{\cos C}{\cos B} = \frac{\cos^2 B - \cos^2 C}{\cos B \cos C} = \frac{\sin^2 C - \sin^2 B}{\cos B \cos C}$$

Retta di Eulero $\quad O, G, H$ sono allineati

$$\det \begin{vmatrix} \cos A & \cos B & \cos C \\ \frac{1}{\cos A} & \frac{1}{\cos B} & \frac{1}{\cos C} \\ \frac{1}{\sin A} & \frac{1}{\sin B} & \frac{1}{\sin C} \end{vmatrix} = 0$$

$a = 2R \sin A$

$2bc \cos A = b^2 + c^2 - a^2$

$$\det \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - dbi$$

Regola di Sarrus

$$\det \begin{vmatrix} a & \\ b & \\ c & M_a \\ d & \end{vmatrix} = a \det(M_a) - b \cdot \det(M_b) + c \cdot \det(M_c) - d \det(M_d)$$

sviluppo di Laplace

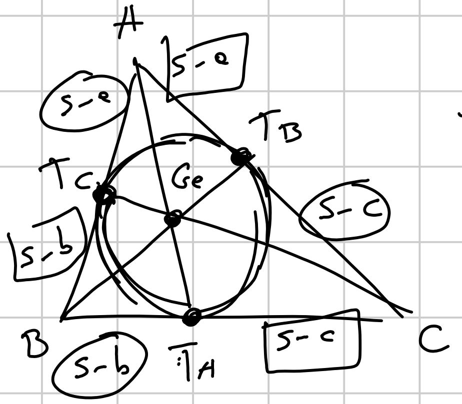
$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & & & \\ a_{N1} & \dots & \dots & a_{NN} \end{pmatrix}$$

$$\det M = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{k=1}^{n} a_{k,\sigma(k)}$$

volume orientato della "scatola" generata dai vettori riga

Esercizio : I, G, Na, Sp sono allineati $\quad$ (O, G, H sono allin.)

I incentro, G baricentro, Na Nagel, Sp Spieker $\quad$ Ge Gergonne

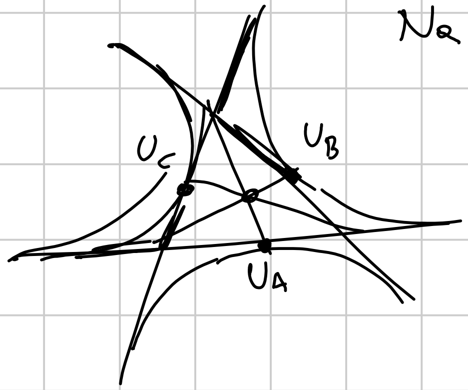


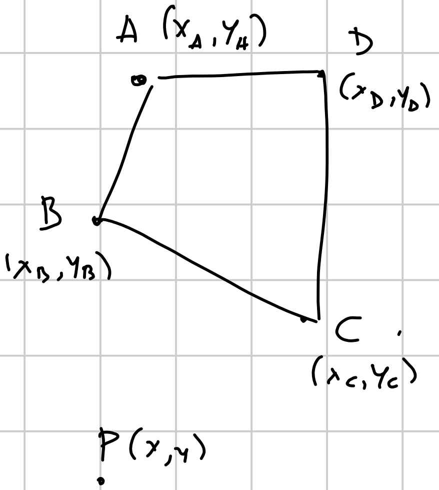
Baricentriche di Ge

$$T_A = B + (C - B) \cdot \frac{s-b}{a}$$

$$\frac{A\,Ge}{Ge\,T_A} = \frac{s-a}{s-b} + \frac{s-a}{s-c}$$

Na è coniug. isotomico di Gergonne





Luogo di punti P per cui

$$PA^2 + PB^2 + PC^2 + PD^2 = 1000000$$

$$\sum_{cyc} (x - x_A)^2 + (y - y_A)^2 = 1'000'000$$

$$x^2 + y^2 = -3$$

$$x^2 + y^2 = 0$$

$$f(P) = PA^2 + PB^2 + PC^2 + PD^2$$

dove si trova il minimo di f?

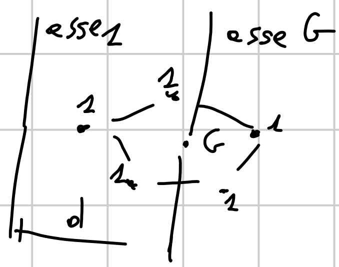
$$f(x) = x^2 - \underline{17}x + 8$$

$$f(x, y) = 4x^2 + 4y^2 - 2x \sum_{cyc} x_A + \sum_{cyc} x_A^2 - 2y \sum_{cyc} y_A + \sum_{cyc} y_A^2$$

$$\arg\min f = \left(\frac{\sum_{cyc} x_A}{4}, \frac{\sum_{cyc} y_A}{4} \right) = G$$

si scrive in soli termini di PG^2

Huygens-Steiner: il momento di inerzia rispetto a $asse_1$ è pari al momento di inerzia rispetto a $asse_G$ + $d^2 \cdot$ (massa del sistema).



Distanza IG.

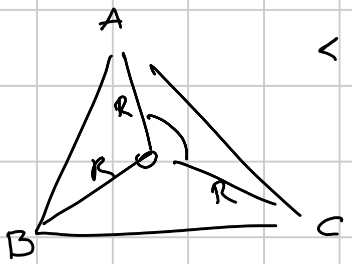
$$I = \frac{aA + bB + cC}{a+b+c} \qquad G = \frac{A+B+C}{3}$$

$$I - G = x_A A + x_B B + x_C C$$

$$IG = |I - G|$$

$$IG^2 = \langle I-G, I-G \rangle$$

In un sistema di rif centrato nel circocentro



$$\langle A, C \rangle = R^2 \cos(2B)$$

OH^2 si scrive solo in termini di R^2 e $(a^2+b^2+c^2)$.

Utilizzo di $\mathbb{C}$ in geometria.

$\mathbb{R}^2 \subseteq \mathbb{P}^2(\mathbb{R})$

$\mathbb{R}^2 \subseteq \mathbb{C}$

Teo fond Algebra : $\mathbb{C}$ è algebricamente chiuso, ossia

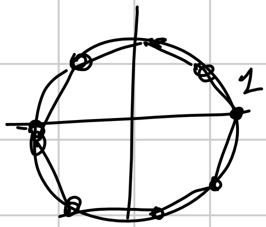
$$\forall p(x) \in \mathbb{C}[x], \ \deg p \geq 1$$
$$\exists \xi \in \mathbb{C} : \ p(\xi) = 0 .$$

(+ Ruffini) Ogni $p(x) \in \mathbb{C}[x]$ con $\deg p \geq 1$ ha tante radici complesse (contate con molt.) quante il suo grado.

Gli zeri di $z^n - 1$ sono detti radici n-esime dell'unità.

$z^n = 1 \qquad |z|^n = 1 \qquad |z| = \sqrt{z \cdot \bar{z}}$

$|z| = 1 \qquad \overline{a+bi} = a - bi$



ξ radice <u>primitiva</u> n-esima dell'unità se è radice n-esima ma non è radice d-esima per un qualche $d \mid n$, $d < n$.

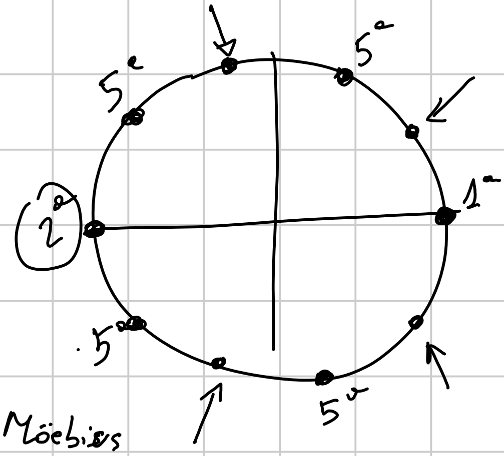
Quante sono le radici primitive n-esime? $\varphi(n)$

Sono tutte radici di $\Phi_n(x)$, n-esimo polinomio ciclotomico

$$\Phi_3(x) = \frac{x^3-1}{x-1} = x^2+x+1 \in \mathbb{Z}[x] \text{ ed è monico}$$

$$\Phi_{10}(x) = \frac{x^{10}-1}{x^5-1} \cdot \frac{x^2-1}{x-1}$$

$\varphi(10) = (2-1)(5-1)$



$$\Phi_n(x) = \prod_{d \mid n} (x^d - 1)^{\mu\left(\frac{n}{d}\right)}$$

μ f. di Möebius

Se ζ è radice di $z^n - 1$, vale $\zeta = \exp\left(\frac{2\pi i}{n} \cdot k\right)$

$$= \cos\left(\frac{2\pi}{n} k\right) + i \sin\left(\frac{2\pi}{n} k\right)$$

$$e^x = \sum_{n \geq 0} \frac{x^n}{n!} \qquad e^z \doteq \sum_{n \geq 0} \frac{z^n}{n!}$$

Bin. Newton $\Big\downarrow$

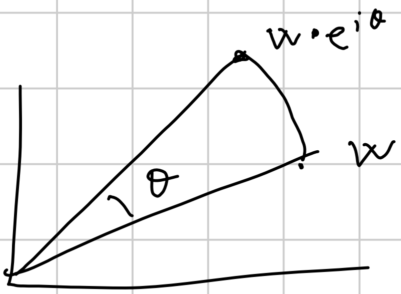
$$e^\alpha \cdot e^\beta = e^{\alpha+\beta}$$

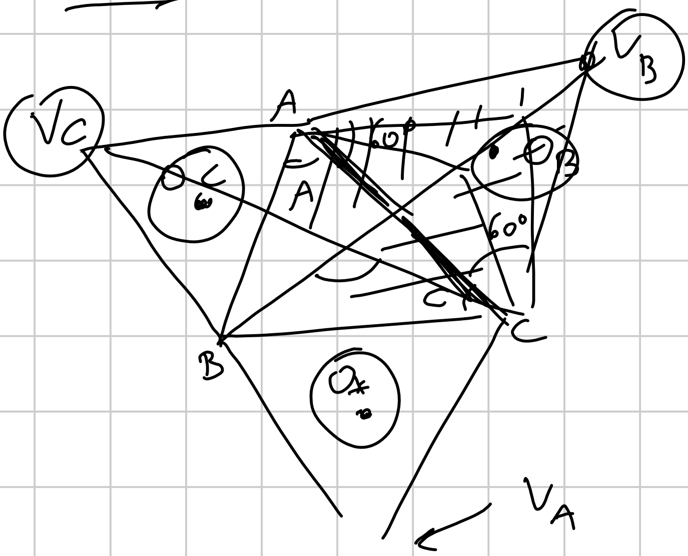
$$\left.\begin{aligned} \sin z &= \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ \cos z &= \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{(2n)!} \end{aligned}\right\} \quad e^{iz} = \cos z + i \sin z$$

$$\boxed{e^{i\pi} + 1 = 0}$$

$\forall z \in \mathbb{C}$, $|z| = 1$, $z = \exp(i\theta)$ per $\theta \in \mathbb{R}$

$$w \longmapsto w \cdot \exp(i\theta) = w(\cos\theta + i \sin\theta)$$





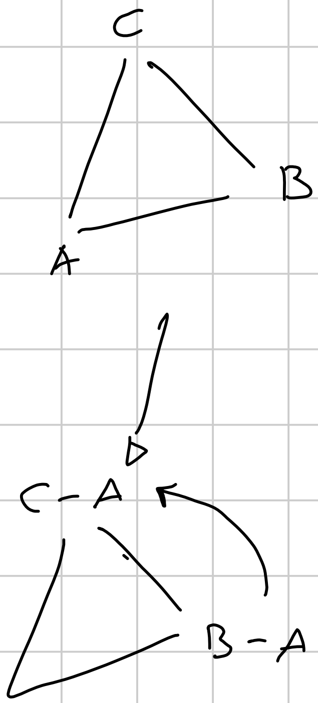
Ex 1. AV_A etc concorrono

$$\frac{[ABV_B]}{[BCV_B]} = \frac{c}{b} \cdot \frac{\sin(A+60°)}{\sin(C+60°)}$$

($\exists$ Iperbole di Kiepert)

$\boxed{O_A O_B O_C \text{ è equilatero.}}$

$AV_A \cap BV_B$ = punto di Fermat-Torricelli



A - A = 0

è equilatero se e solo se una rotazione di 60° attorno ad A porta B in C

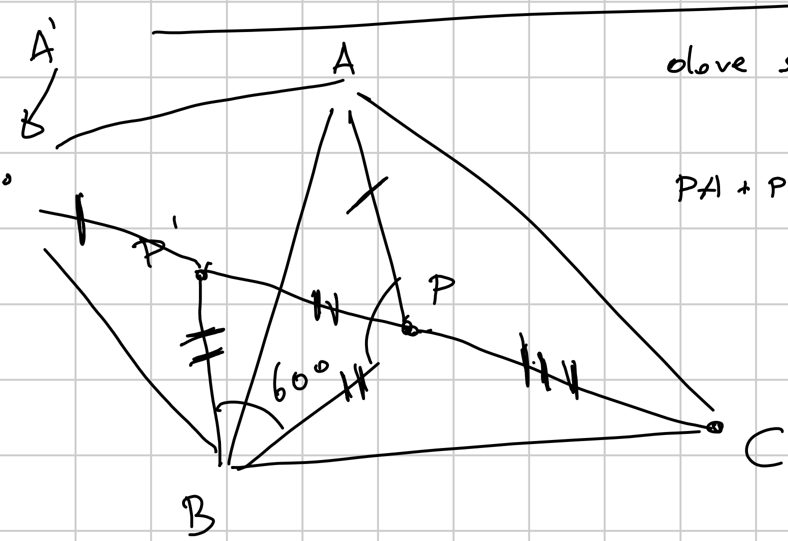
$$\omega = \exp\left(\frac{i\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1+i\sqrt{3}}{2}$$

$$(B-A)\omega = (C-A)$$

$$B\omega - A\omega + A - C = 0$$

$$A + \omega B + \omega^2 C = 0$$



dove si trova P che minimizza $PA + PB + PC$

$$PA + PB + PC = CP + PP' + P'A' \geq CA'$$

$$\widehat{BPC} = \widehat{APB} = \widehat{CPA} = 120°$$

si può dimostrare anche minimizzando $PA + PB + PC$ via Moltiplicatori di Lagrange

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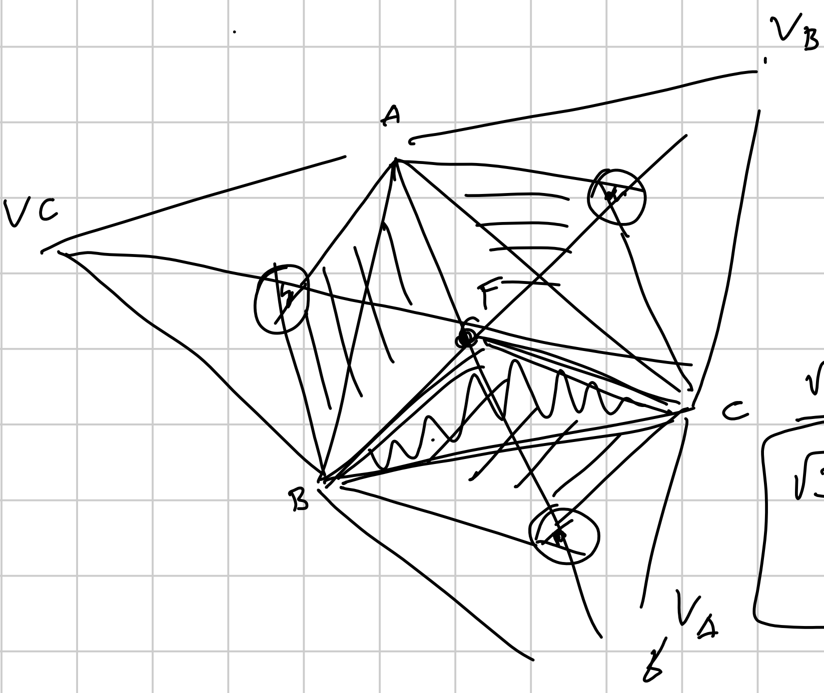
Disug di Weitzenbock

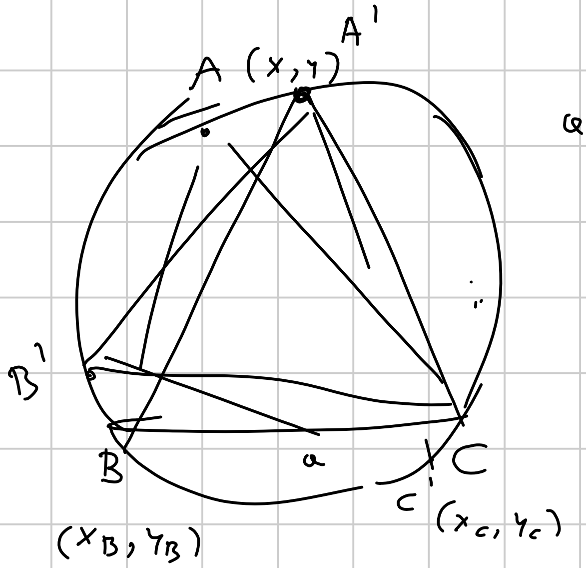


$$\Delta + \frac{\sqrt{3}}{4}(a^2+b^2+c^2) \geq 4\Delta$$

$$\sqrt{3}(a^2+b^2+c^2) \geq 12\Delta$$

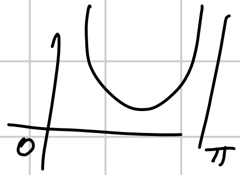
$$\boxed{\sqrt{3}(ab+ac+bc) \geq 12\Delta}$$

(Roland)



$$a^2 + (x - x_B)^2 + (y - y_B)^2 + (x - x_c)^2 + (y - y_c)^2$$
$$= k$$

$abc = 4R\Delta$



$\frac{1}{\sin\theta}$ è convessa su $(0, \pi)$

vale la disug di Jensen.

www.oliforum.it
www.matemate.it → Appunti & Mat. Did.
math.stackexchange.com
www.cut-the-knot.org

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