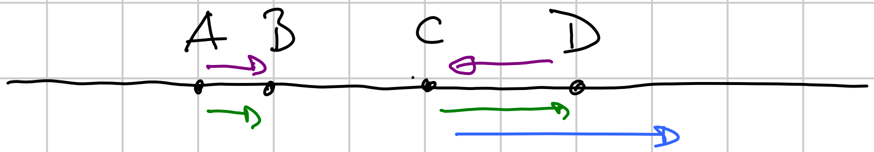
G2 medium - Proiettive - Sam

Titolo nota 05/09/2018

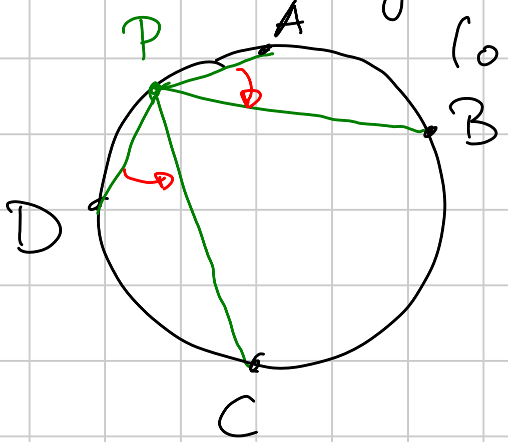
0) Segmenti e archi orientati



$AB > 0 \qquad BA < 0$

$$\frac{AB}{CD} > 0 \qquad \frac{DC}{AB} < 0$$

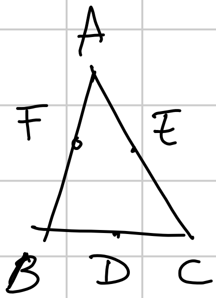
Stessa cosa con gli archi di circonferenza (o le corde)



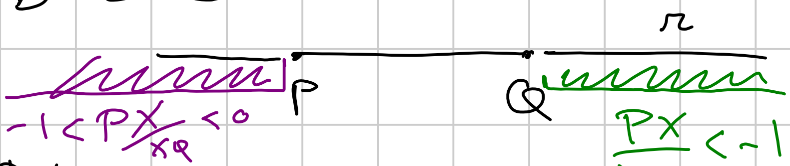
$$\frac{AB}{DC} < 0 \qquad \frac{DB}{BC} > 0$$

$AB \sim A\hat{P}B < 0$

$DC \sim D\hat{P}C > 0$



$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = \begin{cases} 1 & \text{se } AD, BE, CF \text{ concorrono} \\ -1 & \text{se } D, E, F \text{ sono allineati.} \end{cases}$$



$$\frac{PX}{XQ} = 0 \iff X \equiv P$$

$X \in r$

$\frac{PX}{XQ}$ è $\begin{cases} > 0 & \text{se } X \text{ sta nel seg. } PQ \\ < 0 & \text{se } X \text{ sta fuori} \end{cases}$

1) non fa mai -1

2) in $X = Q$ non è definita

Risolviamo 2) dicendo che $\frac{PX}{XQ} = \infty$ quando $X = Q$.

Risolviamo 1) aggiungendo un punto in più alla retta, detto punto all'infinito, cioè l'unico punto X_∞ t.c. $\frac{PX_\infty}{X_\infty Q} = -1$.

Tutte le rette parallele a r passano per X_∞.

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1) Birapporti

Dati A, B, C, D su una retta r, il birapporto è $(A,B;C,D) = \frac{AC}{CB} \cdot \frac{BD}{DA}$

$$\parallel$$

$$\frac{AC}{CB} \Big/ \frac{AD}{DB}$$

Oss: 1) $(A,B;C,D) = 1 \iff C \equiv D$ o $A \equiv B$

2) $\left.\begin{matrix} (B,A;D,C) \\ (C,D;A,B) \\ (D,C;B,A) \end{matrix}\right\}$ sono tutti uguali a $(A,B;C,D) = \lambda$

$(B,A;C,D) = \frac{1}{\lambda}$ $\quad$ $(C,B;A,D) = \frac{\lambda}{1-\lambda}$

$(D,B;C,A) = 1-\lambda$

$$\boxed{\lambda, \frac{1}{\lambda}, \frac{\lambda}{1-\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{1-\lambda}{\lambda}}$$

3) r (con anche il pt all'∞) $\longrightarrow \mathbb{R} \cup \{\infty\}$

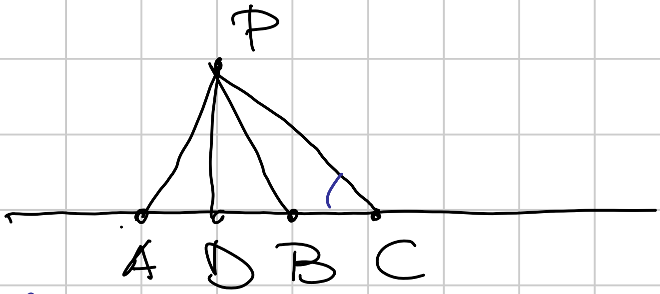
$$D \longmapsto (A,B;C,D)$$

fisso A,B,C su r — questa funzione è bigettiva.

$$A \longmapsto \infty$$
$$B \longmapsto 0$$
$$C \longmapsto 1$$
$$X_\infty \longmapsto -\frac{AC}{CB}$$

Lemma 1 $A,B,C,D \in r$, $P \notin r$

Allora $(A,B;C,D)$ si scrive in termini degli angoli in P.



dim: $A\hat{P}C = \alpha$ $\quad A\hat{P}D = \gamma$

$C\hat{P}B = \beta$ $\quad D\hat{P}B = \delta$

Teo dei seni in $\overset{\triangle}{APC}$: $\dfrac{AC}{\sin\alpha} = \dfrac{AP}{\sin A\hat{C}P}$

" " " " $\overset{\triangle}{BPC}$: $\dfrac{CB}{\sin\beta} = \dfrac{BP}{\sin B\hat{C}P}$

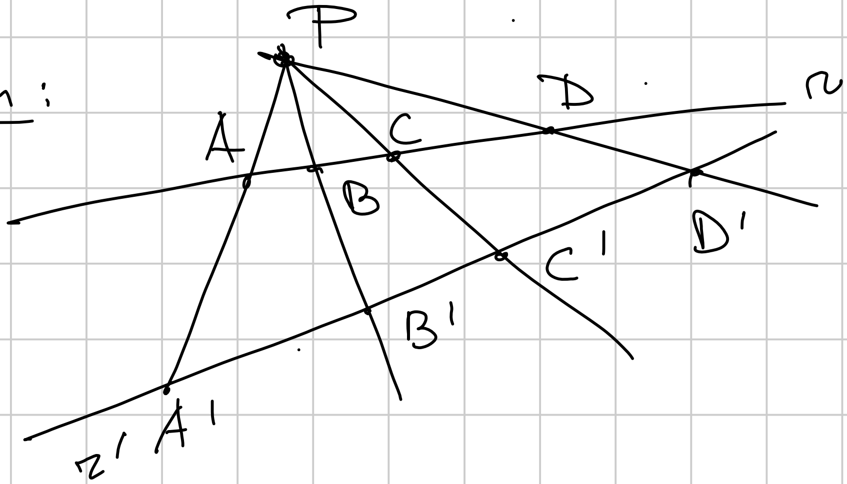
" " " $\overset{\triangle}{BPD}$: $\dfrac{BD}{\sin\delta} = \dfrac{BP}{\sin B\hat{D}P}$

" " " $\overset{\triangle}{DPA}$: $\dfrac{DA}{\sin\gamma} = \dfrac{AP}{\sin A\hat{D}P}$

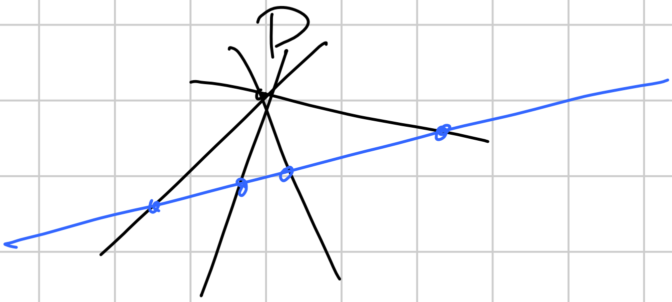
$$\frac{AC}{CB}\cdot\frac{BD}{DA} = \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\delta}{\sin\gamma}$$

Cor 1: Se $A, B, C, D \in r$ e AA', BB', CC', DD' concorrono,
$A', B', C', D' \in r'$
allora $(A, B; C, D) = (A', B'; C', D')$.

dim:



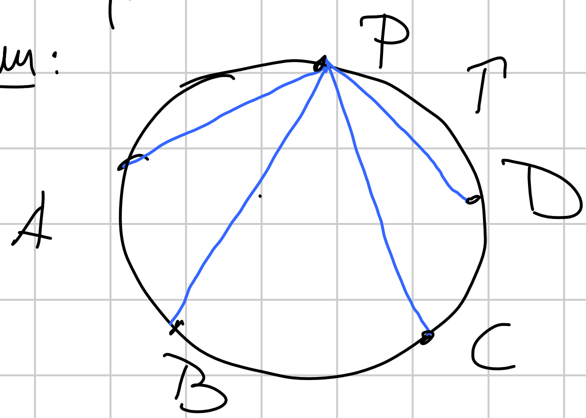
Def: Date 4 rette r_1, r_2, r_3, r_4 concorrenti in un punto P,
$(r_1, r_2; r_3, r_4) = (A, B; C, D)$
con $A = r_1 \cap r$, $B = r_2 \cap r$, $C = r_3 \cap r$, $D = r_4 \cap r$
per una retta r che NON PASSA per P



Notazione: $(A, B; C, D)_P =$
$= (PA, PB; PC, PD)$

Cor 2: Se A, B, C, D sono concicliche su Γ,
allora $(A, B; C, D)_P$ ha lo stesso valore
quando P varia su Γ.

dim:



corde orientate

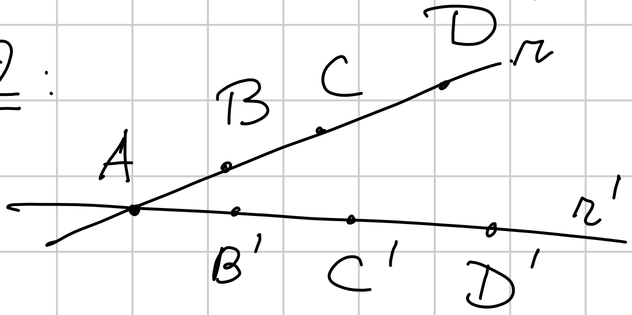
$$\frac{\sin \hat{APC}}{\sin \hat{CPB}} \cdot \frac{\sin \hat{BPD}}{\sin \hat{DPA}} = \frac{AC}{CB} \cdot \frac{BD}{DA}$$

teo delle corde

Def: A, B, C, D su Γ $\quad (A,B;C,D)_\Gamma = (A,B;C,D)_P$
per un qualsiasi $P \in \Gamma$.

Nota: $P = A$
Cosa vuol dire $(AA, AB; AC, AD)$??
AA è la retta tangente a Γ in A.

Prop 2:



BB', CC', DD' concorrono se e solo se $(A,B;C,D) =$
$= (A,B';C',D')$

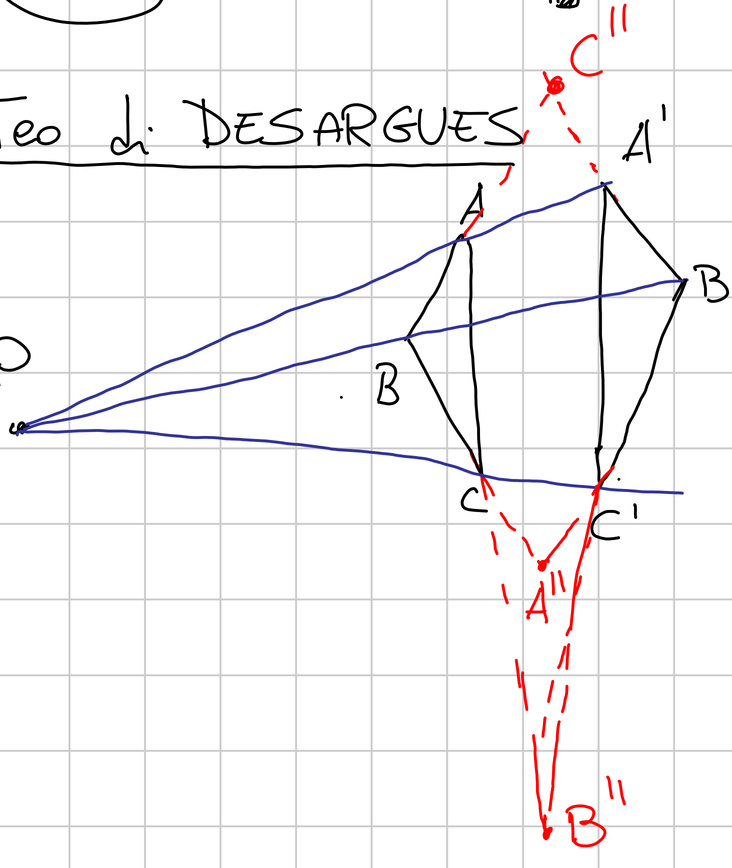
dim: $\Leftarrow$ $\quad P = BB' \cap CC'$

$$(A,B';C',D') = (A,B;C,D) = (PA,PB;PC,PD) \overset{\text{interseco con } r'}{\simeq} (A,B';C',X)$$
$$X = PD \cap r'$$

per l'iniettività del birapporto $X = D'$.
$\Rightarrow$ concorrono.

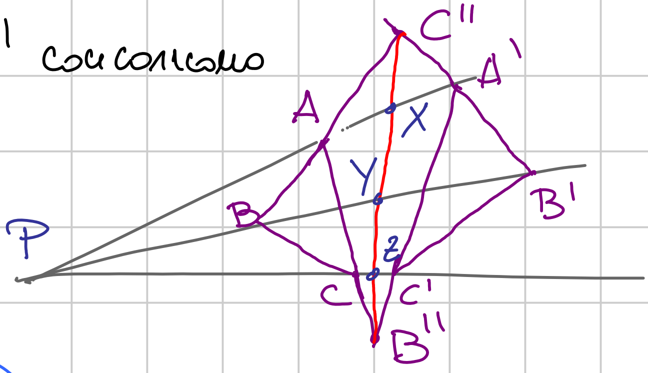
$\Rightarrow$ x esercizio. ∎

Teo di DESARGUES



AA', BB', CC' concorrono
$\Downarrow$
$AB \cap A'B' = C''$
$AC \cap A'C' = B''$
$BC \cap B'C' = A''$
sono allineati

dim: Voglio dire che BC, $B'C'$, $B''C''$ concorrono



Chiamo $X = AA' \cap B''C''$

$Y = BB' \cap B''C''$

$Z = CC' \cap B''C''$

$$(P,A;X,A') \underset{\text{da } C''}{=} (P,B;Y,B')$$

da B'' $\Vert$

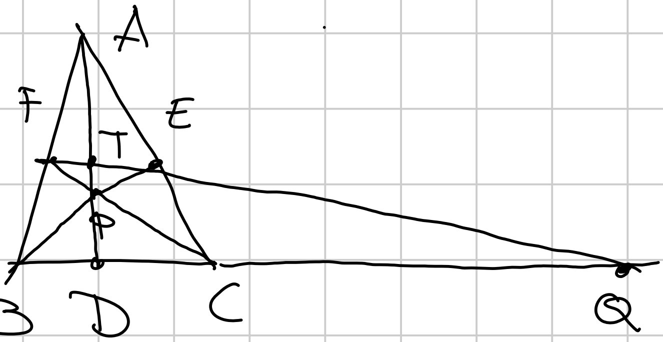
$$(P,C;Z,C')$$

$\Rightarrow$ BC, YZ, $B'C'$ concorrono per la Prop 2.

Es: Se A'', B'', C'' sono allineati, allora AA', BB', CC' concorrono.

Esempi di birapporti

(1)



$$(B,C;D,Q) \underset{\substack{\text{proiez da } A\\ \text{su } EF}}{=} (F,E;T,Q) \underset{\substack{\text{proiez da } P\\ \text{su } BC}}{=} (C,B;D,Q)$$

$$\Rightarrow (B,C;D,Q) = \frac{1}{(B,C;D,Q)}$$

non può fare 1 $\Rightarrow$ $(B,C;D,Q) = -1$

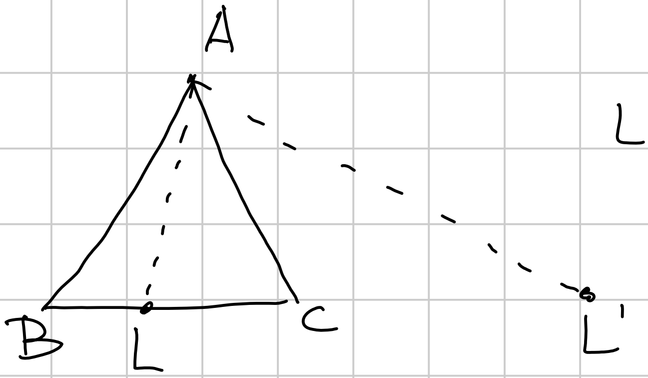
Oppure

Per menelao E,F,Q allineati $\Rightarrow \frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CQ}{QB} = 1$

Per Ceva, AD, BE, CF concorrenti $\Rightarrow \frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA}=1$

$\Rightarrow$ Facendo il rapporto $\boxed{\frac{BD}{DC}\cdot\frac{CQ}{QB}=-1}$.

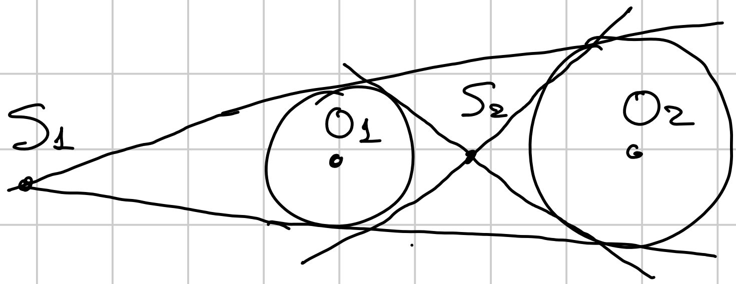
(2)



L, L' piedi di bisettrice interna e esterna

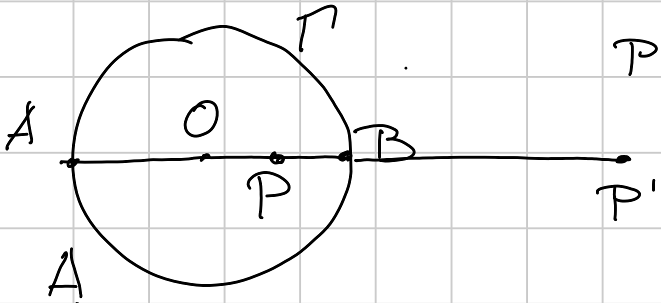
$(B, C; L, L') = ?$

(3)



$(O_1, O_2; S_1, S_2) = ?$

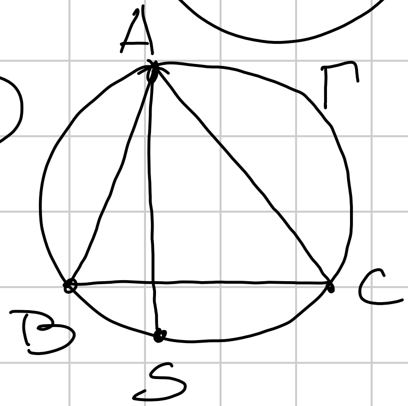
(4)



P' inverso di P in Γ

$(A, B; P, P') = ?$

(5)



AS simmediana

$(A, S; B, C) = ?$

Soluzioni

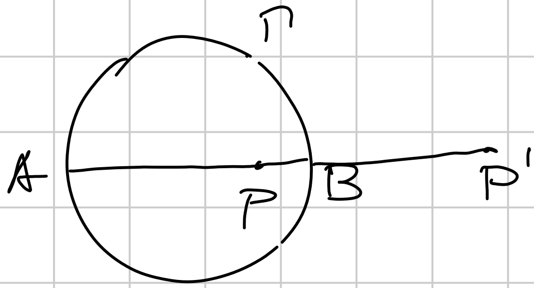
(2) Teo delle bisettrice : $\frac{BL}{LC} = \frac{BA}{AC}$ $\quad \frac{BL'}{L'C} = -\frac{BA}{AC}$

$\Rightarrow (B, C; L, L') = -1$

(3) S_1, S_2 centri di similitudine

$$\Rightarrow \quad \frac{O_1S_1}{S_1O_2} = -\frac{R_1}{R_2} \qquad \frac{O_1S_2}{S_2O_2} = \frac{R_1}{R_2} \quad \Rightarrow (O_1,O_2;S_1,S_2) = -1$$

(4)



(a) faccio i conti

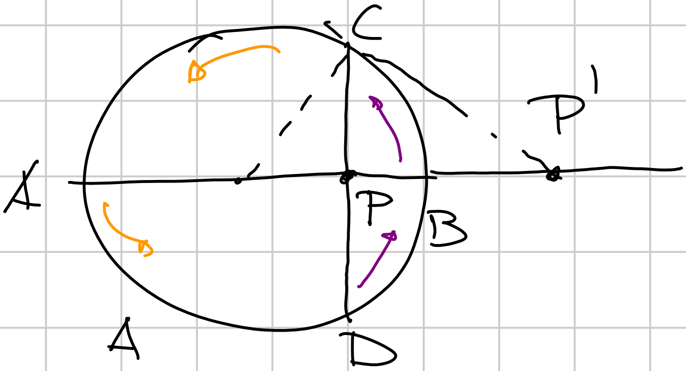
$$X, Y \xrightarrow[\text{inversione}]{} X', Y'$$

$$X'Y' = R^2 \frac{XY}{OX\cdot OY}$$

$$BP' = \frac{BP}{\cancel{OB}\cdot OP}\cdot R^{\cancel{2}}$$

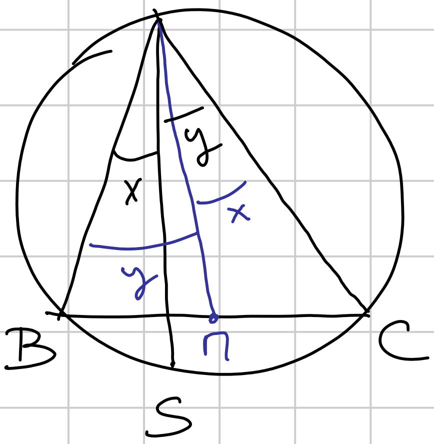
$$AP' = \frac{AP}{\cancel{OA}\cdot OP}\cdot R^{\cancel{2}} \qquad \Rightarrow (A,B;P,P') = -1$$

(b)



$$(A,B;P,P') = (CA,CB;CP,CP') = (A,B;D,C)_\Gamma = \frac{AD}{DB}\cdot\frac{BC}{CA} = -1$$

(5)



$(A,S;B,C)$

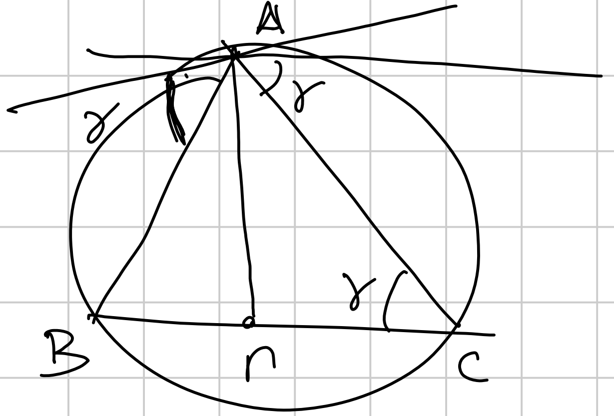
(a)
$$\frac{MC}{\sin x} = \frac{AC}{\sin A\hat{M}C} \qquad \frac{MB}{\sin y} = \frac{AB}{\sin A\hat{M}B} \quad \Rightarrow \frac{\sin x}{\sin y} = \frac{AB}{AC}$$

$$\frac{BS}{SC} = \frac{\sin x}{\sin y} = \frac{AB}{AC} \quad \Rightarrow (A,S;B,C) = -1$$

(b) - $(A,S;B,C)_\Gamma = (AA,AS;AB,AC) =$ simmetrica risp. alla bisett. di $B\hat{A}C$

$= (r, AM; AC, AB) = (X_\infty, M; C, B) = -1$

interseco con BC



r è la $\parallel$ a BC per A

X_∞ = pt all'∞ di r

Lemma 3: I birapporti si conservano sotto inversione

dim: $A, B, C, D \longrightarrow A', B', C', D'$

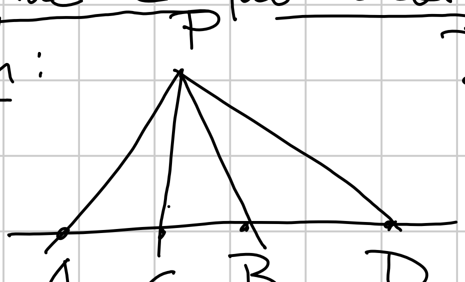
$$\left|\frac{A'C'}{C'B'}\right| = \left|\cancel{R^2}\frac{AC}{OA \cdot \cancel{OC}} \cdot \frac{\cancel{OC} \cdot OB}{BC \cdot \cancel{R^2}}\right| = \left|\frac{AC}{BC}\right| \cdot \left|\frac{OB}{OA}\right|$$

$$\left|\frac{B'D'}{D'A'}\right| = \left|\frac{BD}{DA}\right| \cdot \left|\frac{OA}{OB}\right|$$

e l'inversione mantiene l'ordine tra i punti. □

2) Quaterne e quadrilateri armonici

Lemma 4:



Due delle seguenti implicano la terza:

(i) $(A, B; C, D) = -1$

(ii) PC biseca $A\hat{P}B$

(iii) $PC \perp PD$

(i) + (ii) $\Rightarrow$ PD bisett. esterna $\Rightarrow PD \perp PC$

(ii) + (iii) $\Rightarrow$ PD bisett. esterna $\Rightarrow (A, B; C, D) = -1$

(i) + (iii) $\Rightarrow$ $A\hat{P}D = A\hat{P}C + \frac{\pi}{2}$

$B\hat{P}D = B\hat{P}C + \frac{\pi}{2}$

$$\frac{\sin A\hat{P}C}{\sin C\hat{P}B} = \frac{\sin(A\hat{P}D)}{\sin(D\hat{P}B)} = \frac{\cos(A\hat{P}C)}{\cos(C\hat{P}B)} \Rightarrow \operatorname{tg} A\hat{P}C = \operatorname{tg} C\hat{P}B$$

$\Rightarrow \widehat{APC} = \widehat{CPB} \Rightarrow PC$ bisettrice..

Def: $(A,B;C,D) = -1$ si dicono quaterna armonica

D quarto armonico

Lemma 5: O pt. medio di AB, allora le seguenti sono equivalenti

(i) $(A,B;C,D) = -1$

(ii) $\dfrac{2}{AB} = \dfrac{1}{AC} + \dfrac{1}{AD}$

(iii) $CA \cdot CB = CO \cdot CD$

(iv) $OC \cdot OD = OA^2$

(v) $\dfrac{OC}{OD} = \left(\dfrac{AC}{AD}\right)^2 = \left(\dfrac{BC}{BD}\right)^2$

Def: A,B,C,D su Γ con $(A,B;C,D)_\Gamma = -1$ si dicono quadrilatero armonico

Prop 6: A,B,C,D su Γ, allora le seguenti sono equivalenti

(i) $AB \cdot CD = BC \cdot AD$

(ii) BD simmediana di $\overset{\Delta}{ABC}$

(iii) le tangenti a Γ in A e C si incontrano su BD

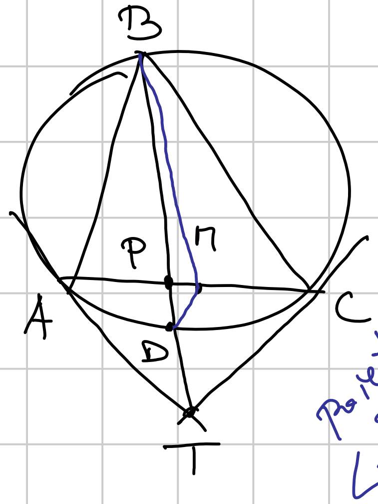
(iv) $(A,C;B,D)_\Gamma = -1$

(v) M pt. medio di AC, allora MB, MD simmetriche rispetto ad AC.

(vi) la bisettrice di $\widehat{ABC}$ e quella di $\widehat{ADC}$ si intersecano su AC

(vii) $\dfrac{AB^2}{AD^2} = \dfrac{MB}{MD}$.

(v)



AC bisett. di $B\hat{M}D$

$\Updownarrow$

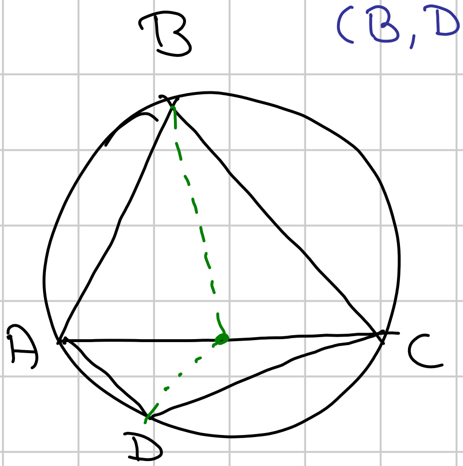
$(B,D;P,T) = -1$

e

$P\hat{M}T = \frac{\pi}{2}$ ✓ ovvio

$(B,D;A,C)_T = -1 \iff$ BD simmediana

(vi)



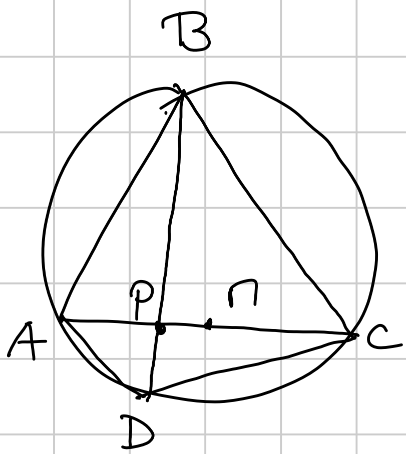
bisett. di $A\hat{B}C = BL$ $\quad L, L' \in AC$

bisett di $A\hat{D}C = BL'$

$\frac{AL}{LC} = \frac{AB}{BC} \qquad \frac{AL'}{L'C} = \frac{AD}{DC}$

$L = L' \iff \frac{AB}{BC} = \frac{AD}{DC} \iff (A,C;B,D) = -1$

(vii)



$$\frac{BP}{PD} = \frac{AP}{\sin A\hat{B}D} \cdot \sin B\hat{A}P \left(\frac{1}{AP}\frac{\sin A\hat{D}B}{\sin P\hat{A}D}\right)$$

$$= \frac{\sin B\hat{A}C}{\sin A\hat{B}D} \cdot \frac{\sin A\hat{D}B}{\sin C\hat{A}D} =$$

$$= \frac{BC}{AD}\frac{AB}{CD}$$

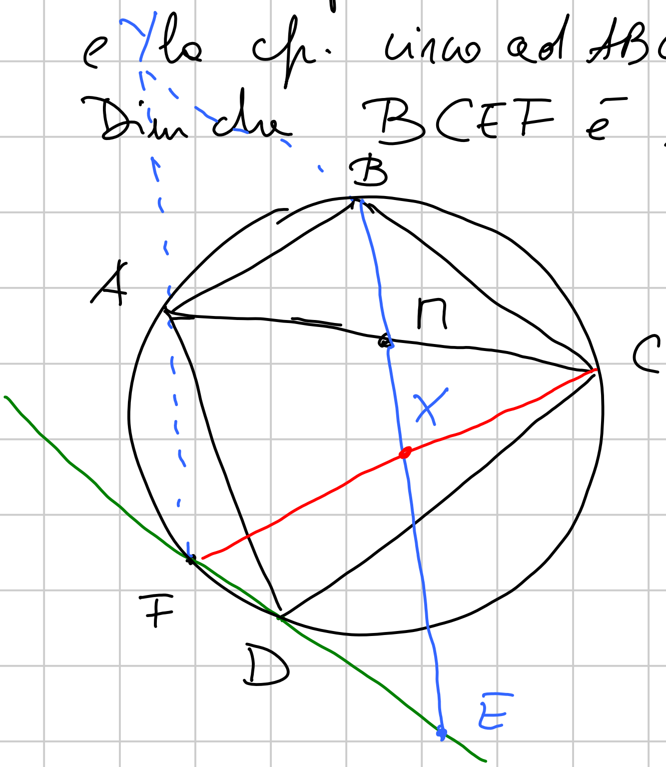
$$AB \cdot CD = BC \cdot AD \iff \frac{AB}{AD} = \frac{BC}{CD} \iff \frac{BP}{PD} = \left(\frac{AB}{AD}\right)^2$$

$\Updownarrow$

MP bisett. di $B\hat{M}D \iff \frac{BP}{PD} = \frac{BM}{MD}$

$$AB \cdot CD = BC \cdot AD \implies \frac{BM}{MD} = \left(\frac{AB}{AD}\right)^2.$$

Es: ABCD ciclico, bisettrici di $\hat{ABC}$ e $\hat{ADC}$ si intersecano su AC. Sia M pt. med di AC.
La retta parallela a BC per D incontra BM in E e la cfr. circo ad ABCD in $F \neq D$.
Dim che BCEF è un parallelogramma



dim: ABCD armonico

$(B, D; A, C) = -1$

$BE \cap CF = X$

$AF \cap BC = Y$

$(FB, FD; FY, FC) = -1$

interseco con BC

$(B, X_\infty; Y, C) = -1$

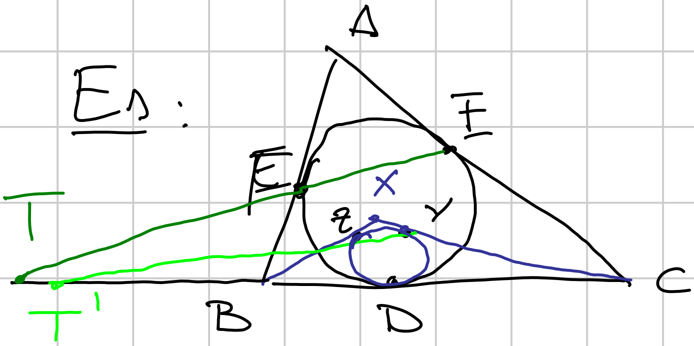
X_∞ pt all'∞ di BC

$YB = BC$

$\Downarrow$

$FY \parallel BM \Rightarrow FX = XC$ ☒

Es:



X t.c. l'incentro in $X\hat{B}C$ tange BC in D.
Allora EFZY è ciclico.

Hope: $FE \cap ZY$ sta su BC.

$FE \cap BC = T$ $\quad (B, C; D, T) = -1$
perché AD, BE, CF concorrono

$ZY \cap BC = T'$ $\quad (B, C; D, T') = -1$
perché XD, BY, CZ concorrono.

$\Rightarrow T = T'$

$TD^2 = TY \cdot TZ$

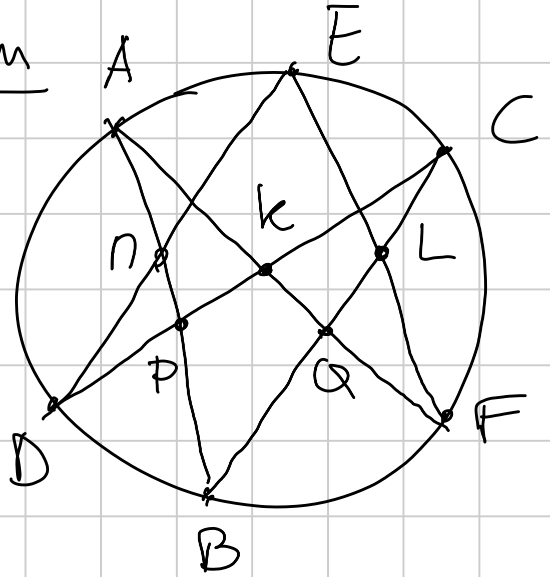
$TD^2 = TE \cdot TF$ $\Rightarrow TE \cdot TF = TY \cdot TZ \Rightarrow$ ciclico.

Teo (Pascal)

Γ cf. A, B, C, D, E, F su Γ

$\Rightarrow$ $AB \cap DE = M$, $BC \cap EF = L$, $CD \cap FA = K$ sono allineati

dim



da F su Γ

$$(C, L; Q, B) = \text{da } D \text{ su } AB$$

$$= (C, E; A, B)_\Gamma = \text{da } K \text{ su } CB$$

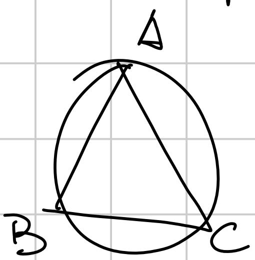
$$= (P, M; A, B) =$$

$$= (C, MK \cap BC; Q, B)$$

$MK \cap BC = L \Rightarrow M, K, L$ allineati. $\square$

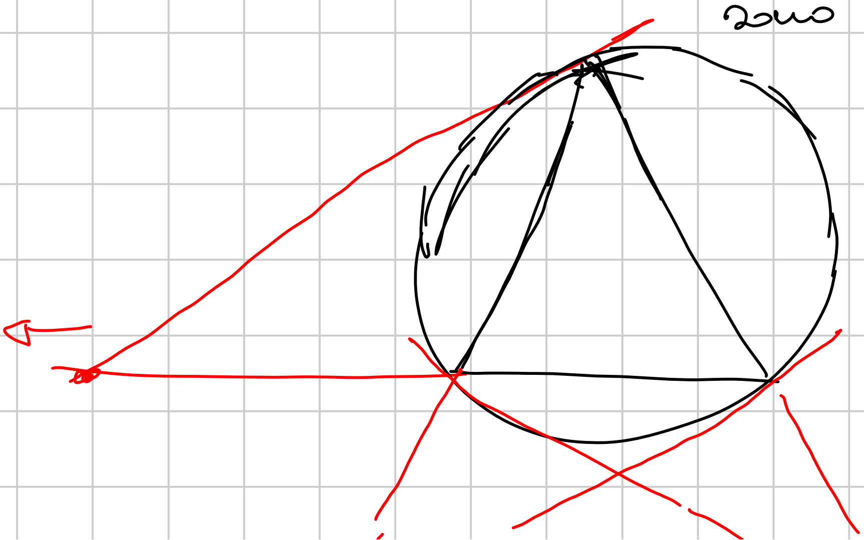
Oss: Se due punti coincidono, si usa la tangente

Es:



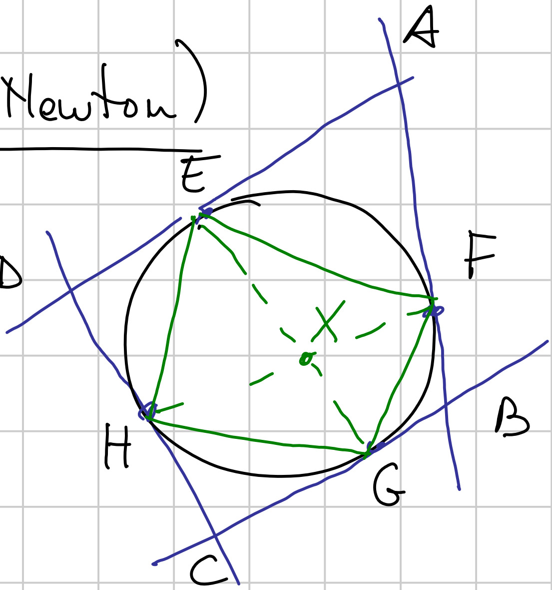
$AABBCC$

$AA \cap BC$, $AB \cap CC$, $BB \cap CA$ sono allineati.



queste rette si chiamano asse di Lemoine

Teo (Newton)



Allora AC, BD, EG, FH concorrono

dim: $EG \cap FH = X \qquad EH \cap FG = Y$

Pascal su $EGGFHH$

$$\left.\begin{array}{l} EG \cap FH = X \\ GG \cap HH = C \\ GF \cap EH = Y \end{array}\right\} \text{sono allineati}$$

Pascal su $EEHFFG$

$$\left.\begin{array}{l} EE \cap FF = A \\ EH \cap FG = Y \\ HF \cap GE = X \end{array}\right\} \text{sono allineati}$$

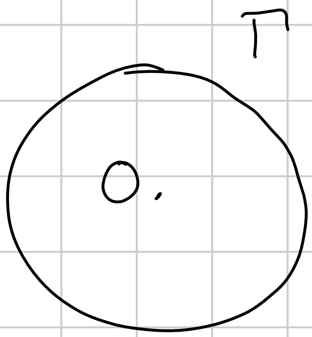
$\Rightarrow$ A, C, X, Y sono allineati

Definisco $Z = EF \cap HG$ e dimostro che B, D, X, Z sono allineati ancora con 2 pascal. $\square$

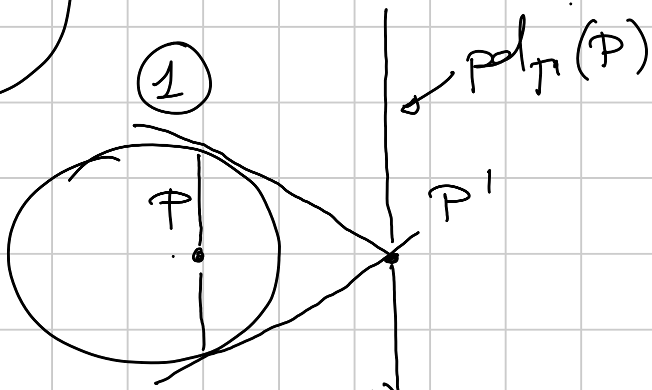
Cor: A, C, X, Y
B, D, X, Z allineati.

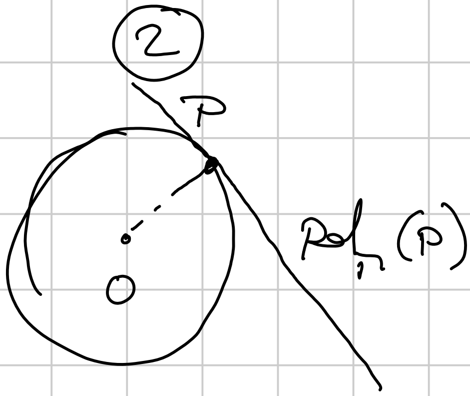
Teo (Brianchon): $ABCDEF$ circoscritto a Γ allora AD, BE, CF concorrono.

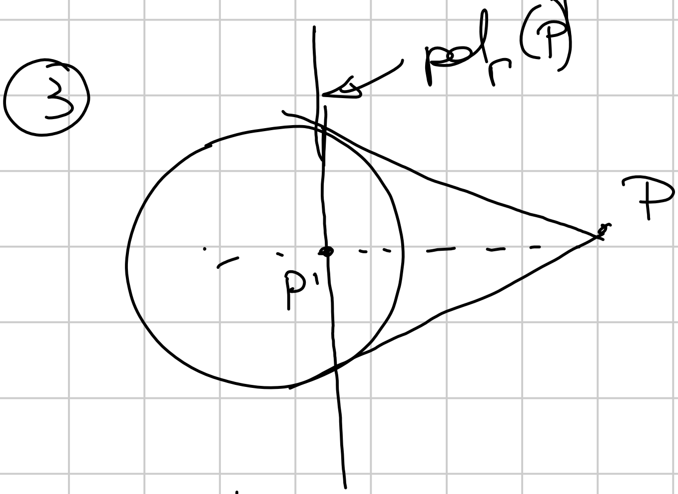
3) Poli e polari



La polare rispetto a Γ di un punto P è la retta $\perp$ a OP che passa per l'inverso di P.





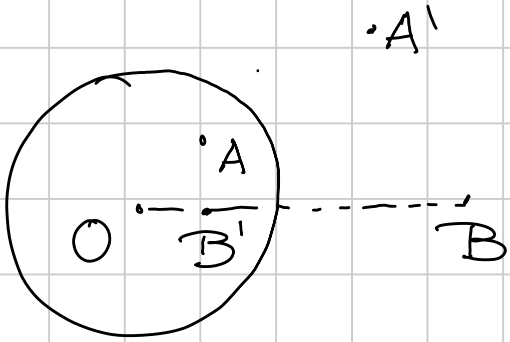


Γ cir. r retta $P = pol_\Gamma(r) \iff r = pol_\Gamma(P)$

Proprietà

1. $A \in pol_\Gamma(B) \iff B \in pol_\Gamma(A)$
2. $pol_\Gamma(r \cap s) =$ retta per $pol_\Gamma(r)$ e $pol_\Gamma(s)$
3. $pol_\Gamma(A) \cap pol_\Gamma(B) = pol_\Gamma(AB)$

dim: 1

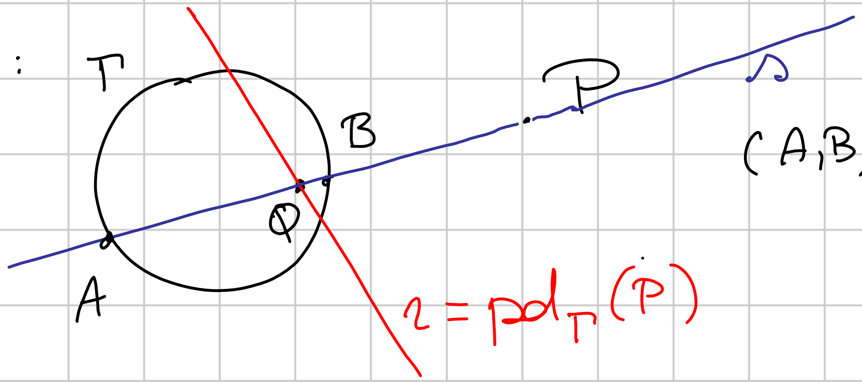


$$A \in pol_\Gamma(B) \iff \widehat{AB'O} = \frac{\pi}{2} \iff \widehat{BA'O} = \frac{\pi}{2} \iff B \in pol_\Gamma(A)$$

2, 3 esercizio

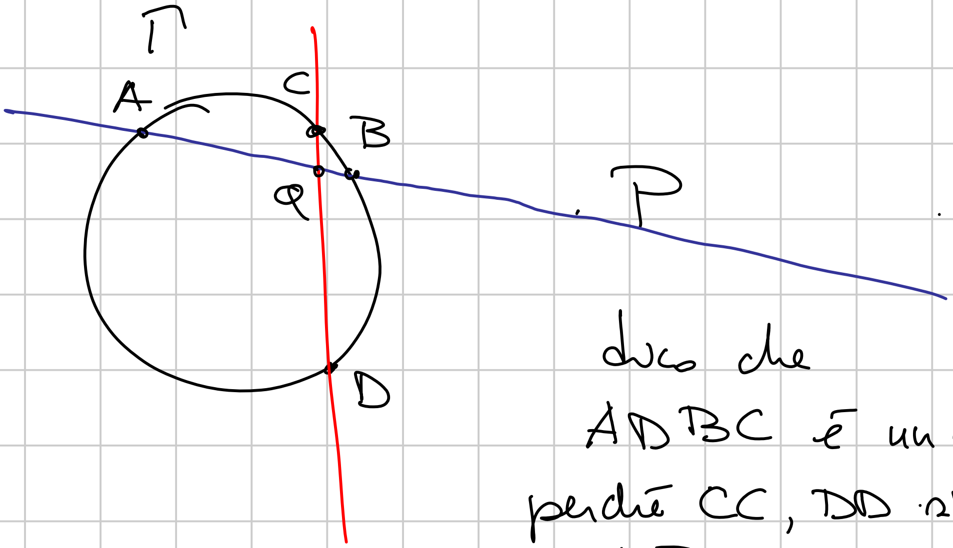
Oss: $\mathrm{pol}_\Gamma(P) = \{ \mathrm{pol}_\Gamma(r) \mid P \in r \}$

Prop:



$(A, B; Q, P) = -1$

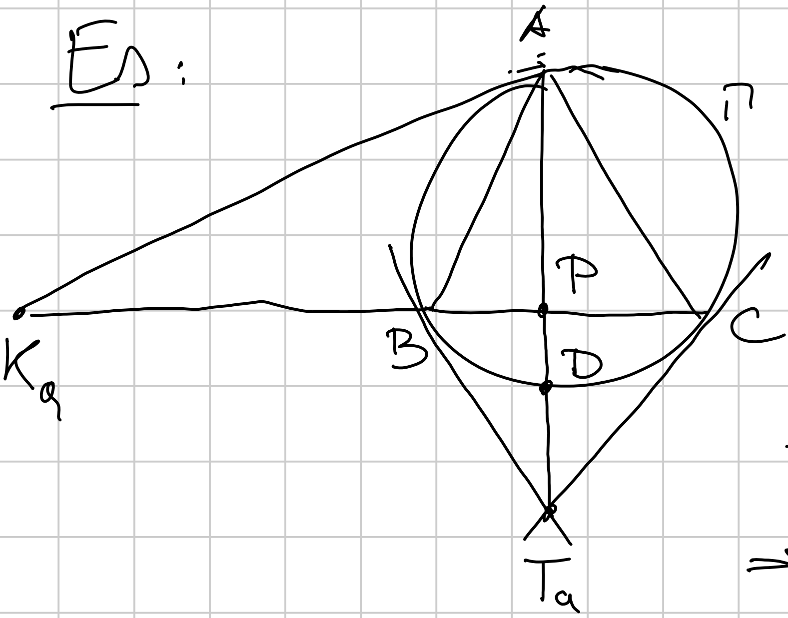
dim:



dico che $ADBC$ è un quad. armonico perché CC, DD si intersecano in P

$\Rightarrow (A, B; C, D) = -1$
$\qquad\qquad\quad \shortparallel$
$\quad (A, B; Q, P)$.

Es:



$BC = \mathrm{pol}_\Gamma(T_a)$

$K_a \in \mathrm{pol}_\Gamma(T_a)$

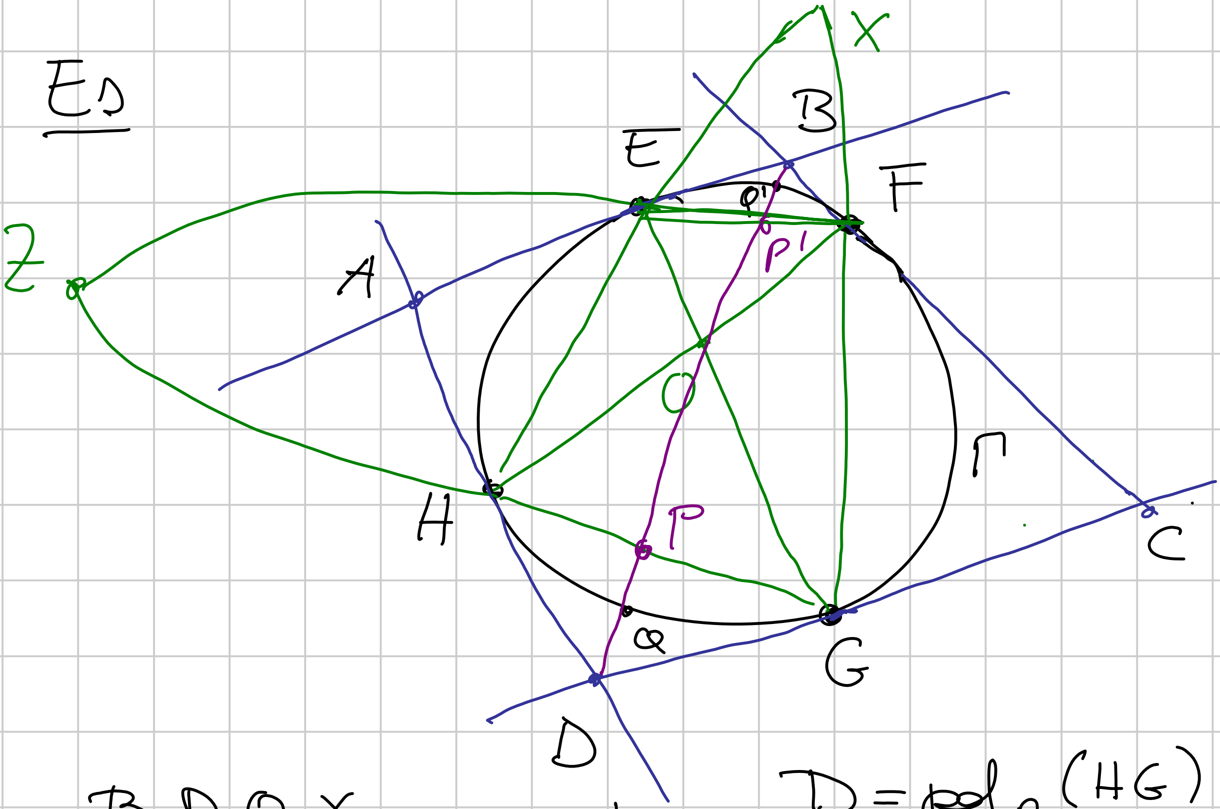
$\Rightarrow T_a \in \mathrm{pol}_\Gamma(K_a)$

e $A \in \mathrm{pol}_\Gamma(K_a)$

$\Rightarrow T_a A = \mathrm{pol}_\Gamma(K_a)$

$\Rightarrow (B, C; P, K_a) = -1$

Es



B, D, O, X
Z, A, O, C allineati

$D = \mathrm{pol}_\Gamma(HG)$
$B = \mathrm{pol}_\Gamma(EF)$

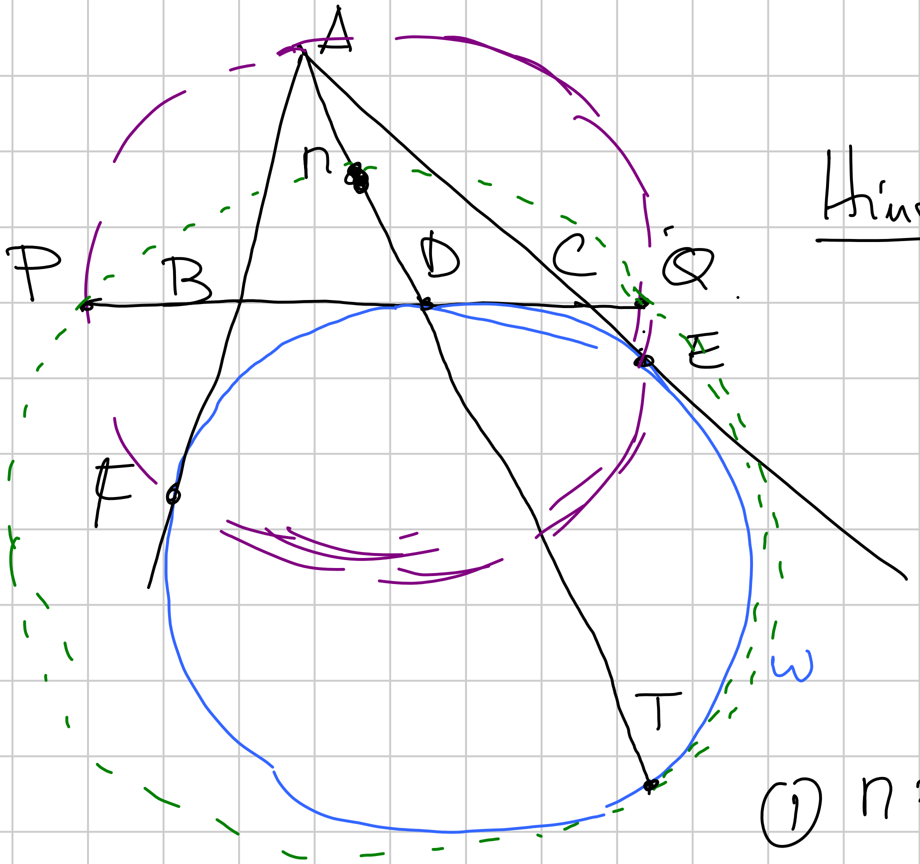
$\Longrightarrow Z = \mathrm{pol}_\Gamma(BD)$

$P = BD \cap HG \quad P' = BD \cap EF$

$$(H, G; P, Z) = -1$$
$$(E, F; P', Z) = -1$$

Romanian TST 2018

ABC triangolo, ω excerchio opposto ad A
D, E, F tangenze di ω con BC, CA, AB
La circonf per EAF interseca BC in P e Q.
M pt. medio di AD
Dim che la circoscritta a MPQ è tangente a ω.



Hint: DETF armonico
⇓
BC
EF
tang a ω in T
concorrono in R

① M P T Q ciclico

② Tangenza in T calcolando le potenze da R