ALGEBRA 2 - Medium Senior 2019

Titolo nota 08/09/2019

Disuguaglianze.

1) Riarrangiamento $a_1, a_2, a_3, \dots, a_n, b_1, b_2, b_3, \dots, b_n \in \mathbb{R}$

Se $a_1 \geq a_2 \geq \dots \geq a_n$ e $b_1 \geq b_2 \geq \dots \geq b_n$

allora $\forall\ \sigma \in S_n$ (gruppo delle permutazioni)

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n \geq a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \dots + a_n b_{\sigma(n)}$$

$\Rightarrow$ Chebyshev
$$\frac{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}{n} \geq \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \cdot \left(\frac{b_1 + b_2 + \dots + b_n}{n}\right)$$

Dim.: Somme Riarrangiamento su TUTTE LE PERMUTAZIONI σ oppure
Somme di le permutazioni del tipo $\sigma(i) = i + k \pmod n$
con $k = 0, \dots, n-1$

Dim (riarrangiamento) STEP FONDAMENTALE $n=2$

$a_1 \geq a_2 \quad b_1 \geq b_2$

$$a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1$$

$$a_1 (b_1 - b_2) \geq a_2 (b_1 - b_2)$$

$$(a_1 - a_2)(b_1 - b_2) \geq 0.$$

$e_1 = \max\{a_1, \ldots, a_n\}$

$b_{\sigma(1)}$ $\exists$ $j\ (=\sigma^{-1}(1))$ t.c. a destra ho il termine

$a_j b_1$ $j \neq 1$

$$\underbrace{a_1 b_{\sigma(1)} + a_j b_1 \leq a_1 b_1 + b_{\sigma(1)} a_j}$$

$a_1 \geq a_j$

$b_1 \geq b_{\sigma(1)}$

$$a_1 b_1 + a_2 b_{\tau(2)} + \cdots a_n b_{\tau(n)} \geq a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \cdots + a_n b_{\sigma(n)}$$

$\Uparrow$ Hp induzione

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n$$

$$\underbrace{a_1 \cdot \frac{1}{a_2}}_{b_1} + \underbrace{a \cdot \frac{1}{a_3}}_{b_2} + \cdots + \underbrace{a_n \cdot \frac{1}{a_1}}_{b_n} \geq n$$

without loss of generality

$a_1 \geq a_2 \geq \cdots \geq a_n$?

$b_1 = \frac{1}{a_n}$ $b_2 = \frac{1}{a_{n-1}}$ — $b_n = \frac{1}{a_1}$

$c_1 = \frac{1}{a_1}$ - — — — — $c_n = \frac{1}{a_n}$? $c_1 \leq c_2 \leq \cdots \leq c_n$

Cosa dice il riarrangiamento ?

$$\frac{a_1}{a_n} + \frac{a_2}{a_{n-1}} + \cdots + \frac{a_n}{a_1} \geq \text{membro sinistro del testo}$$

1b) Se $a_1 \geq a_2 \cdots \geq a_n$ e $b_1 \geq \cdots \geq b_n$ $\quad c_i = -b_{n-i+1}$

$c_1 \geq \cdots \geq b_n$

$$a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1 \leq a_1 b_{\sigma(1)} \cdots + a_n b_{\sigma(n)}$$

$$\parallel \qquad\qquad \parallel$$

$$-a_1 c_1 - a_2 c_2 - \cdots - a_n c_n \leq -a_1 c_{\sigma(1)} \cdots$$

+ se orientati in modo inverso

$$n = a_1 c_1 + \cdots + a_n c_n \leq a_1 c_2 + \cdots + a_{n-1} c_n + a_n c_1$$

$$\parallel$$

LHS del testo

ATTENZIONE a "riordinare le variabili"

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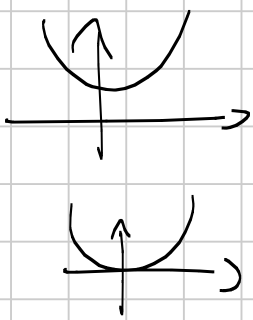
3) <u>CS</u>

$$(a_1^2 + a_2^2 + \cdots a_n^2)(b_1^2 + \cdots b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$

I) $P(x) = \sum_{i=1}^{n} (a_i + x b_i)^2 \geq 0$



$\Delta < 0$

$\Delta = 0$

$\Rightarrow \quad \Delta \leq 0$

$$\frac{\Delta}{4} = \left(\sum_i a_i b_i\right)^2 - \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) \leq 0$$

II) SOS $\left(\sum_i a_i^2\right)\left(\sum b_j^2\right) - \left(\sum_i a_i b_i\right)\left(\sum_j a_j b_j\right) =$

$$= \sum_i \sum_j \left(a_i^2 b_j^2 - a_i b_i a_j b_j\right)$$

$$= \sum_i \sum_j \left(a_j^2 b_i^2 - a_i b_i a_j b_j\right)$$

$$= \sum_i \sum_j \frac{\left(a_i^2 b_j^2 + a_j^2 b_i^2 - 2 a_i b_i a_j b_j\right)}{2} = \sum_{i,j} \left(a_i b_j - a_j b_i\right)^2 \geq 0$$

Importante se uso SOS è che i quadrati che "creo" è bene che si annullino nei casi di uguaglianza conosciuti

$$\frac{a^2+b^2}{2} \geq ab \leftarrow \text{ho uguaglianza per } a=b$$

$$\frac{(a+b)^2}{2} \geq 2ab$$

$\nwarrow \neq 0$ se $a=b$

$$f(a,b,c) \geq g(a,b,c) \qquad = \text{ se } a=b=c$$

$$f(a,b,c) - g(a,b,c) = f_1(a,b,c)^2 + f_2(a,b,c)^2 + f_3(a,b,c)^2 \geq 0$$

$$0 = f(a,a,a) - g(a,a,a) = f_1(a,a,a)^2 + f_2(a,a,a)^2 + f_3(a,a,a)^2 \geq 0$$

III) OMOGENIZZAZIONE

un'espressione $f(a_1, \dots, a_n)$ si dice OMOGENEA di grado d se

$$f(\lambda a_1, \dots, \lambda a_n) = \lambda^d f(a_1, \dots, a_n)$$

$$f(x,y) = x^2 + y^2 + xy \quad \leftarrow \quad d = 2$$

$$f(x,y,z) = x^3 + y^3 + z^3 - 3xyz \quad \leftarrow \quad d = 3$$

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \quad \leftarrow \quad d = 0$$

$$f(a_1, \dots, a_n, b_1, \dots, b_n) = \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) - \left(\sum_i a_i b_i\right)^2 \quad \leftarrow \quad d = 4$$

$$f(a_1, \dots, a_n) = \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) - \left(\sum_i a_i b_i\right)^2 \quad \leftarrow \quad \text{ora } d = 2$$

POSSO fissare un vincolo / posso levare un vincolo.

⚠ SOLO SE $d \neq 0$ (grado del vincolo)

f omogenea di grado d

$$f(a_1, \dots, a_n) \geq 0 \qquad a_1, \dots, a_n \geq 0$$

$$\Updownarrow$$

$$f(b_1, \dots, b_n) \geq 0 \quad \text{se} \quad b_1 + \dots + b_n = 1 \qquad b_1, \dots, b_n \geq 0$$

$a_1 + \dots + a_n = S$

$b_i = \lambda a_i$ $\qquad \sum_i b_i = \lambda S$ $\qquad \lambda = \frac{1}{S}$

$g(b_1, \dots, b_n) = 1$

$g(\lambda a_1, \dots; \lambda a_n) = \lambda^{\tilde{d}} g(a_1, \dots, a_n) = 1$ $\quad \leftarrow \neq 0$

$f(b_1, \dots, b_n) \geq 0$

$\lambda^d f(a_1, \dots, a_n) \geq 0$

$\uparrow$

$\lambda \neq 0$

$$(a_1 b_1 + \dots + a_n b_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$$

$$\boxed{\begin{matrix} a_1^2 + \dots + a_n^2 = 1 \\ b_1^2 + \dots + b_n^2 = 1 \end{matrix} \Rightarrow a_1 b_1 + \dots + a_n b_n \leq 1}$$

$\Updownarrow$

Equiv. a CS.

$a_1 b_1 \leq \frac{a_1^2 + b_1^2}{2}$ $\qquad \sum_i a_i b_i \leq \frac{\sum_i a_i^2 + \sum_i b_i^2}{2} = 1$

$A_i = \frac{a_i}{\sqrt{\sum_i a_i^2}}$ $\qquad B_i = \frac{b_i}{\sqrt{\sum_i b_i^2}}$

$$A_1 B_1 \le \frac{A_1^2 + B_1^2}{2}$$

$$\frac{a_1 b_1}{\sqrt{\sum_i a_i^2}\sqrt{\sum_i b_i^2}} \le \frac{\frac{a_1^2}{\sum_i a_i^2} + \frac{b_1^2}{\sum_i b_i^2}}{2}$$

$$\frac{\sum_i a_i b_i}{\sqrt{\sum_i a_i^2}\sqrt{\sum_i b_i^2}} \le \frac{\cancel{\frac{\sum_i a_i^2}{\sum_i a_i^2}}^{1} + \cancel{\frac{\sum_i b_i^2}{\sum_i b_i^2}}^{1}}{2} = 1$$

$$a_1 b_1 c_1 \le \frac{a_1^3 + b_1^3 + c_1^3}{3} \qquad \sum_i a_i^3 = 1 \qquad \sum_i b_i^3 = 1 \qquad \sum_i c_i^3 = 1$$

$$\sum_i a_i b_i c_i \le \frac{\sum_i a_i^3 + \sum_i b_i^3 + \sum_i c_i^3}{3} = 1$$

$$= \left(\sum_i a_i^3\right)^{1/3} \left(\sum_i b_i^3\right)^{1/3} \left(\sum_i c_i^3\right)^{1/3} \quad *$$

Ho dimostrato $*$ sotto il vincolo

$$\sum_i a_i^3 = 1$$

Ma poiché $\sum_i a_i b_i c_i - \left(\sum_i a_i^3\right)^{1/3}\left(\sum_i b_i^3\right)^{1/3}\left(\sum_i c_i^3\right)^{1/3} = f((a_i),(b_i),(c_i))$

f è 1-omogenea rispetto a (a_i)

$$\sum_i a_i^3 = \lambda^3 \qquad \tilde{a}_i = \frac{a_i}{\lambda} \qquad \sum_i \left(\frac{a_i}{\lambda}\right)^3 = 1$$

$$\frac{1}{\lambda} f((a_i),(b_i),(c_i)) \overset{\text{omogeneità}}{=} \sum_i \frac{a_i}{\lambda} b_i c_i - \left(\sum_i \left(\frac{a_i}{\lambda}\right)^3\right)^{1/3} \ldots - \ldots \le 0$$

OMOGENEIZZAZIONE

(funzione se ho $f \geq 0$ "disuguaglianza" con constraint $g = 1$, g omogenea di grado $d \neq 0$)

$$f(a,b,c) = \frac{a}{b^2+2} + \frac{b}{c^2+2} + \frac{c}{a^2+2} \geq 1$$

es.

con vincolo $\frac{a+b+c}{3} = 1$

$$1 = V\frac{d^{\sim}}{d}$$

$d^{\sim}$ ← grado da dare, assegnare

d ← grado del vincolo

$$\frac{a}{\underbrace{b^2 + 2\left(\frac{a+b+c}{3}\right)^2}_{\text{omogeneo di grado } (-1)}} + \text{—} \text{—} \text{—} \geq 1 = \frac{3}{a+b+c}$$

omogeneo di grado 0

$$\frac{a}{b^2+2\left(\frac{a+b+c}{3}\right)^2} + \frac{b}{c^2+2\left(\frac{a+b+c}{3}\right)^2} + \frac{c}{a^2+2\left(\frac{a+b+c}{3}\right)^2} \geq \frac{3}{a+b+c}$$

$$\left(\frac{a^2+2b^2+3c^2}{3} \geq \left(\frac{a+\sqrt{2}b+\sqrt{3}c}{3}\right)^2\right)$$

Hölder, Hölder e p-V spazi

$$a_1 b_1 \leq \frac{a_1^2 + b_1^2}{2} \quad \to \quad CS$$

$$a_1 b_1 c_1 \leq \frac{a_1^3 + b_1^3 + c_1^3}{3} \quad \to \quad \text{Holder} \quad \frac{1}{3} \; \frac{1}{3} \; \frac{1}{3}$$

$$a_1 b_1 \leq \frac{a_1^p}{p} + \frac{b_1^q}{q} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right) \qquad \left(\sum_i a_i^p\right) = 1$$

$$\left(\sum_i b_i^q\right) = 1$$

$$\sum_i a_i b_i \leq 1 = \left(\sum_i a_i^p\right)^{\frac{1}{p}} \left(\sum_i b_i^q\right)^{\frac{1}{q}}$$

AM-GM

$$\frac{na + mb + kc}{n + m + k} \geq \left(a^n b^m c^k\right)^{\frac{1}{n+m+k}} \qquad \underbrace{a, a, \dots}_{n}, \underbrace{b, \dots, b}_{m}\ c\ \dots_{k}$$

$$w_a a + w_b b + w_c c \geq a^{w_a} b^{w_b} c^{w_c}$$

AM-GM pesata

$$0 \leq w_a \leq 1 \qquad \boxed{w_a + w_b + w_c = 1}$$

$$w_a = \frac{1}{p_a} \qquad w_b = \frac{1}{p_b} \qquad w_c = \frac{1}{p_c}$$

$$a \to A^{p_a}$$

$$\frac{1}{p_a} A_i^{p_a} + \frac{1}{p_b} B_i^{p_b} + \frac{1}{p_c} C_i^{p_c} \geq ABC \qquad \frac{1}{p_a} + \frac{1}{p_b} + \frac{1}{p_c} = 1$$

$$\sum_i A_i B_i C_i \leq \left(\sum_i A_i^{p_a}\right)^{\frac{1}{p_a}} \left(\sum_i B_i^{p_b}\right)^{\frac{1}{p_b}} \left(\sum_i C_i^{p_c}\right)^{\frac{1}{p_c}}$$

$$\left(\sum_i A_i\right)^{\alpha_1}\left(\sum_i B_i\right)^{\alpha_2}\left(\sum_i C_i\right)^{\alpha_3} =$$

$$= \left[\left(\sum_i A_i\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}}\left(\sum_i B_i\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}}\left(\sum_i C_i\right)^{\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}}\right]^{\alpha_1+\alpha_2+\alpha_3}$$

$$\geq \left[\sum_i A_i^{\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}} B_i^{\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}} C_i^{\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}}\right]^{\alpha_1+\alpha_2+\alpha_3}$$

$$\prod\left(a_i^2+b_i^2\right) \geq \left(\left(\prod a_i\right)^{\frac{2}{n}}+\left(\prod b_i\right)^{\frac{2}{n}}\right)^n.$$

IMO '01/2

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

① MAGIA! $\quad \dfrac{a}{\sqrt{a^2+8bc}} \geq \dfrac{a^\lambda}{a^\lambda+b^\lambda+c^\lambda}$ $\quad \lambda$ per cui sia vera

$\lambda = \frac{4}{3}$

$$\cancel{a}\left(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}\right) \geq a^{1/3}\sqrt{a^2+8bc}$$

$$a + \frac{b^{4/3}+c^{4/3}}{a^{1/3}} \geq \sqrt{a^2+8bc}$$

$$\cancel{a^2} + 2a^{2/3}\left(b^{4/3}+c^{4/3}\right) + \frac{\left(b^{4/3}+c^{4/3}\right)^2}{a^{2/3}} \geq \cancel{a^2} + 8bc$$

$$\left(b^{4/3}+c^{4/3}\right)\left(2a^{4/3}+b^{4/3}+c^{4/3}\right) \underset{\substack{\uparrow \\ \text{AM-GM}}}{\geq} 8bca^{2/3}$$

② $\left(\sum_{cy} \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{cy} \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{cy} (a^2+8bc)a\right) \geq \left(\sum_{cy} a\right)^3$

$$T^2 \geq \frac{\left(\sum_{cy} a\right)^3}{\left(\sum_{cy} a^3 + 24abc\right)} \overset{?}{\geq} 1$$

sicuramente vero

$$\left(\sum_{cy} a\right)^3 = a^3+b^3+c^3 + \underbrace{\ldots \;-\; -}\ \geq a^3+b^3+c^3+24abc$$

③ + lungo da fare

$$\left(\sum_{cy} \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{cy} a\sqrt{a^2+8bc}\right) \geq \left(\sum_{cy} a\right)^2$$

$$\left(\sum_{cy} a\right)^2 \geq \sum_{cy} a\sqrt{a^2+8bc}$$

↓ LAVORO DA FARE

↓ $2bc \leq b^2+c^2$ ... etc

IMO 05/3

$x, y, z > 0$ se $xyz \geq 1$ allora

$$\sum_{cy} \frac{x^5-x^2}{x^5+y^2+z^2} \geq 0$$

$$\Updownarrow$$

$$\sum_{cy} \frac{\cancel{(x^5-x^2)} - \cancel{x^5} - y^2 - z^2}{x^3+y^2+z^2} \geq -3$$

$$\sum_{cyc} \frac{x^2+y^2+z^2}{x^5+y^2+z^2} \le 3 \quad \Big\} \quad \underline{\circ\ \text{dd.}}$$

$xyz \geq 1$

$xyz > 1$

posiz. pend.

$x \to \lambda x$

$\lambda \leq 1$

$$\le \sum_{cyc} \frac{x^2+y^2+z^2}{\frac{x^5}{xyz}+y^2+z^2} \le 3$$

$$(x^5+y^2+z^2)\left(\frac{1}{x}+y^2+z^2\right) \geq (x^2+y^2+z^2)^2$$

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2} \le \frac{\frac{1}{x}+y^2+z^2}{x^2+y^2+z^2} \le \frac{yz+y^2+z^2}{x^2+y^2+z^2}$$

$$\sum \qquad xy+yz+zx \le x^2+y^2+z^2 \quad \checkmark$$

(AM-GM)

(Riarrangiamento)

(CS)

$$(x^2+y^2+z^2)(y^2+z^2+x^2) \geq (xy+yz+zx)^2.$$

$$\frac{a}{b^2+2}+\frac{b}{c^2+2}+\frac{c}{a^2+2} \overset{?}{\geq} 1 \qquad a+b+c=3$$

$$\text{CS} \quad \Big((b^2+2)a+(c^2+2)b+(a^2+2)c\Big)\ T \geq (a+b+c)^2$$

$$\frac{1}{3}(a+b+c)^3 \geq ab^2+bc^2+ca^2+\frac{2}{9}(a+b+c)^3$$

$$\frac{(a+b+c)^3}{9} \geq ab^2+bc^2+ca^2 \quad ?$$

$$a^3+b^3+c^3+3\sum_{sym} ab^2+6abc \geq 9\sum_{cyc} ab^2$$

SHUR? $a^3+b^3+c^3+3abc \geq \sum_{sym} a^2b$

SHUR $\to$ $a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \geq 0$

VORNICU-SHUR $x(a-b)(a-c)+y(\cdot\ -\ \ )\ z(\cdot\ ,\ --) \geq 0$

se x,y,z e a,b,c sono ordinati allo stesso modo

$$\left(a^3+b^3+c^3+3abc-\sum_{sym}\right)+3abc+4\sum_{sym}-9\sum_{cyc} \geq 0$$

$\downarrow$

SHUR $\quad + \left(4\sum_{sym}-8\sum_{cyc}\right)+3abc\sum_{sym}$

$\downarrow \quad \downarrow \quad \downarrow$

$$a^2b+b^2c+c^2a-ab^2-bc^2-ca^2$$

$$\underbrace{a(a-b)(a-c)+b(b-c)(b-a)+c(c-b)(c-a)}_{=0}+4\underbrace{(a-b)(b-a)(c-a)}_{=0}+\underbrace{3abc-a^2b-b^2c-c^2a}_{=-b^3} \overset{?}{\geq} 0$$

$a=0 \quad b=c$

$$\frac{a}{b^2+2}+\frac{b}{c^2+2}+\frac{c}{a^2+2} \geq 1 \qquad a+b+c=3$$

$a \geq b \qquad a-\varepsilon,\ b+\varepsilon,\ c$

$$\frac{a-\varepsilon}{(b+\varepsilon)^2+2}+\frac{b+\varepsilon}{c^2+2}+\frac{c}{(a-\varepsilon)^2+2}$$

$$-\frac{\varepsilon}{b^2+2} - \frac{\varepsilon\cdot a}{(b^2+2)^2}\cdot 2b + \frac{\varepsilon}{c^2+2} + \frac{\varepsilon\, c\, 2a}{(a^2+2)}$$

① ② ①b ②b

$$\frac{2\cancel{a}b}{(b^2+2)^2} \gtrless \frac{2\cancel{a}c}{(a^2+2)^2}$$

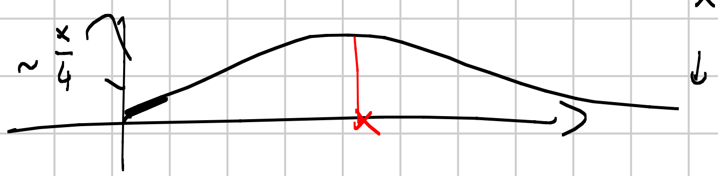
① ≤ ①b se $b \le c$

② ≤ ②b se $c \geq a$

$$\frac{\cancel{2}b}{(b^2+2)^2} \gtrless \frac{\cancel{2}c}{(a^2+2)^2} \le \frac{c}{(c^2+2)^2} \le \frac{b}{(b^2+2)}$$

$\sqrt{\frac{2}{3}} \le b \le c$

$$f(x) = \frac{x}{(x^2+2)^2} \sim \frac{1}{x^3}$$



$$f(x) = \frac{1}{\left(\frac{x^2+2}{\sqrt{x}}\right)^2}$$

$$g(x) = x^{3/2} + \frac{2}{\sqrt{x}}$$

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

$$g'(x) = \frac{3}{2}x^{1/2} - \frac{1}{x^{3/2}} < 0$$

$$\frac{3}{2}x^{1/2} = \frac{1}{x^{3/2}}$$

$$x^2 = \frac{2}{3}$$

$$x = \sqrt{\frac{2}{3}}$$

$a, b, c \in (0, 3)$ "tipicamente" ~ 1

Funzioni convesse, massimi e minimi

f convessa in I intervallo se

$\forall\ x, y \in I$ e $\forall\ \lambda \in (0,1)$ si verifica

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

- vera sse $\lambda = \frac{1}{2}$ $\forall\ x, y \in I$ $\Leftarrow$

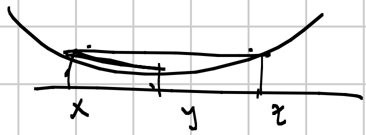
(i) $f\left(\sum_i \lambda_i x_i\right) \leq \sum_i \lambda_i f(x_i)$ (induzione)

$\sum_i \lambda_i = 1$, $\lambda_i \in (0,1)$, $x_i \in I$

(ii) massimo si trova ai bordi dell'intervallo

esiste un unico minimo all'interno $(f'(x) = 0)$

<u>Def. equivalenti</u> ① Rapporti incrementali crescenti



$x \leq y \leq z$

$$\frac{\Delta f}{\Delta x}(y,x) \leq \frac{\Delta f}{\Delta x}(z,x) \leq \frac{\Delta f}{\Delta x}(z,y)$$

$$\frac{\Delta f}{\Delta x}(a,b) = \frac{f(b) - f(a)}{b-a}$$

② se $\exists\ f'$ allora f' è crescente

③ se $\exists\ f''$ allora $f'' \geq 0$.

o

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

o - omogenea

$a+b+c=1$

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3}{2}$$

$$f(x) = \frac{x}{1-x}$$

f convessa

$$= -1 + \frac{1}{1-x}$$

$f+\lambda$ è convessa

$f(x-x_0)$ è convessa

$f(g(x))$ g conv, f conv e crescente

$\frac{1}{x^\alpha}$ $\forall \alpha > 0$

x^α convessa $\alpha \geq 1$

x^α concava $0<\alpha<1$

Es. cmsh max e min $\sin\alpha + \sin\beta + \sin\gamma$ α, β, γ angoli di un triangolo

$\cos\alpha + \cos\beta + \cos\gamma$

$\sin(x)' = \cos(x)$

$\cos(x)' = -\sin(x)$

$$\frac{1}{3} f(a) + \frac{1}{3} f(b) + \frac{1}{3} f(c) \geq f\left(\frac{a+b+c}{3}\right)$$

$$f(a) + f(b) + f(c) \geq 3 f\left(\frac{1}{3}\right) = 3 \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{3}{2}$$

SL 2015 / A3 $-1 \leq x_i \leq 1$ trovare il massimo di $i = 1, \dots, 2n$

$$\sum_{1 \leq r < s \leq 2n} \underbrace{(s-r-n)}_{\in [-3n+1,\, n-1]} x_r x_s$$

1) riduzione $x_i \in \{\pm 1\}$ $\leftarrow 1$

2) $y_i = \sum_{j \leq i} x_j - \sum_{j > i} x_j$ somme di $2n$ numeri ± 1

$$y_i^2 = \underbrace{\sum x_k^2}_{= 2n} + 2\sum_{k<j\leq i} x_k x_j + 2\sum_{i<k<j} x_k x_j - 2\sum_{k\leq i<j} x_j x_k$$

$$\sum y_i^2 = 4n^2 + 2\sum_{k<j} x_k x_j \,(2n - 2\text{\# quanti negativi})$$

$j - k$

$$4n^2 + 2\sum_{k<j} x_k x_j \left(2n - 2(j-k)\right)$$

$1 \quad 2 \quad 3 \quad \ldots \quad k \quad \ldots \quad j \quad i \quad \ldots \quad 2n$

$$\sum_i y_i^2 = 4n^2 + 4\sum_{k<j} x_k x_j (n - j + k)$$

$$= 4n^2 - 4T$$

$$4T = 4n^2 - \sum_i y_i^2$$ massimizzare

Massimo di T è in meno di

$$\frac{\sum_i y_i^2}{4} - n^2$$

y_i è pari e $y_i - y_{i+1} = 2x_i \in \{\pm 2\}$

$$(y_1, y_2, \ldots, y_{2n}) = (0, 2, 0, 2, 0, 2 \ldots)$$

$$\min \overset{?}{=} \frac{4n}{4} - n^2 = n - n^2$$

$$\leadsto \max T = n^2 - n$$

$$x_i = (-1)^i$$

$\sum_1 x_i - \sum_{i<j} x_i x_j$ $\qquad a, b, c \in [-1, 1]$

$a+b+c - ab - bc - ca$

<u>Teo.</u> Weierstrass

$f(x_1, \dots, x_n)$ $\qquad f$ CONTINUA

$(x_1, x_2, \dots, x_n) \in U$ $\qquad U$ CHIUSO $\rightarrow$ opportunamente se ci sono $=$ o $\leq$

LIMITATO

ALLORA $\exists\ (\bar{x}_1, \dots, \bar{x}_n) \in U$ di MASSIMO DELLA funzione

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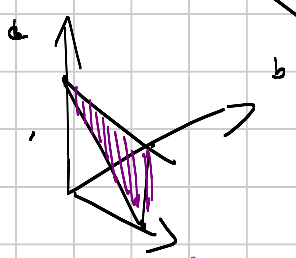
$a+b+c=1$ $\qquad a \geq 0 \quad b \geq 0 \quad c \geq 0$



LIMITATO!

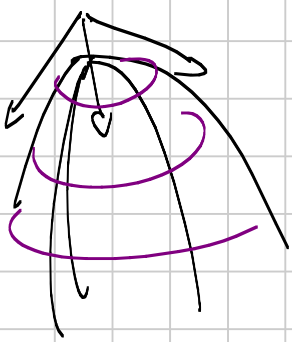
$a^2+b^2+c^2=1$

$a \geq 0 \quad b \geq 0 \quad c \geq 0$

LIMITATO!

$abc=1$

$a \geq 0 \quad b \geq 0 \quad c \geq 0$



$\leftarrow$ ILLIMITATO!

- U limitato se $\exists\, M > 0$ t.c. $|x_i| \leq M$

$\forall (x_1, \dots, x_n) \in U$

$a + b + c = 1$ $\quad 0 \leq a \leq 1$

$a \geq 0 \quad b \geq 0 \quad c \geq 0$

$a^2 + b^2 + c^2 = 1 \qquad |a| \leq 1$

$abc = 1 \qquad M > 0 \qquad a = M + 1 \qquad b = \dfrac{1}{M+1} \qquad c = 1$

NON LIMITATO!

$$1 \leq \frac{a^3 + b^3 + c^3}{3} \qquad abc = 1$$

$U = \{abc = 1\}$

$$B = \left\{ \begin{array}{l} a \leq 10 \\ b \leq 10 \\ c \leq 10 \end{array} \right\} \qquad f(a,b,c) = \frac{a^3 + b^3 + c^2}{3}$$

ha un minimo

$$\leq \frac{1^3 + 2^3 + \left(\frac{1}{2}\right)^3}{3}$$

cosa succede ad $f(a,b,c)$ fuori da B?

fra $U \setminus B \qquad a \geq 10 \quad o \quad b \geq 10 \quad o \quad c \geq 10$

Ma allora $f(a, b, c) \geq \dfrac{10^3}{3} \geq \dfrac{1^3 + 2^3 + \left(\frac{1}{2}\right)^3}{3}$

(1) Esiste il minimo in $abc = 1$ di $f(a,b,c)$

(2) questo minimo sta in $B \cap U$

Serve a giustificare l'esistenza e come ci comportiamo con i minimi (tipo, c'è un minimo/massimo stretto in un certo modo).

$$\frac{a}{\sqrt{a^2+8bc}} \geq \frac{a^\lambda}{a^\lambda+b^\lambda+c^\lambda} \qquad \text{come individuare } \lambda:$$

c'è sicuramente uguaglianza in $a=b=c=1$

$b=c=1$

$$\frac{a}{\sqrt{a^2+8}} \geq \frac{a^\lambda}{a^\lambda+2} \qquad \text{per quali } \lambda \text{ è vero}$$

$$f_\lambda(a) = \left(\frac{a}{\sqrt{a^2+8}} - \underbrace{\frac{a^\lambda}{a^\lambda+2}}_{=1-\frac{2}{a^\lambda+2}}\right) \geq 0 \qquad =0 \text{ in } 1=a \ \forall \lambda$$

$$f_\lambda'(a) = 0$$

$$\frac{1}{\sqrt{a^2+8}} - \frac{a^2}{(a^2+8)^{3/2}} - \frac{\lambda 2 a^{\lambda-1}}{(a^\lambda+2)^2} = 0$$

$$\frac{1}{3} - \frac{1}{27} - \frac{2\lambda}{9} = 0$$

$$\frac{9}{27} - \frac{1}{27} - \frac{6\lambda}{27} = 0 \qquad \lambda = \frac{8}{6} = \frac{4}{3}.$$