A1-advanced – scambret

Titolo nota 07/09/2019

1) x, y, z reali positivi $\quad xyz = 1$

$$\Rightarrow \sum \left(\frac{x}{x-1}\right)^2 \geq 1$$

2) $a_1, \dots, a_n$ e $b_1, \dots, b_n$ reali

$$\sum a_i^2 = 1, \quad \sum b_i^2 = 1, \quad \sum a_i b_i = 0$$

$$\Rightarrow \left(\sum a_i\right)^2 + \left(\sum b_i\right)^2 \leq n$$

3) Trovare una costante C tale che $\forall n$ e per ogni n-upla di interi positivi $a_1, \dots, a_n$ vale

$$\sum H(a_i) \leq C \sqrt{\sum i a_i}$$

dove $H(n) = \sum_{k=1}^{n} \frac{1}{k}$

4) Trovare la più piccola costante k tale che

$$\sum \left(\frac{2a}{a-b}\right)^2 + k \geq 4 \sum \left(\frac{2a}{a-b}\right)$$

5) Trovare il miglior $\boxed{k \in \mathbb{N}}$ t. c.
$\forall a,b,c \quad abc = 1$ vale

$$\left(\sum \frac{1}{a}\right) + \frac{k}{a+b+c+1} \geq 3 + \frac{k}{4}$$

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1) $\sum \left(\frac{x}{x-1}\right)^2 \geq 1$ $\qquad$ $\underline{xyz = 1}$

$$\left(\frac{x-1}{x}\right)^{-1} = \left(1 - \frac{1}{x}\right)^{-1}$$ $\qquad$ $1 + \frac{1}{x-1}$

$a = \frac{x}{x-1} \Rightarrow 1 - \frac{1}{a} = \frac{1}{x}$

$\sum a^2 \geq 1$ $\qquad$ $\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) = 1$

$(a-1)(b-1)(c-1) = abc$

$\Rightarrow \boxed{ab+bc+ca = a+b+c-1}$ $\qquad$ $a+b+c = S$

$$\sum ab = \frac{S^2 - \sum a^2}{2}$$

$$\Rightarrow S^2 - \sum a^2 = 2S - 2$$

$$S^2 - 2S + 2 = \sum a^2$$

$$\Rightarrow (S-1)^2 + 1 = \sum a^2 \geq 1$$

Correzione

4) Trovare k migliore t.c.

$$\sum \left(\frac{2a}{a-b}\right)^2 + k \geq 4 \sum \frac{2a}{a-b}$$

$$x = \frac{2a}{a-b} \qquad \frac{2}{x} = \qquad \frac{a-b}{a} = \left(1 - \frac{b}{a}\right)$$

$$\frac{b}{a} = 1 - \frac{2}{x} \implies 1 = \prod\left(1 - \frac{2}{x}\right)$$

$$\implies \sum xy = 2\sum x - 4$$

$$\boxed{\left(\sum x^2\right) + k \geq 4 \sum x}$$ Testo

$$\implies \text{(Vincolo)} \qquad \sum xy = \frac{\left(\sum x\right)^2 - \sum x^2}{2}$$

$$\implies \boxed{\left(\sum x\right)^2 - \sum x^2 = 4\sum x - 8}$$ Vincolo

$$\left(\sum x\right)^2 - 4\sum x + 8 + k \geq 4\sum x$$

$$\implies \left(\sum x - 4\right)^2 + k - 8 \geq 0 \implies k \geq 8$$

2) $a_1, \dots, a_n$
$b_1, \dots, b_n$ reali

$$\sum a_i^2 = 1 \qquad \sum b_i^2 = 1 \qquad \overset{(3)}{\sum a_i b_i = 0}$$

$$\Rightarrow \left(\sum a_i\right)^2 + \left(\sum b_i\right)^2 \le n$$

$\vec{a}, \vec{b}$

$$\|\vec{a}\| = 1$$
$$\|\vec{b}\| = 1$$
$$\langle \vec{a}, \vec{b} \rangle = 0$$

$$\sum a_i = \langle \vec{a}, \vec{1} \rangle$$

$$\underline{\langle \vec{a}, \vec{1} \rangle^2} + \langle \vec{b}, \vec{1} \rangle^2 \le \langle \vec{1}, \vec{1} \rangle$$

$$\vec{1} = \mu_1 \vec{a} + \mu_2 \vec{b} + \vec{r}$$

$$\langle \vec{a}, \vec{1} \rangle = \mu_1 + \mu_2 \underbrace{\langle \vec{a}, \vec{b} \rangle}_{0} + \underbrace{\langle \vec{a}, \vec{r} \rangle}_{0}$$

$$\langle \vec{a}, \vec{1} \rangle = \mu_1$$

$$\vec{1} = \langle \vec{a}, \vec{1} \rangle \vec{a} + \langle \vec{b}, \vec{1} \rangle \vec{b} + \vec{r}$$

$$\langle \vec{a}, \vec{1} \rangle = \mu_1 \qquad \langle \vec{b}, \vec{1} \rangle = \mu_2$$

$$\langle \vec{1}, \vec{1} \rangle = \langle \mu_1 \vec{a} + \mu_2 \vec{b} + \vec{r}, \mu_1 \vec{a} + \mu_2 \vec{b} + \vec{r} \rangle$$

$$= \underline{\mu_1^2 + \mu_2^2 + \langle \vec{r}, \vec{r} \rangle}$$

$$\langle \vec{a}, \vec{1} \rangle^2 + \langle \vec{b}, \vec{1} \rangle^2 \leq \langle \vec{1}, \vec{1} \rangle$$

$$\cancel{\mu_1^2} + \cancel{\mu_2^2} \leq \cancel{\mu_1^2} + \cancel{\mu_2^2} + \underbrace{\langle \vec{r}, \vec{r} \rangle}_{\geq 0}$$

3) $\sum H(a_i) \leq C \underbrace{\sqrt{\sum i a_i}}$ $\qquad H(x) = \underbrace{\sum_{k=1}^{x} \frac{1}{k}}$

$$\sum_i \sum_{k=1}^{i} \frac{1}{k} = \sum_k \sum_i$$

$$H(x) \leq 1 + \ln x$$

$$\sum H(a_i) \leq n + \sum \ln a_i = \left(n + \ln(\Pi a_i)\right)$$

$$C\sqrt{\sum i a_i} \geq \qquad \frac{a_1 + 2a_2 + 3a_3 + \ldots + n a_n}{n} \geq \sqrt[n]{n!\, a_1 \ldots a_n}$$

$$\left[C\sqrt{\sum i a_i}\right]^2 \geq C^2\, n \sqrt[n]{n!\, a_1 a_2 \ldots a_n} \geq \underline{C^2\, n \sqrt[n]{\Pi a_i}\, \frac{n}{e}}$$

- $\underline{n! \geq \left(\frac{n}{e}\right)^n \alpha}$

$m = n$

$$C^2 \frac{n^2}{e} \sqrt[n]{\prod a_i} \ge \left(\cancel{n} + \ln\left(\prod a_i\right)\right)^2$$

$$e^x \ge 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$e^x = \left(e^{\frac{x}{2}}\right)^2 \ge \left(1 + \frac{x}{2}\right)^2 \Leftarrow$$

$$C = 2\sqrt{e}$$

5) a, b, c $\quad abc = 1$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{k}{a+b+c+1} \ge 3 + \frac{k}{4}$$

Sol: $k = 13$

$\left(t, t, \frac{1}{t^2}\right)$ $\quad \frac{1}{t}, \frac{1}{t}, t^2$

$$k \le \frac{4(t+2)(2t^3+t^2+1)}{t(2t+1)} \qquad t = \frac{2}{3}$$

$$= 4\left(t^2 + 2t + \frac{2}{t} - \frac{3}{2t+1}\right)$$

Lenze conti :)

$k \geq 13$

$x = \frac{\alpha}{\beta}$

$x = \frac{\alpha^2}{\beta\gamma}$

$xyz = 1$

$$\frac{\left(\sum \alpha^3\beta^3 - 3\alpha^2\beta^2\gamma^2\right)}{\alpha^2\beta^2\gamma^2} \geq \frac{13}{4} \quad \frac{\left(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma\right)}{\alpha^3+\beta^3+\gamma^3+\alpha\beta\gamma}$$

$$12\left(\alpha^3+\beta^3+\gamma^3\right) \geq 13\gamma^2(\alpha+\beta)+\alpha\beta\gamma$$

γ sia il massimo

$$\left(\sum \alpha\beta\right)^2 \geq 3\alpha\beta\gamma(\alpha+\beta+\gamma)$$