A? - Advanced (??)

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Generating functions

$$(a_n)_{n\geq 0} \longmapsto f(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n \cdot x^n$$

FUNZ. GENERATRICE (ORDINARIA) di $(a_n)_{n\geq 0}$.

Es: $a_n = 1 \quad \forall n \in \mathbb{N}$

$\hookrightarrow f(x) = 1 + x + x^2 + x^3 + \dots \boxed{= \frac{1}{1-x}}$ (avrebbe senso solo per $|x|<1$)

Derivate "formale"

$$\frac{d}{dx}(x^n) = n x^{n-1} \qquad \frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

"passo al limite"

Es: $\frac{d}{dx}\left(1 + x + x^2 + x^3 + \dots\right) = \sum_{n=0}^{\infty} (n+1) x^n$

Es 2: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$

Es F: $\begin{cases} F_n = F_{n-1} + F_{n-2} \\ F_1 = 1 \\ F_0 = 0 \end{cases}$ Fibonacci

$$f(x) = \sum_{n=0}^{\infty} F_n x^n = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \dots$$

$$F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \dots \quad -$$

$$0 + F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \dots - \quad -$$

$$0 + 0 + F_0 x^2 + F_1 x^3 + F_2 x^4 + \cdots =$$

$$F_0 + (F_1 - F_0)x + 0 + 0 + 0 + \cdots$$

$$f(x) - x f(x) - x^2 f(x) = F_0 + (F_1 - F_0)x = x$$

$$\Rightarrow f(x) = \frac{x}{1-x-x^2} = \frac{A}{P(x)} + \frac{B}{Q(x)} \qquad \deg P, Q = 1$$

$$A, B \in \mathbb{C}$$

$$x^2 + x - 1 = \left(x - \frac{-1+\sqrt{5}}{2}\right)\left(x - \frac{-1-\sqrt{5}}{2}\right)$$

$$AQ(x) + BP(x) = x$$

$$A\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) + B\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right) = -x$$

$$(A+B)x + \frac{A+B}{2} - \frac{A-B}{2}\sqrt{5} = -x$$

$$\begin{cases} A+B = -1 \\ \dfrac{A+B}{2} = \dfrac{A-B}{2}\sqrt{5} \end{cases} \qquad \begin{cases} A+B = -1 \\ A-B = -\dfrac{1}{\sqrt{5}} \end{cases}$$

$$A = \frac{-1-\frac{1}{\sqrt{5}}}{2} \qquad B = \frac{-1+\frac{1}{\sqrt{5}}}{2}$$

$$= -\frac{\sqrt{5}+1}{2\sqrt{5}} \qquad = \frac{1-\sqrt{5}}{2\sqrt{5}}$$

$$f(x) = -\frac{\sqrt{5}+1}{2\sqrt{5}} \frac{1}{x+\frac{1}{2}+\frac{\sqrt{5}}{2}} + \frac{1-\sqrt{5}}{2\sqrt{5}} \frac{1}{x+\frac{1}{2}-\frac{\sqrt{5}}{2}} =$$

$$= -\frac{\cancel{\sqrt{5}+1}}{\cancel{2}\sqrt{5}} \frac{\cancel{2}}{\cancel{1+\sqrt{5}}} \frac{1}{1-\left(-\frac{2x}{1+\sqrt{5}}\right)} + \frac{\cancel{1-\sqrt{5}}}{\cancel{2}\sqrt{5}} \frac{\cancel{2}}{\cancel{(1-\sqrt{5})}} \frac{1}{1-\frac{2x}{\sqrt{5}-1}} =$$

$$= -\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(-\frac{2x}{1+\sqrt{5}}\right)^n + \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{2x}{\sqrt{5}-1}\right)^n =$$

$$= \sum_{n=0}^{\infty} x^n \left(\frac{2^n}{(\sqrt{5}-1)^n} \cdot \frac{1}{\sqrt{5}} - \frac{(-2)^n}{(1+\sqrt{5})^n} \frac{1}{\sqrt{5}} \right)$$

$$\Rightarrow F_n = \frac{1}{\sqrt{3}} \left(\frac{2}{\sqrt{5}-1} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{-2}{1+\sqrt{5}} \right)^n$$

Oss: $\frac{x}{(1-x)^2} = \frac{A}{1-x} + \frac{d}{dx}\left(\frac{B}{1-x}\right)$

Es: $\sum_{k=0}^{n} \binom{4n}{4k+1}$

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{k}, \ldots \quad (a_k)_{k \geq 0}$

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

$$f(1) = \sum_{k=0}^{\infty} \binom{n}{k} = (1+1)^n = 2^n$$

$$f(-1) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} = (1-1)^n = 0.$$

Somma dei binom. di posto pari $=$ Somma dei binom. di posto disp. $= 2^{n-1}$

Per sommare uno ogni 4, mi serve "qualcosa" le cui potenze abbiano periodo 4.

$$f(i) = \binom{n}{0} + i\binom{n}{1} - \binom{n}{2} - i\binom{n}{3} + \binom{n}{4} + i\binom{n}{5} - \binom{n}{6} \ldots$$

$$f(-i) = \binom{n}{0} - i\binom{n}{1} - \binom{n}{2} + i\binom{n}{3} + \binom{n}{4} - i\binom{n}{5} - \binom{n}{6} \ldots$$

$$f(i) - f(-i) = 2i\left(\binom{n}{1} + \binom{n}{5} + \binom{n}{8} \cdots\right) - 2i\left(\binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \ldots\right)$$

$$\Rightarrow \sum_{k=0}^{\infty} \binom{4n}{4k+1} = \frac{1}{2}\left(2^{n-1} + \frac{f(i) - f(-i)}{2i}\right) =$$

$$= \frac{1}{2}\left(2^{n-1} + \frac{(1+i)^n - (1-i)^n}{2i}\right) = \ldots$$

Caso generale: $(a_n)_{n \geq 0}$ $\quad \sum_{k=0}^{\infty} a_{m \cdot k} = ?$ $\quad m \in \mathbb{N} \setminus \{0\}$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Allora $\sum_{k=0}^{\infty} a_{m\cdot k} = \frac{1}{m} \sum_{h=0}^{m-1} f(\xi^{h})$

ξ radice primitiva m-esima di 1.

Es: $\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \sum_{\substack{k+l=n \\ k,l\geq 0}} \binom{a}{k}\binom{b}{l}$

$a(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$b(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$

$a(x)\cdot b(x) = a_0 b_0 + (a_0 b_1 + b_0 a_1)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots + \left(\sum_{k+l=m} a_k b_l\right)x^m$

$\Rightarrow \sum_{k+l=n} \binom{a}{k}\binom{b}{l}$ è il coeff. di x^n in $\left(\sum_{k=0}^{\infty}\binom{a}{k}x^k\right)\cdot\left(\sum_{l=0}^{\infty}\binom{b}{l}x^l\right) =$

$= (1+x)^a(1+x)^b = (1+x)^{a+b}$

$\Rightarrow$ è $\binom{a+b}{n}$

IDEE IMPORTANTI

1) Saper riconoscere le serie.

2) Saper interpretare in termini di funzioni le operazioni sulle successioni.

1) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \dots$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} + \dots$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Es: $1 + x + 2x^2 + 3x^3 + 4x^4 + \dots = 1 + \frac{x}{(1-x)^2}$

Es: $(a_k)_{k\geq 0}$ $\quad b_k = k\cdot a_k$

$$f(x) = \sum a_k x^k \qquad g(x) = \sum b_k x^k \qquad g(x) = x\cdot \frac{d}{dx}(f(x))$$

Es (numeri di CATALAN)

$C_n =$ n-esimo num. di Catalan.

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots + C_{n-1} C_0 \qquad \begin{matrix} C_0 = 1 \\ C_1 = 1 \end{matrix}$$

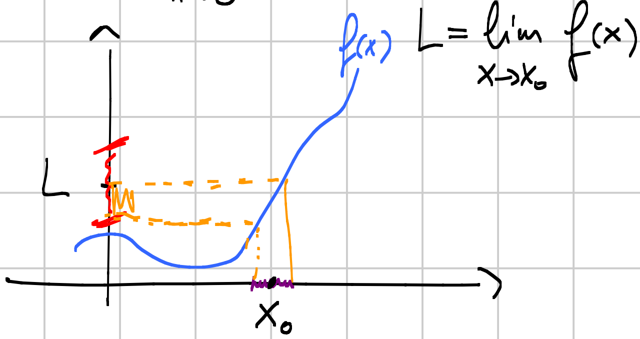
$$f(x) = \sum_{n=0}^{\infty} C_n x^n$$

Sol: $$(f(x))^2 = \left(\sum_{n=0} C_n x^n\right)\left(\sum_{n=0}^{\infty} C_n x^n\right) = \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} C_j C_{k-j} =$$

$$= \sum_{k=0}^{\infty} C_{k+1} x^k = \frac{f(x) - C_0}{x} = \frac{f(x) - 1}{x}$$

$$x(f(x))^2 - f(x) + 1 = 0$$

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x} \qquad C_0 = \lim_{x\to 0} f(x)$$



$$\lim_{x\to 0^+} \frac{1+\sqrt{1-4x}}{2x} = +\infty$$

$$\lim_{x\to 0^-} \frac{1+\sqrt{1-4x}}{2x} = -\infty$$

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = 1 \implies f(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n =$$

$$= \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-3}{2}\right)}{n!} x^n$$

<u>Teo</u>: $(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$

<u>Oss</u>: $\binom{\frac{1}{2}}{n} = (-1)^{n-1} \frac{(2n-3)!!}{2^n n!}$

Usando questo in $f(x)$ si trova che
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

<u>Problema 1</u>: Trovare il num. di modi in cui si possono pagare n euro usando solo monete da 1 o 2 euro, a meno dell'ordine

<u>Problema 2</u>: Trovare il num. dei polinomi $P(x)$ con coeff in $\{0,1,2,3\}$ tali che $P(2) = n$.

<u>Problema 3</u>: $a_0, a_1, a_2, a_3, \ldots$ è una successione crescente di interi non negativi tale che ogni intero non negativo si può scrivere in maniera unica come $a_i + 2a_j + 4a_k$ dove i, j, k non sono necessariamente distinti. Determinare a_{2019}.

$\boxed{P1}$ $a_n = \#$ modi di scrivere n come somma di 1 e 2
$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad n = h + 2k \quad h, k \in \mathbb{N}.$$

il coeff di x^n in $\left(\sum_{h=0}^{\infty} x^h\right)\left(\sum_{k=0}^{\infty} x^{2k}\right)$ è proprio a_n $\Rightarrow$ $f(x) = \sum_{h=0}^{\infty} x^h \cdot \sum_{k=0}^{\infty} x^{2k} =$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x^2} + \frac{1}{(1-x)^2}\right) = \frac{1}{2}\left[(1 + x^2 + x^4 + x^6 \ldots) + (1 + 2x + 3x^2 + \ldots)\right]$$

$$= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \ldots \qquad a_n = \left[\frac{n}{2}\right] + 1$$

$\boxed{P2}$ $a_n = \#$ polinomi che fanno quello che voglio.
$$f(x) = \sum a_n x^n$$

$$P(x) = c_0 + c_1 x + \dots + c_k x^k \qquad P(2) = n$$

$$c_0 + 2c_1 + 4c_2 + \dots + 2^k c_k = n \qquad c_j \in \{0,1,2,3\}$$

$$x^{c_0} \cdot x^{2c_1} \cdot x^{4c_2} \cdot \dots x^{2^k c_k} = x^n$$

$$2^j c_j \in \{0, 2^j, 2^{j+1}, 3\cdot 2^j\}$$

$$f(x) = (1+x+x^2+x^3)(1+x^2+x^4+x^6) \cdots\cdots =$$

$$= \prod_{j=0}^{\infty} \left(1 + x^{2^j} + x^{2^{j+1}} + x^{3\cdot 2^j}\right) = \qquad 1+t+t^2+t^3 = \frac{1-t^4}{1-t}$$

$$= \prod_{j=0}^{\infty} \left(1 + x^{2^j} + (x^{2^j})^2 + (x^{2^j})^3\right) =$$

$$= \frac{\cancel{1-x^4}}{1-x} \cdot \frac{\cancel{1-x^8}}{1-x^2} \cdot \frac{\cancel{1-x^{16}}}{\cancel{1-x^4}} \cdot \frac{\cancel{1-x^{32}}}{\cancel{1-x^8}} \cdots\cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^2}$$

$$a_n = \left[\frac{n}{2}\right] + 1$$

$\boxed{P3}$ ogni intero non neg. è scritto <u>in maniera unica</u> come $a_i + 2a_j + 4a_k$

$$f(x) = \sum_{n=0}^{\infty} x^{a_n} \qquad f(x) f(x^2) f(x^4) = \frac{1}{1-x}$$

$$f(x^2) f(x^4) f(x^8) = \frac{1}{1-x^2}$$

$$f(x) = (1+x) f(x^8)$$

$$f(x) = (1+x)(1+x^8)(1+x^{8^2})(1+x^{8^3}) \cdots$$

$(a_n)_{n\geq 0}$ sono i num. che si possono scrivere con cifre 0, 1 in base 8

$$2019 = 2^0 + 2^1 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}$$

$$a_{2019} = 1 + 8 + 8^5 + 8^6 + 8^7 + 8^8 + 8^9 + 8^{10}$$

IMO '95 p = primo dispari

Quanti sono i sottoinsiemi A di $\{1,2,\dots,2p\}$ t.c.

(i) $|A| = p$

(ii) la somma degli elementi di A è divisibile per p ?

Sol: $f(x,y) = (1+xy)(1+x^2y)(1+x^3y)\cdot\dots\cdot(1+x^{2p}y) = \sum_{h,k} c_{h,k}\, x^h y^k =$

$$= 1 + \sum_{k=1}^{2p}\sum_{h=1}^{\infty} c_{h,k}\, x^h y^k$$

$$C = c_{p,p} + c_{2p,p} + c_{3p,p} + \dots$$

il coeff di y^p: $\sum_{h=1}^{\infty} c_{h,p}\, x^h$ $\qquad$ ξ rad. prim. p-esima di 1

$$\underbrace{\frac{1}{p}\sum_{j=0}^{p-1}\sum_{h=1}^{\infty} c_{h,p}\,\xi^{h\cdot j}}_{\text{il coeff di } y^p \text{ in } \frac{1}{p}\sum_{j=0}^{p-1} f(\xi^j, y)} = C$$

$$\frac{1}{p}\sum_{j=0}^{p-1} f(\xi^j, y) = \frac{1}{p}\sum_{j=0}^{p-1} (1+\xi^j y)(1+\xi^{2j}y)\cdots(1+\xi^{2pj}y) =$$

$j=0 \longrightarrow (1+y)^{2p}$

$1 \le j \le p-1 \longrightarrow (1+y^p)^2$

$$= \frac{1}{p}\left((1+y)^{2p} + (p-1)(1+y^p)^2\right)$$

$$C = \frac{1}{p}\left(\binom{2p}{p} + 2(p-1)\right)$$

IMO 2008-5