A2 - advanced - scambret

1) Trovare tutti i valori possibili di $f(2007)$

$$f:\mathbb{N}_+\to\mathbb{N}_+ \quad f(m+n)\geq f(m)+f(f(n))-1$$

2) $f:\mathbb{R}^+\to\mathbb{R}^+ \quad f(x+f(y))=f(x+y)+f(y)$

3) Trovare tutti i possibili valori di $P(0)$ tale che per ogni $x,y\in\mathbb{R}$ $|y^2-P(x)|\leq 2|x| \iff |x^2-P(y)|\leq 2|y|$ dove $P(\cdot)$ è un polinomio a coefficienti reali

4) $f:\mathbb{R}\to\mathbb{R}$

- $a+b+c\geq 0 \qquad f(a^3)+f(b^3)+f(c^3)\geq 3f(abc)$
- $a+b+c\leq 0 \qquad f(a^3)+f(b^3)+f(c^3)\leq 3f(abc)$

1) Trovare tutti i valori possibili di $f(2007)$

$$f: \mathbb{N}_+ \to \mathbb{N}_+ \quad f(\underline{m+n}) \geq f(\underline{m}) + f(\underline{f(n)}) - 1$$

1) f è non decrescente $\qquad f(m+n) \geq f(m) \quad \forall m,n$

2) $f(n) > n+1$ ~~NON VA~~

3) Trovo $f(2007) = 1, 2, \dots, 2008$

$$\boxed{\begin{array}{ll} f(n) = 1 & \forall n \leq 2006 \\ f(2007) = k & \\ f(n) = n & \forall n \geq 2008 \end{array}}$$

$$\boxed{\begin{array}{ll} f(n) = n & 2007 \nmid n \\ f(n) = n+1 & 2007 \mid n \end{array}}$$

4) $f(n) \leq n+1$

Per assurdo $\quad f(k) = \underline{k+c} \quad$ con $c > 1$

$$f(ak) = \underline{ak} + \underline{a(c-1)+1}$$

$m = ak, \; n = k$

$$f(q) - q \geq k$$

$m = f(q) - q \quad n = q$

$$\cancel{f(f(q))} \geq f(f(q)-q) + \cancel{f(f(q))} - 1$$

$$1 \geq f(k) = k + c \qquad \underline{\text{NO}}$$

$m + n = n \qquad$ NO

$m + n = f(n) \rightarrow m = f(n) - n$

2) $f:\mathbb{R}^+\to\mathbb{R}^+$ $\quad f(x+f(y)) = f(x+y)+f(y)$

1) $x+f(y)\neq x+y \Rightarrow f(y)=0$ NO

2) $x+f(y)\neq y \Rightarrow f(x+y)=0$ NO

3) $f(y)>y$

$h(y) = f(y)-2y$

$g(y) = f(y)-y$

$$g(x+g(y)+y)+\cancel{x}+\cancel{g(y)}+\cancel{y} = g(x+y)+\cancel{x}+\cancel{y}+\cancel{g(y)}+y$$

$$g(\underline{x}+g(y)+\underline{y}) = g(x+y)+y \qquad \forall x,y>0$$

$g:\mathbb{R}^+\to\mathbb{R}^+$

$$\boxed{g(x+g(y)) = g(x)+y \qquad \forall x>y>0}$$

g e' iniettiva

$x \to x+g(z)$

$$g(x+g(y)+g(z)) = g(x+g(z))+y$$
$$= g(x)+y+z$$
$$= g(x+g(y+z))$$

$y,z>0$

x grande

$$g(y)+g(z) = g(y+z) \qquad \forall y,z>0$$

3) Trovare tutti i possibili valori di $P(0)$ tale che per ogni $x, y \in \mathbb{R}$ $|y^2 - P(x)| \leq 2|x| \iff |x^2 - P(y)| \leq 2|y|$ dove $P(\cdot)$ è un polinomio a coefficienti reali

Risposta: $g(0) < 0$ o $g(0) = 1$

Step 1 $P(x) = -x^2 - \alpha$

$$y^2 + x^2 + \alpha > 2|x| \Rightarrow y^2 + (|x| - 1)^2 + \alpha - 1 > 0$$
$$x^2 + y^2 + \alpha > 2|y|$$

$$P(x) = -kx^2 - \alpha$$
$$= -\frac{2}{\varepsilon}x^2 - \varepsilon \qquad P(0) = -\varepsilon$$

$$y^2 + \frac{2}{\varepsilon}x^2 + \varepsilon - 2|x| > 0$$
$$\Rightarrow y^2 + \underbrace{\frac{2}{\varepsilon}\cancel{x^2} + \frac{\varepsilon}{2}}_{} + \frac{\varepsilon}{2} \cancel{-2|x|} > 0 \qquad \phi = \phi$$

$$\Rightarrow P(0) < 0$$

Step 3 $P(x) = ax^2 + bx + 1$ $\quad x^2 + 1 \quad$ $P(0) = 1$

Step 4 Se $P(x)$ ha coefficiente di testa positivo
$\Rightarrow \deg P \leq 2$

$$|x^2 - P(y)| \leq 2|y|$$
$$\iff |y^2 - P(x)| \leq 2|x|$$

Se $P(x) > 0$ $\quad \left|\underline{x^2 - P(\sqrt{P(x)})}\right| \leq 2\sqrt{P(x)}$

se $\deg P > 2$ $\qquad \frac{n}{2}^2 < \frac{n}{2}$

Step 5 Se $P(x)$ ha coefficiente di testa negativo

$P(M) < -2M$ e $M^2 > P(0)$

$$|y^2 - P(M)| > 2M \quad \forall y$$

$$\Rightarrow |M^2 - P(y)| > 2|y| \quad \forall y$$

$\deg P \leq 1$

4) $f:\mathbb{R}\to\mathbb{R}$

•) $a+b+c\geq 0$ $\quad f(a^3)+f(b^3)+f(c^3)\geq 3f(abc)$

•) $a+b+c\leq 0$ $\quad f(a^3)+f(b^3)+f(c^3)\leq 3f(abc)$

⊗ $f(0)=0$ wlog

$a+b+c=0$ $\quad f(a^3)+f(b^3)+f(c^3)=3f(abc)$ $\quad Q(a,b,c)$

f manda $\mathbb{R}^+\to\mathbb{R}^+_0$ e $\mathbb{R}^-\to\mathbb{R}^-_0$

⊗ f dispari

$c=-a-b$ $\quad f(a^3)+f(b^3)+\underline{\underline{3f(ab(a+b))}}=f((a+b)^3)$ $\quad R(a,b)$

$$\underline{f((a+b+c)^3)}=f(a^3)+\underline{f((b+c)^3)}+3f(a(b+c)(a+b+c))$$
$$=\underline{f(a^3)+f(b^3)+f(c^3)}+\underline{3f(bc(b+c))}$$
$$+\underline{\underline{3f(a(b+c)(a+b+c))}}$$

$$f(\underline{\underline{bc(b+c)}})+f(a(b+c)(a+b+c))=f(\underline{\underline{ac(a+c)}})+f(b(a+c)(a+b+c))$$

$S(a,b,c)$

$$I=\begin{cases} ac(a+c)=b(a+c)(a+b+c)=y \\ bc(b+c)=x \\ a(b+c)(a+b+c)=2y-x \Leftarrow \end{cases}$$

$f(x)+f(2y-x)=2f(y)$ $\quad \forall x,y\in I$

$b=\mu a$ $\quad c=\lambda a$ $\quad \lambda=\frac{\mu^2+\mu}{1-\mu}$

$$\frac{x}{y} = \frac{bc(b+c)}{ac(a+c)} = \frac{\mu(\mu+1)}{\lambda(\lambda+1)} = \frac{(\mu-1)^2}{1+\mu^2}$$

$\mu \in [0,1] \Rightarrow \frac{x}{y} \in [0,1]$ $\qquad x<y$

$f(x) + f(2y-x) = 2f(y)$ $\qquad 0<x<y$

Lemma $\quad f(x)+f(y) = f\left(\frac{x+y}{2}\right)\cdot 2$ $\qquad 0<x<y$

$$f(x) = ax$$

$$x = \frac{p}{2^q}$$