A2 - basic

Note Title 9/8/2019

NOTAZIONE

Siano a, b, c numeri, $f(\_,\_,\_)$ funz.

$$\sum_{cyc} f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b)$$

ES

$$\sum_{cyc} \frac{a^2 b}{c} = \frac{a^2 b}{c} + \frac{b^2 c}{a} + \frac{c^2 a}{b}$$

$$\sum_{cyc} a = a + b + c$$

$$\sum_{sym} f(a,b,c) = f(a,b,c) + f(a,c,b) + f(b,a,c) + f(b,c,a) + f(c,a,b) + f(c,b,a)$$

$$\sum_{sym} ab = ab + ac + ba + bc + ca + cb$$

$$\sum_{sym} a = 2 \sum_{cyc} a$$

PROB 1 $\quad n \geq 3$ intero dispari
$x_1, \dots, x_n$ reali non neg. Dim.

$$\min_{i=1,\dots,n} (x_i^2 + x_{i+1}^2) \leq \max_{j=1,\dots,n} (2 x_j x_{j+1})$$

e identifichiamo $x_{n+1} = x_1$

OSS:
$$(x-y)^2 \geq 0 \implies x^2+y^2 \geq 2xy$$

Sol: Basta trovare una i e una j t.c.

$$x_i^2 + x_{i+1}^2 \leq 2 x_j x_{j+1}$$

$\exists i : \begin{cases} x_i \leq x_{i+1} \leq x_{i+2} & (1) \\ x_i \geq x_{i+1} \geq x_{i+2} & (2) \end{cases} \qquad x_{n+2} := x_2$

Supponiamo (2)

$$2x_i x_{i+1} = x_i x_{i+1} + x_i x_{i+1} \geq$$
$$x_{i+1} x_{i+1} + x_{i+2} x_{i+2} =$$
$$x_{i+1}^2 + x_{i+2}^2$$

(1) : ESERCIZIO

PROB 2 :

$$P(x) = x^6 - 6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1$$

Dimostrare che non può avere tutte le radici positive reali

MEDIA ARITM. AM

Sol: Siano $\alpha_1, \dots, \alpha_6$ le radici:

$$(*) \quad \sum_{i=1}^{6} \alpha_i = 6 \qquad \Rightarrow \qquad \frac{\sum_{i=1}^{6} \alpha_i}{6} = 1$$

$$\alpha_1 \cdots \alpha_6 = 1 \qquad \qquad \sqrt[6]{\alpha_1 \cdots \alpha_6} = 1$$

MEDIA GEOM GM

Disug AM-GM: $AM \geq GM$

"=" vale sse $\alpha_1 = \dots = \alpha_6$

$AM = GM \Rightarrow \alpha_1 = \dots = \alpha_6$

$(*) \Rightarrow \alpha_1 = \dots = \alpha_6 = 1$

Ma $P(1) \neq 0$ ↯ ASSURDO

Ripasso

Sia $p \in \mathbb{Z}$ e $p \neq 0$, $a_1, \dots, a_k \in \mathbb{R}^+$

Def. la media p-esima

$$M_p(a_1, \dots, a_k) = \left(\frac{\sum_{i=1}^{k} a_i^p}{k} \right)^{\frac{1}{p}}$$

$$M_0(a_1, \dots, a_k) = \sqrt[k]{a_1 \cdots a_k}$$

MEDIA GEOM.

$$AM = \frac{\sum_1^k a_i}{k}$$ MEDIA ARITM. $p = 1$

$$QM = \sqrt{\frac{\sum_1^k a_i^2}{k}}$$ $p = 2$ MEDIA QUAD.

$$HM = \frac{k}{\sum_1^k \frac{1}{a_i}}$$ $p = -1$ MEDIA ARM.

Th: Siano $a_1, \dots, a_k \in \mathbb{R}^+$, $p < q$, $p, q \in \mathbb{Z}$

$$\underset{M_{-\infty}}{\overset{\shortparallel}{\min(a_1, \dots, a_k)}} \leq M_p(a_1, \dots, a_k) \leq M_q(a_1, \dots, a_k) \leq MAX(a_1, \dots, a_k) = M_{+\infty}$$

Inoltre vale l'uguale sse $a_1 = \dots = a_n$

PROB 3 : $a, b, c \in \mathbb{R}^+$ t.c. $a \cdot b \cdot c = 1$

Dimostrare che:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Th [Cauchy - Schwarz (C-S)]

$x_1, \dots, x_m, y_1, \dots, y_m \in \mathbb{R}$

$$\left(\sum_{i=1}^{m} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{m} x_i^2\right)\left(\sum_{i=1}^{m} y_i^2\right)$$

e l'"=" vale sse $\exists \lambda \in \mathbb{R}$, $\lambda \neq 0$ t.c. $x_i = \lambda y_i$ $(i = 1, \dots, m)$

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

$$x_i^2 = \frac{1}{a^3(b+c)} \implies x_i = \frac{1}{a\sqrt{a(b+c)}}$$

$y_1^2 = a(b+c)$ $\qquad$ $y_1 = \sqrt{a(b+c)}$

e x_2, x_3, y_2, y_3 sono le cicliche

$$\left(\sum_{cyc} \frac{1}{a^3(b+c)}\right)\left(\sum_{cyc} a(b+c)\right) \geq$$

$$\geq \left(\sum_{cyc} \frac{1}{a\cancel{\sqrt{a(b+c)}}} \cdot \cancel{\sqrt{a(b+c)}}\right)^2 =$$

$$= \left(\sum_{cyc} \frac{1}{a}\right)^2$$

$$\sum_{cyc} \frac{1}{a^3(b+c)} \geq \frac{\left(\sum_{cyc} \frac{1}{a}\right)^2}{\sum_{cyc} ab+ac} =$$

$$= \left(\frac{bc+ac+ab}{abc}\right)^2 \cdot \frac{1}{ab+ac+bc+ba+ca+cb}$$

$$= \frac{(bc+ac+ab)^{\cancel{2}}}{2\cancel{(ab+ac+bc)}} = \frac{3}{2}\underbrace{\frac{\sum_{cyc} ab}{3}}_{AM} \geq$$

$$\geq \frac{3}{2}\underbrace{\sqrt[3]{a^2b^2\cdot c^2}}_{GM} = \frac{3}{2}\sqrt[3]{(a\cdot b\cdot c)^2} = \frac{3}{2}$$

PROB 4 $\quad x, y, z \in \mathbb{R}^+$ Dimostra

$$\sum_{cyc} \frac{y^2+z^2}{x} \geq 2(x+y+z)$$

In C-S Prendiamo $x_i = \dfrac{a_i}{\sqrt{b_i}}$, $y_i = \sqrt{b_i}$

dove $a_1, \dots, a_n \in \mathbb{R}$, $b_1, \dots, b_n \in \mathbb{R}^+$

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i^2}{b_i}\right) \cdot \left(\sum_{i=1}^{n} b_i\right)$$

Cor [TITU]:

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}$$

Sol: $\displaystyle \sum_{cyc} \frac{y^2+z^2}{x} = \sum_{sym} \frac{y^2}{x} \geq$

$$\geq \frac{\left(\sum_{sym} x\right)^{\cancel{2}}}{\cancel{\sum_{sym} x}} = \sum_{sym} x = 2 \sum_{cyc} x$$

$= 2(x+y+z)$

PROB 5 Siano $a, b, c, d \in \mathbb{R}^+_0$ t.c.

$\boxed{a+b+c+d=4}$. Dimostrare che

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4$$

Th [DISEQ. DI RIARRANGIAMENTO]:

Siano $x_1 \geq x_2 \geq \dots \geq x_n$ reali
$y_1 \geq y_2 \geq \dots \geq y_n$

Allora

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq$$
$$\geq \sum_{i=1}^{n} x_i y_{n-i+1}$$

Per ogni permutazione σ

Sol:

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4$$

$\{p, q, r, s\} = \{a, b, c, d\}$ t.c.

$$p \geq q \geq r \geq s$$

$$p \cdot q \cdot r \geq p \cdot q \cdot s \geq p \cdot r \cdot s \geq q \cdot r \cdot s$$

$$a^2bc + b^2cd + c^2da + d^2ab =$$

$$a(abc) + b(bcd) + c(cda) + d(dab)$$

$$\leq p(pqr) + q(pqs) + r(prs) + s(qrs)$$

$$= \left(\sqrt{(pq+rs)(pr+qs)}\right)^2 = (GM)^2$$

AM-GM

$$\leq \left(\frac{pq+rs+pr+qs}{2}\right)^2 =$$

$$= \frac{1}{4}\left((p+s)(q+r)\right)^2 \leq$$

AM-GM

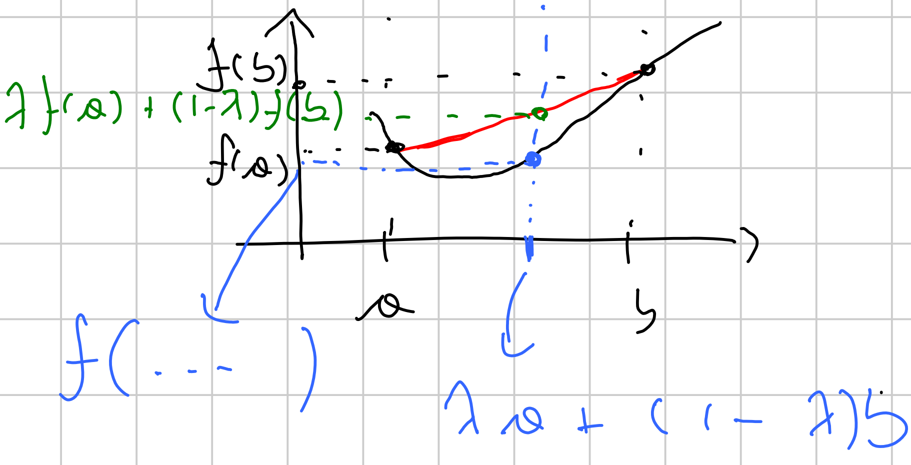
$$\leq \frac{1}{4}\left(\left(\frac{p+q+r+s}{2}\right)^2\right)^2 =$$

$$= \frac{1}{4} 2^4 = 4$$

Def: $f : [a, b] \to \mathbb{R}$

CONVESSA: $\underline{\lambda f(x) + (1-\lambda) f(y)} \geq f(\underline{\lambda x + (1-\lambda) y})$

$\forall x, y \in [a, b]$, $\forall \lambda \in [0, 1]$



$$\boxed{\begin{array}{l} f''(x) \geq 0 \\ \forall x \in [a, b] \end{array}}$$

CONCAVA

$\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y)$

$\forall x, y \in [a, b]$, $\forall \lambda \in [0, 1]$

Th [DIS. DI JENSEN]

$x_1, \dots, x_n \in \mathbb{R}$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+_0$

t.c. $\sum_{i=1}^{n} \alpha_i = 1$

- Se F è convessa allora

$$F(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 F(x_1) + \dots + \alpha_n F(x_n)$$

- Se F è concava allora

$$F(\alpha_1 x_1 + \dots + \alpha_n x_n) \geq \alpha_1 F(x_1) + \dots + \alpha_n F(x_n)$$

Oss 1
Con $n = 2$ è la def di funzione concava e convessa

Oss 2
Se $\alpha_1 = \alpha_2 = \dots = \alpha_n = \frac{1}{n}$

$$F\left(\frac{x_1 + \dots + x_n}{n}\right) \leq \frac{F(x_1) + \dots + F(x_n)}{n}$$

se F è convessa

PROB 6, $a, b, c \in \mathbb{R}^+$, Dimostrare

deg 1

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1$$

deg 1

$$\lambda(a, b, c) = (\lambda a, \lambda b, \lambda c \qquad (\lambda \in \mathbb{R}^+$$

$$\frac{\lambda a}{\sqrt{\lambda^2 a^2 + 8\lambda^2 bc}} = \frac{a}{\sqrt{a^2+8bc}}$$

Se prendo $\lambda = \dfrac{1}{a+b+c}$

$(a', b', c') = (\lambda a, \lambda b, \lambda c)$ è t.c.

$$a'+b'+c' = \frac{a+b+c}{a+b+c} = 1$$

- WLOG $a+b+c=1$
- $f(x) = \dfrac{1}{\sqrt{x}}$ CONVESSA (esercizio)

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq \frac{1}{\sqrt{}}$$

$x_1 = a^2+8bc$

x_2, x_3 le ciclate

ESERCIZI

1. Scrivi esplicitamente $\sum_{cyc} \frac{a^2}{a+c}$, $\sum_{sym} \frac{a^2}{a+c}$

2. Calcola il minimo di $a^4 + b^2 + c$ quando $a, b, c \in \mathbb{R}^+$ e $abc = 1$

3. $a, b, c \in \mathbb{R}^+$ dimostra $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$

4. Calcola il minimo di $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$

   sapendo che $x + y + z = 2$

5. $a, b, c \in \mathbb{R}^+$ con $a + b + c = 3$
   Dim che $\frac{a+3}{3a+bc} + \frac{b+3}{3b+ca} + \frac{c+3}{3c+ab} \geq 3$

6. $a, b, c, d \in \mathbb{R}^+$ t.c. $2(a+b+c+d) \geq abcd$
   Dim che $a^2 + b^2 + c^2 + d^2 \geq abcd$

7. $a, b, c \in \mathbb{R}$ t.c. $(a, b, c) \neq (0, 0, 0)$
   Trova il massimo valore di
   $$\frac{(a^2 b + b^2 c + c^2 a)^2}{(a^2 + b^2 + c^2)^3}$$

8. Finisci il problema 6

$$a+b+c=3$$

$$\sum_{cyc} \frac{a+3}{3a+bc} \geq 3$$

$$\sum_{cyc} \frac{2a+b+c}{a^2+ab+ac+bc} =$$

$$= \sum_{cyc} \frac{a+b+a+c}{(a+b)(a+c)} =$$

$$= \sum_{cyc} \left(\frac{1}{a+b} + \frac{1}{a+c} \right) \overset{C-S}{\geq}$$ dettagli $\hookrightarrow$ per es.

$$\frac{(2a+2b+2c)^2}{\sum_{cyc}((a+b)+(c+a))} = \frac{36}{4 \cdot 3} = 3$$

ES 7.
$$\frac{(a^2b+b^2c+c^2a)^2}{(a^2+b^2+c^2)^3} \leq K$$ $\shortparallel$ costante

$$(a^2b+b^2c+c^2a)^2 \overset{C-S}{\leq}$$

$$\left[(a^4+b^4+c^4)(a^2+b^2+c^2)\right]$$

$$\cdot (a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)$$

$$k(a^2b^2+b^2c^2+a^2c^2) \overset{?}{\leq}$$

$$(a^2+b^2+c^2)^2$$

$$-k(a^2b^2+b^2c^2+a^2c^2)+(a^2+b^2+c^2)^2 = (*)$$

$$\geq 0$$

$$k=\frac{1}{3}$$

conti ...

$$(*)=\frac{1}{2}\left[(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2\right]$$

$$\geq 0$$

$$a=b=c=1 \qquad \checkmark$$

PROB 6 (CONTINUO)

$$a+b+c=1$$

$$\sum_{cyc}\frac{a}{\sqrt{a^2+8bc}} \geq \frac{1}{\sqrt{\sum_{cyc} a(a^2+8bc)}} =$$

$$= \frac{1}{\sqrt{a^3+b^3+c^3+24abc}} \overset{?}{\geq} \mathcal{I}$$

$$1 = (a+b+c)^3 =$$

$$= a^3+b^3+c^3+6abc+3\sum_{sym} a^2b$$

AM - GM

$$\geq a^3+b^3+c^3+6abc+18abc$$

$$= a^3+b^3+c^3+24abc$$

6. $a, b, c, d \in \mathbb{R}^+$ t.c. $2(a+b+c+d) \geq abcd$

Dim che $a^2+b^2+c^2+d^2 \geq abcd$

CASO 1 : $a+b+c+d \geq 8$

$$\sum_{cyc} (a^2+4) \overset{\text{AM-GM}}{\geq} 4\sum_{cyc} a$$

$$\underline{\sum_{cyc} a^2} + \sum_{cyc} 4 = \sum_{cyc} a^2 + 16$$

$$\sum_{cyc} a^2 \geq 4\sum_{cyc} a - 16 \geq 4 \cdot 8 - 16$$

$$\geq 16$$

CASO 2 $\quad a+b+c+d \leq 8$

$$\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right)^2 \overset{AMGM}{\geq}$$

$$\sqrt[4]{a^2 \cdot b^2 \cdot c^2 \cdot d^2} \cdot \left(\sqrt[4]{a \cdot b \cdot c \cdot d}\right)^2 =$$

$$64\,(a \cdot b \cdot c \cdot d)$$

$$\sum_{cyc} a^2 \geq \frac{64\, a \cdot b \cdot c \cdot d}{\left(\sum_{cyc} a\right)^2} \geq$$

$$\geq \frac{\cancel{64} \cdot a \cdot b \cdot c \cdot d}{\cancel{64}}$$

$$2\sum_{cyc} a \geq abcd \qquad \text{vincolo}$$

$$\sum_{cyc} a^2 \overset{?}{\geq} abcd$$

$$2\sum a = abcd \;, \quad 1 = 2\frac{\sum a}{abcd}$$

Non posso fare solo così

$$\frac{2\sum_{cyc} a}{abcd} \geq 1$$

per dim: $\left(\sum_{cyc} a^2\right)^3 \overset{?}{\geq} (abcd)^3$ ⑥

Hope!

uso l'ipotesi

dimostro quella + forte: $\left(\sum_{cyc} a^2\right)^3 \geq \left(2\sum_{cyc} a\right)^2 (abcd) \geq$ ⑥

Ora si conclude il conto e si termina per AM-GM pesata o Bunching

Abbiamo speranza perchè a LHS i termini hanno il grado + concentrato

Infatti: $\sum_{cyc} a^4 \underset{AM \quad GM}{\geq} 4abcd = \sum_{cyc} abcd$

i gradi sono $(4,0,0,0)$ $\qquad (1,1,1,1)$

x casa finite l'esercizio chiave

$$\sum_{sym} a^3 b \geq \sum_{sym} a^2 bc$$