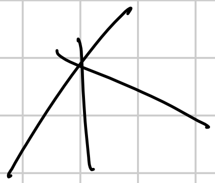
# G2 - MEDIUM PROIETTIVA

Titolo nota 09/09/2019

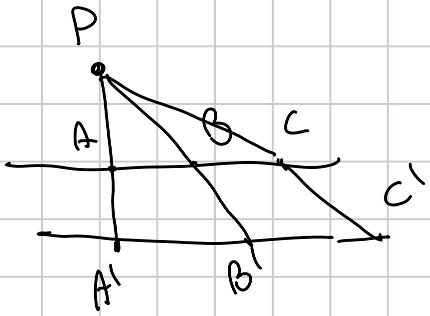
• $r, s, t$ concorrenti oppure $r, s, t$ parallele



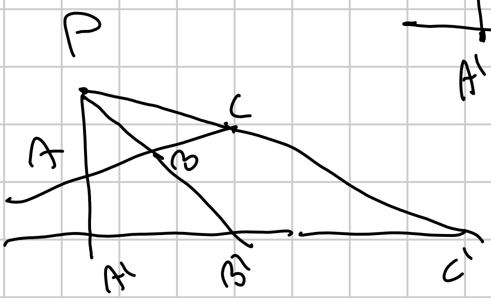
rette proiettive = rette $\cup$ {punto all'infinito}

piano proiettivo = piano $\cup$ retta all'infinito

• Omotetie



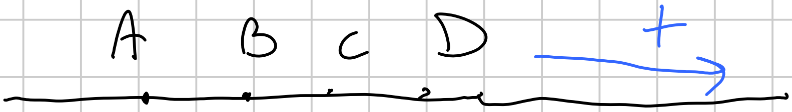

$AB \parallel A'B'$ $\qquad \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$




$AB = BC$

$\times$ $A'B' = B'C'$ NO!

Birapporto




$$(A,B;C,D) = \frac{\frac{AC}{AD}}{\frac{BC}{BD}} = \frac{AC \cdot BD}{BC \cdot AD}$$

I segmenti vanno presi con segno!

$AB > 0 \quad AB = -BA$
$BA < 0$

($\forall$ $A,B,C$ $AB+BC=AC$ con segm. orientati)

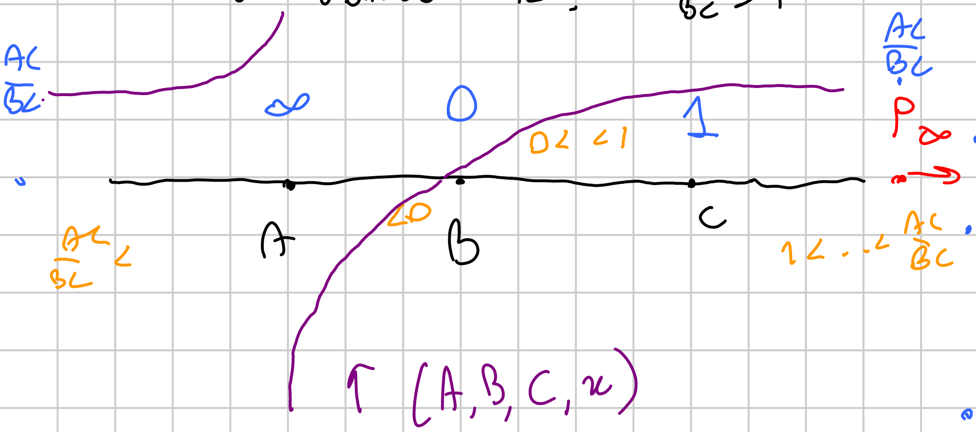
1° OSS $A,B,C,D_1$ e $D_2$ su una retta $(A,B;C,D_1) = (A,B;C,D_2)$

$\iff D_1 = D_2$

Dim $\dfrac{AC \cdot BD_1}{BC \cdot AD_1} = \dfrac{AC \cdot BD_2}{BC \cdot AD_2} \Rightarrow \dfrac{BD_1}{AD_1} = \dfrac{BD_2}{AD_2} \to$ si verifica che $D_1 = D_2$

2° OSS fisso $A,B,C$ su una retta. Come varia $(A,B,C,D)$

Cambiando D? $\frac{AC}{BC} > 1$




$\frac{AC}{BC}$

- Se $D = C$: $\frac{AC \cdot BC}{BC \cdot AC} = 1$
- Se $D = B$: $\frac{AC \cdot BB}{BC \cdot AB} = 0$
- Se $D = A$: $\frac{AC \cdot BA}{AA \cdot BC} = \infty$
- Se $D = P_\infty$: $\frac{BD}{AD} \to 1$, $(A,B,C,D) = \frac{AC}{BC}$

- $(A,B,C,x)$ è iniettiva
- $(A,B,C,x)$ è suriettiva

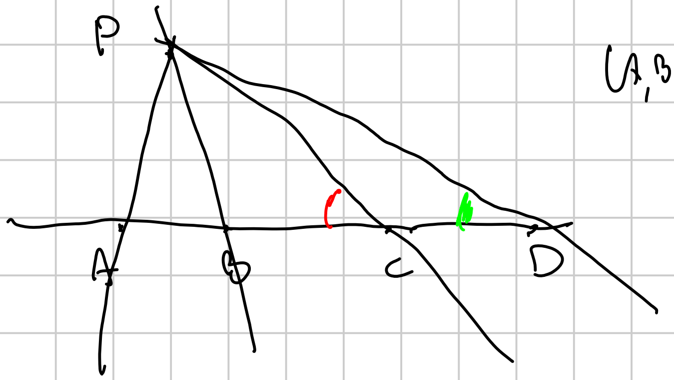
come funzione da retta $\cup P_\infty \to \mathbb{R} \cup \{\infty\}$

<u>OSS 3</u> Cosa succede se permuto l'ordine?

$$\boxed{(A,B,C,D) = K \qquad (A,B,D,C) = \frac{1}{K}}$$

<u>Ex</u> Cosa succede con le altre $4! - 2 = 22$ permutazioni?

<u>Lemma</u> Birapporto su fascio di rette




$(A,B;C,D) = \frac{AC \cdot BD}{AD \cdot BC} =$

Teorema dei seni su $\triangle APC$

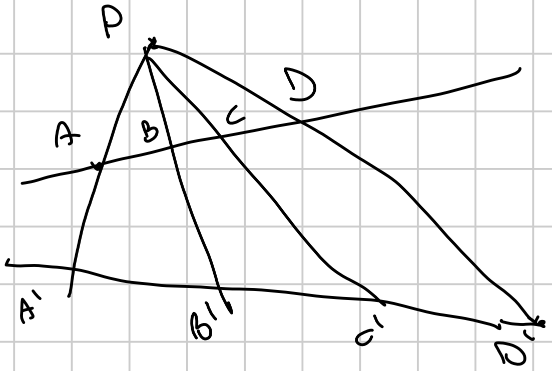
$$\frac{\boxed{AC}}{\sin APC} = \frac{AP}{\sin ACP}$$

(su $\triangle ABP$, $\triangle BPD$, $\triangle APD$)

$$(A,B;C,D) = \frac{\cancel{AP} \cdot \frac{\sin \boxed{APC}}{\sin ACP} \cdot \cancel{BP} \cdot \frac{\sin \boxed{BPD}}{\sin BDP}}{\cancel{AP} \cdot \frac{\sin \boxed{APD}}{\sin ADP} \cdot \cancel{BP} \cdot \frac{\sin \boxed{BPC}}{\sin BCP}} = \frac{\sin APC \cdot \sin BPD}{\sin APD \cdot \sin BPC}$$

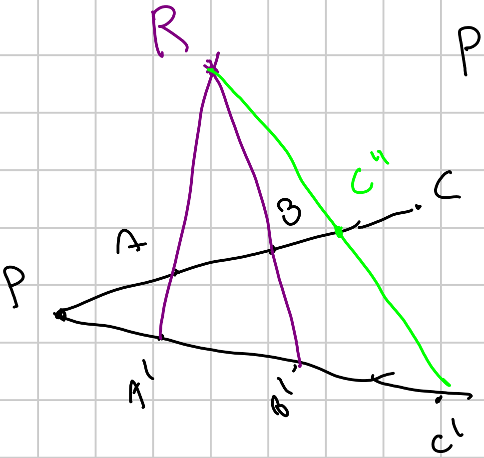
Birapporto, 4 pt su retta $\Rightarrow$ 4 rette passanti per un punto

- Il birapporto è invariante per proiezione




$$(A,B;C,D) = \frac{\sin APC \cdot \sin BPD}{\sin APD \cdot \sin BPC} = (A',B',C',D')$$

## Lemma converso




P, A, B, C su r

P, A', B', C' su S

$(P,A;B,C) = (P,A';B',C')$ $\Updownarrow$ AA', BB', CC' concorrono

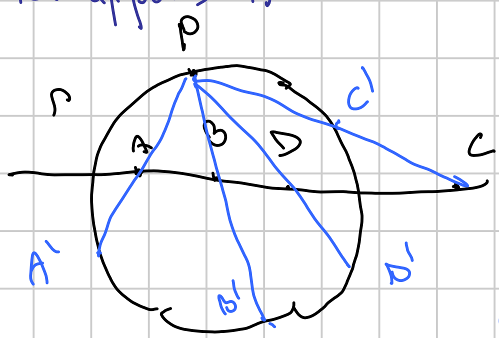
$\Uparrow$ già fatto

$\Downarrow$ $R = AA' \cap BB'$

$C'' = RC' \cap r$

$$(P,A,B,C'') \overset{R}{=} (P,A',B',C') \underset{hp}{=} (P,A,B,C) \Rightarrow C = C''$$

• Birapporto su cerchio $A',B',C',D' \in \Gamma$, $\boxed{P \in \Gamma}$



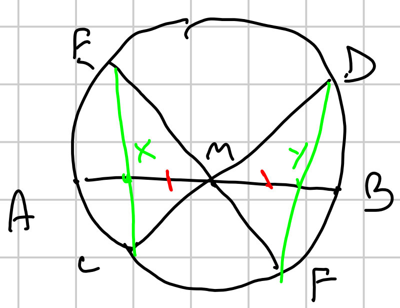

$(A',B',C',D') \overset{def}{=}$ Birapporto delle rette passanti per P

$$= (A,B;C,D) = \frac{\sin A'PC' \cdot \sin B'PD'}{\sin B'PC' \cdot \sin A'PD'}$$

Se P varia su Γ gli angoli sono gli stessi

⚠ P deve stare su Γ

Ex: teorema della farfalla:



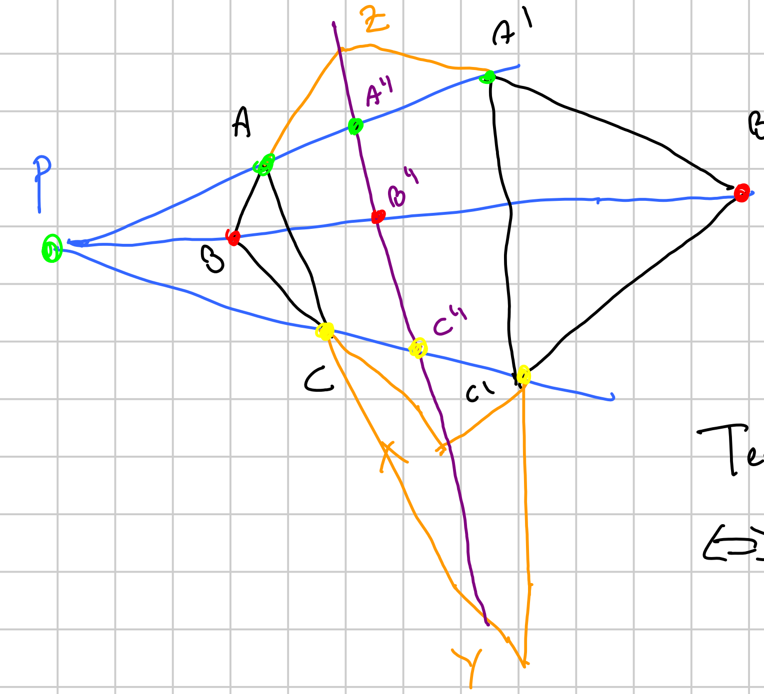

$AM = MB$

$X = EC \cap AB$

$Y = DF \cap AB$

Tesi: $MX = MY$

# Teorema di Desargues




$ABC$ $A'B'C'$ triangoli

$Z = AB \cap A'B'$

$X = BC \cap B'C'$

$Y = AC \cap A'C'$

Tesi: $AA'$, $BB'$, $CC'$ concorrono in $P$

$\Longleftrightarrow$ $X\,Y\,Z$ allineati

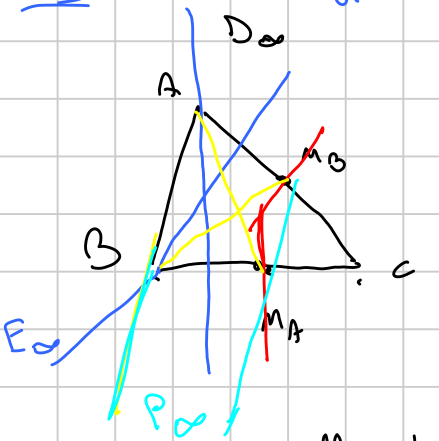
Dim: $l = XZ$ $A'' = l \cap AA'$ e ciclic.

$Y \in l \Leftrightarrow l, AC, A'C'$ concorrono $AC, A'C', A''C''$ concorrono

$$(P, A, A', A'') \underset{Z}{\barwedge} (P, B, B', B'') \underset{X}{\barwedge} (P, C, C', C'')$$

per il lemma, $AC$, $A'C'$, $A''C''$ concorrono.

Es: retta di Eulero




$D_\infty$ = pt all'infinito di $AH$

$E_\infty$ = pt all'infinito di $BH$

$\triangle A M_A D_\infty$ , $\triangle B M_B E_\infty$

$AD_\infty \cap BE_\infty = H$ $M_A D_\infty \cap M_B E_\infty = O$

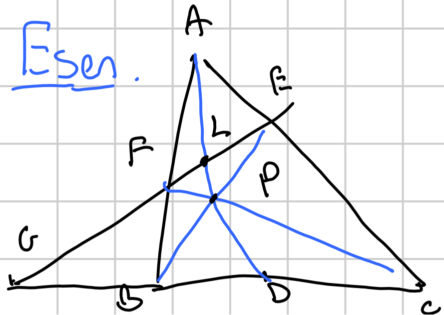
$AM_A \cap BM_B = G$

$G, H, O$ allineati $\underset{\text{Desargues}}{\Longleftrightarrow}$ $AB$, $\underbrace{M_A M_B}_{\text{sono parallele}}$, $D_\infty E_\infty$ concorrono

$AB \cap M_A M_B = P_\infty$

$P_\infty, D_\infty, E_\infty$ sono allineati sulla retta all'$\infty$

Esercizio.




$D, E, F$ sui lati

$G = EF \cap BC$

Tesi: $(B, C; D, G) = -1$

$\Longleftrightarrow$ $AD$, $BE$, $CF$ concorrono

1) Ceva + Menelao (Cristo!)

2) AD, BE, CF concorrono in P $\qquad L = AD \cap EF$

$$(B, C; D, G) \overset{P}{=} (E, F; L, G) \overset{A}{=} (C, B; D, G)$$

$K \qquad\qquad \frac{1}{K}$

$K = \frac{1}{K}$

$K^2 = 1$

$K = 1$ solo se il birapporto è degenere $(C = D)$

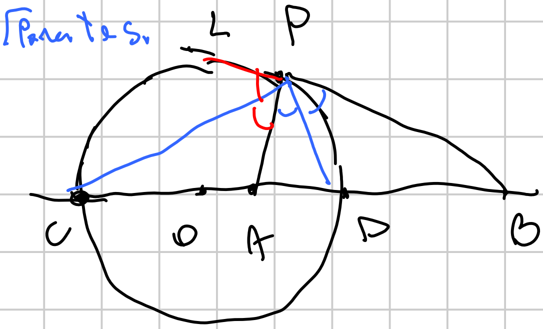
$K = -1 = (B, C; D, G)$

Oss: $(A, B; C, D) = (B, A; C, D) \iff \boxed{(A, B; C, D) = -1}$

Quaterna armonica

$$\frac{AC \cdot BD}{BC \cdot AD} = -1 \qquad \left|\frac{AC}{BC}\right| = \left|\frac{AD}{BD}\right| \qquad \text{Circonferenza di Apollonio}$$

Parentesi:




$\left|\frac{AP}{PB}\right| = K$ costante?

È cerchio che ha centro su AB

$\left|\frac{AP}{PB}\right| = \left|\frac{AC}{CB}\right| = \left|\frac{AD}{DB}\right| \Rightarrow C, D$ piedi delle bisettrici di $\Delta APB$

$\Rightarrow \angle CPD = \frac{1}{2} \cdot \angle BP\measuredangle = 90^\circ$

Es: A, B, C, D su retta, P punto fuori, due cose implicano la 3ª:

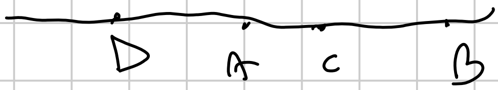
① $(A, B; C, D) = -1$ $\qquad 1 + 2 \Rightarrow 3$

② PC biseca $\widehat{APB}$ (o PD biseca $\widehat{APB}$)

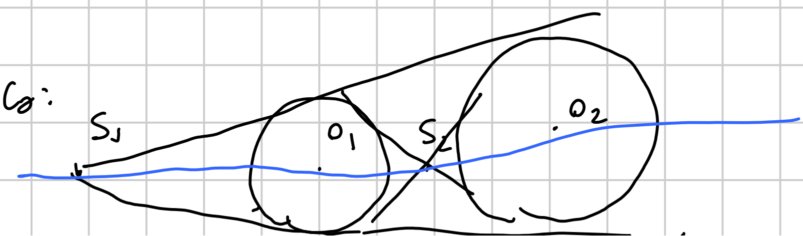
③ $\angle CPD = 90^\circ$

$$\frac{AC}{CB} = \frac{AD}{DB} \Rightarrow \left|\frac{\overset{+}{AC} \cdot \overset{-}{BD}}{\underset{-}{AD} \cdot \underset{-}{BC}}\right| = 1$$

Segno −




Altri esempi di quaterna armonica:



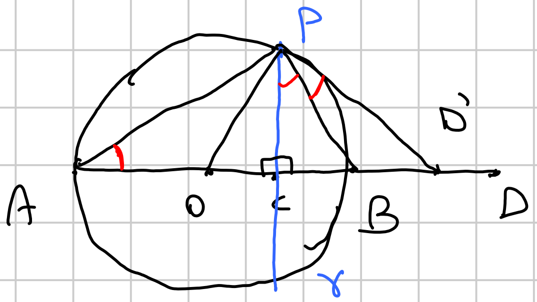

Es: $(O_1, O_2; S_1, S_2) = -1$

hint: $S_1$ e $S_2$ sono centri di similitudine

• $(A, B, C, D) = -1$, O pt medio di AB, N pt medio di CD

① $OC \cdot OD = OA^2$

C e D sono inversi rispetto alla circonf. di diametro AB, $\gamma$




$\angle PCA = 90°$

tangente in P ∩ AB in D'

$OC \cdot OD' = r^2 = OP^2$

$\angle D'PB = \angle PAB \Rightarrow \Delta APD'$ e $\Delta BPD'$ simili

$\parallel$

$\angle CPB$ perché $\Delta APB$ è rettangolo

$\Rightarrow$ PB è bisettrice $\Rightarrow (A, B; C, D')$ è armonico $\Rightarrow D' = D$

e $OC \cdot OD = r^2$

2) $DC \cdot DO = DB \cdot DA =$

$\parallel$

$DP^2$ per Euclide su $\Delta OPD$

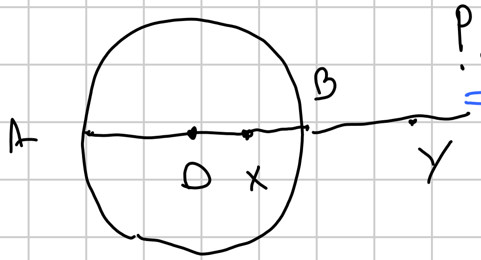
$DP^2 = DB \cdot DA$ per potenza di D rispetto $\gamma$

3) $\dfrac{OC}{OD} = \dfrac{AC^2}{AD^2} = \dfrac{BC^2}{BD^2}$

4) $AB^2 + CD^2 = 4ON^2$ (N pt medio CD)

5) $\dfrac{2}{AB} = \dfrac{1}{AC} + \dfrac{1}{AD}$

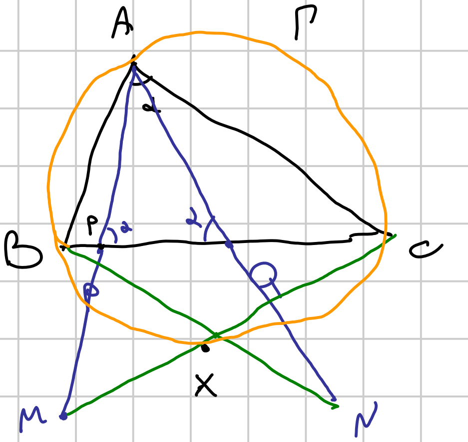
} x Esercizio




$P_\infty$ X e Y sono inversi

$\Rightarrow (A, B, O, P_\infty) = -1$

IMO 2014-4




P, Q su BC

$\angle AQB = \angle APC = \alpha$

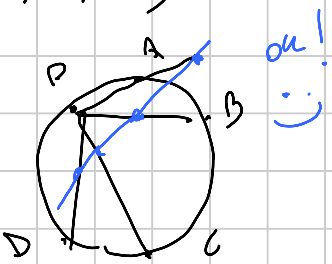
$MP = PA$ $NQ = QA$

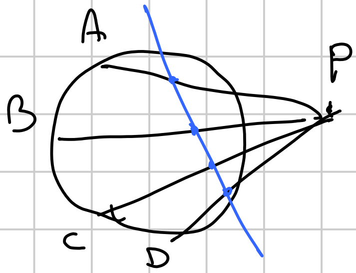
Tesi: $MC \cap BN$ sta sulla circonferenza di ABC

$X = MC \cap \Gamma$ Vogliamo dimostrare che $X \in BN$ $Y = BN \cap \Gamma$

$(A, X; B, C)$

Nope: $(A, X, B, C) \overset{?}{=} (A, Y, B, C)$




NO! non funziona.

Proietto da C su AM

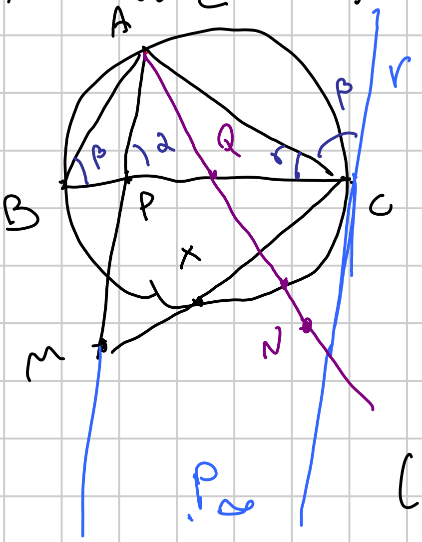
$A \to A$

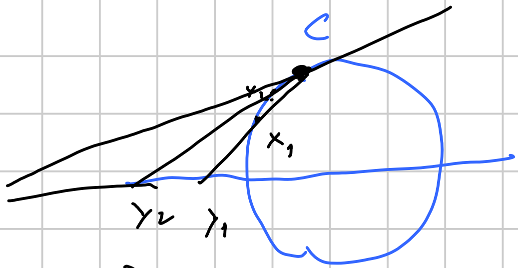
$B \to P$

$X \to M$

$C \to r \cap AM$

$C \to P_\infty$ di AM




proiezione C da C

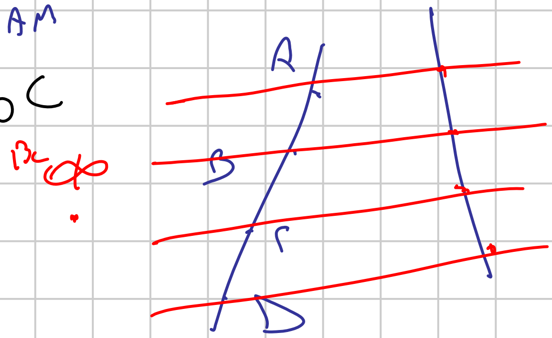
= AM ∩ tangente in C

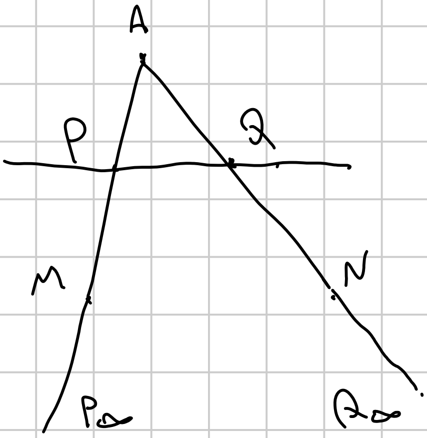
tg in C // AM

r // AM r

$$(A, X; B, C) = (A, \underbrace{M}_{\text{Sy}}, \underbrace{P}_{\text{pt medio } AM}, P_\infty) = -1$$

Proietto $(A, M, P, P_\infty)$ da Punto all'infinito di BC




Proietto da $R_\infty \in BC$

$R_\infty$

$A \to A$

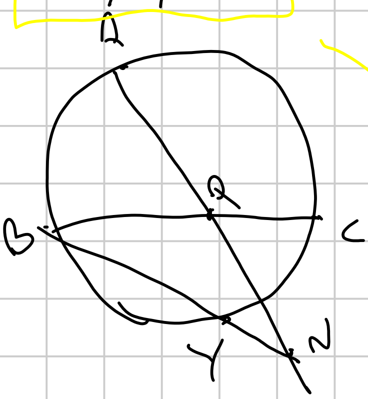
$P \to Q$ $PQ // MN$

$M \to N$

$P_\infty \to Q_\infty$

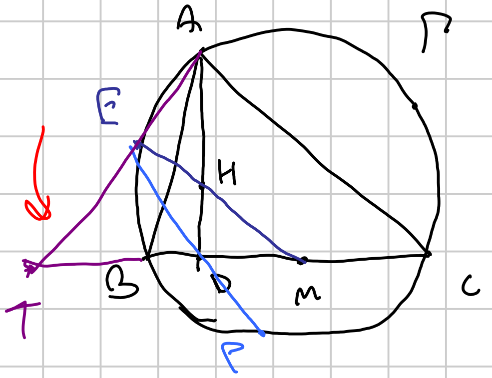
$$\boxed{(A, X, B, C)} = (A\ M, P, P_\infty) = (A, N, Q, Q_\infty) = (A, Y, C, B) = -1$$

$$= \boxed{(A, Y, B, C)}$$




$X = Y$

APMO 2012 - 4 (WC 2013)




AD altezza, M pt medio BC

$MH \cap \Gamma = E$

$ED \cap \Gamma = F$ Tesi: $\dfrac{BF}{FC} = \dfrac{AB}{AC}$

$\dfrac{|BF \cdot AC|}{|FC \cdot AB|} = 1$

segmenti veri

$(A,F;B,C) \overset{?}{=} \dfrac{\sin APB \cdot \sin FPC}{\sin FPB \cdot \sin APC} = \dfrac{\frac{AB}{2R} \cdot \frac{FC}{2R}}{\frac{FB}{2R} \cdot \frac{AC}{2R}}$

$|\sin APB| = \left|\dfrac{AB}{2R}\right|$

Ricordarsi i segni e l'ordine dei punti

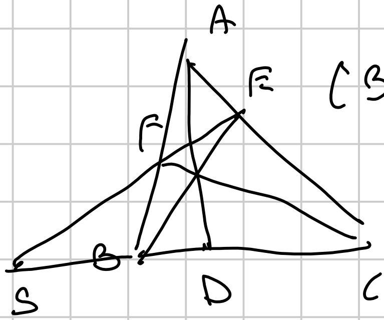
$= \dfrac{\overset{+}{AB} \cdot \overset{+}{FC}}{\underset{-}{FB} \cdot \underset{+}{AC}} = -1$

segmenti orientati

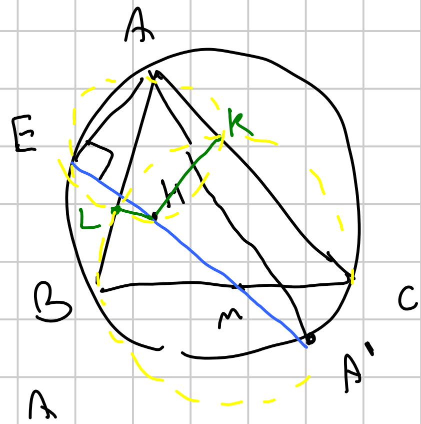
Tesi $\Longleftrightarrow (A,F;B,C) = -1$. Proietto da E

$(A,F;B,C) \overset{E}{=} (T,D;B,C) \overset{?}{=} -1$

Obiettivo: T sta sulla retta per i piedi delle altezze




$(B,C;D,S) = -1$

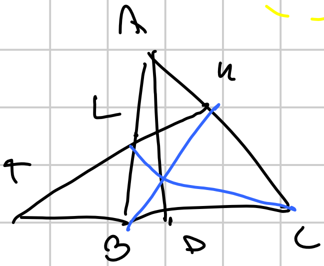



A' diam opposto di A

A'MH E allineati $\Rightarrow \angle AEH = 90°$

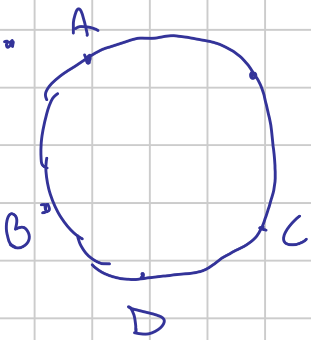
- AKHLE è ciclico di diametro AH
- BCKL è ciclico

AE, LK, BC sono gli assi radicali $\Rightarrow$ concorrono in T



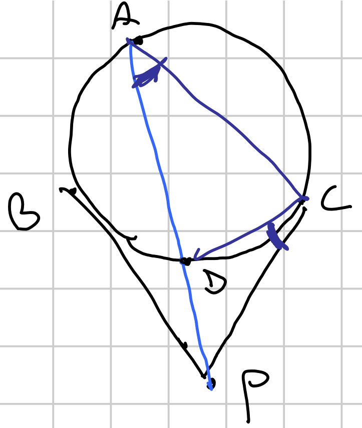

$\Rightarrow$ per il lemma $(B,C,D,T) = -1 \Longleftrightarrow$ AD, BK, CL concorrono

□




$(A,B,C,D) = -1 \Longleftrightarrow \dfrac{AC}{AB} = \dfrac{DC}{BD} \Longleftrightarrow AC \cdot BD = AB \cdot CD$

A, B, C, D sono un quadrilatero ARMONICO




Simmediana $\quad P = \cap$ tangenti da $B$ e $C$

$D = AP \cap \Gamma \qquad$ Tesi: $A, D, B, C$ è armonico

- $\triangle CDP \sim \triangle ACP$

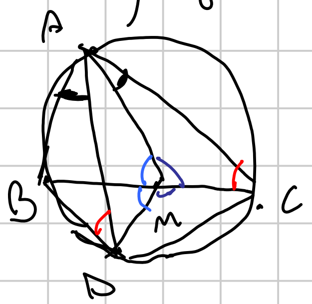
$$\frac{AC}{AP} = \frac{DC}{CP} \qquad \frac{AC}{DC} = \frac{AP}{CP} \overset{BP,\,CP\ tg}{=} \frac{AP}{BP} = \frac{AB}{BD}$$

$\triangle ABP \sim \triangle BDP$

$$AC \cdot BD = AB \cdot DC$$

- $AP$ è simmediana in questa situazione ($Inv + simm$ di $\sqrt{AB\,AC}$ in $A$)

- $A, D, B, C$ su $\Gamma$ quadrilatero armonico:
- $(A, D, B C) = -1$
- $AC \cdot BD = AB \cdot DC$
- $AD$ è simmediana di $\triangle ABC$
  - $DA$ è " di $\triangle DBC$
  - $CB$ è " di $\triangle CAD$
- $AD$, $tg$ in $B$, $tg$ in $C$ concorrono
- $BC$, $tg$ in $A$, $tg$ in $D$ concorrono




$M$ pt medio $BC \Rightarrow \angle AMB = \angle BMD$

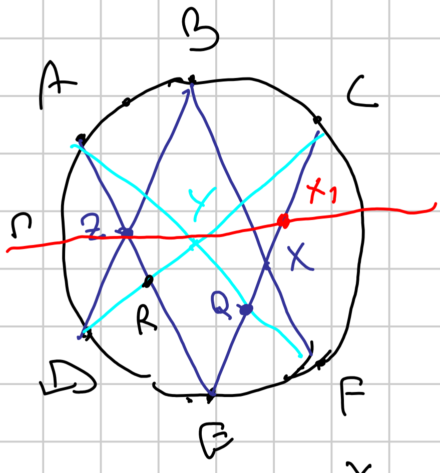
$\angle AMC = \angle CMD$

$\angle BAD = \angle MAC$ perché $AD$ simmediana

$\Rightarrow \triangle ABD, \triangle AMC$ simili

$\triangle ABD, \triangle DMC$ simili

## Teorema di Pascal




$A, B, C, D, E, F$ su $\Gamma$

$AE \cap BD = Z$

$BF \cap CE = X$

$AF \cap CD = Y$

$\}$ $X Y Z$ sono allineati

DIM: $YZ \cap CE = X_1$

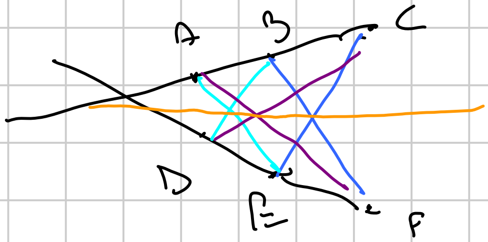
$CE$, prendo $C, X_1, E$, $Q = AF \cap CE$

$R = CD \cap AE$

$$\underline{(C, X_1, E, Q)} \underset{\text{in } AE}{\overset{Y}{=}} (R, Z, E, A) \underset{\text{in } \Gamma}{\overset{D}{=}} (C, B, E, A) \underset{\text{in } CE}{\overset{F}{=}} \underline{(C, X, E, Q)}$$

$\Rightarrow X_1 = X$

**Es**: Teorema di Pappo: come sopra, ma $A, B, C, D, E, F$ sono su due rette




La dimostrazione è identica!

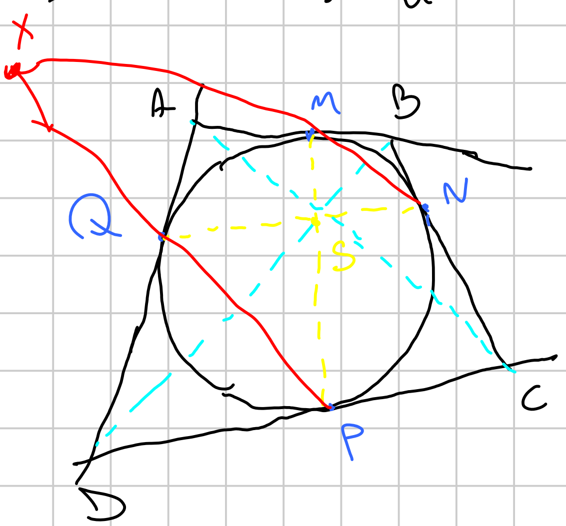
Teo Pascal ++ $A, B, C, D, E, F \to X\,Y\,Z$ intersecando

allora $X, Y, Z$ sono allineati $\iff$ $A, B, C, D, E, F$ stanno su una CONICA

Errore comune è: $ABCDEF$ è ciclico $\Rightarrow$ $XYZ$, dimostro che sono allineati

$\Rightarrow$ NO, non serve ricordare che $ABCDEF$ sono su un cerchio, solo su una CONICA

• **Es**. teorema di Newton:




$\gamma$ inscritto in $ABCD$

$M, N, P, Q$ pt tangenza.

**Tesi**: $BD, AC, MP, NQ$ concorrono

• Pascal su
$$\begin{matrix} M & Q & N \\ Q & M & P \end{matrix}$$

tg su $M$    tg su $Q$

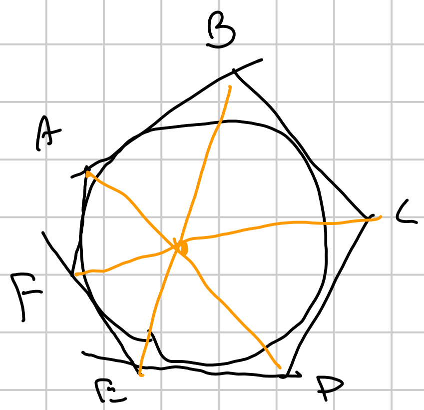
$MM \cap QQ = A$     $QP \cap NM = X$

$MP \cap NQ = S$ $\Rightarrow$ $A, S, X$ allineati

Pascal su
$$\begin{matrix} N & P & Q \\ P & N & M \end{matrix}$$
$\Rightarrow$ $S, X, C$ allineati

$\Rightarrow$ $AC, QN, PM$ concorrono

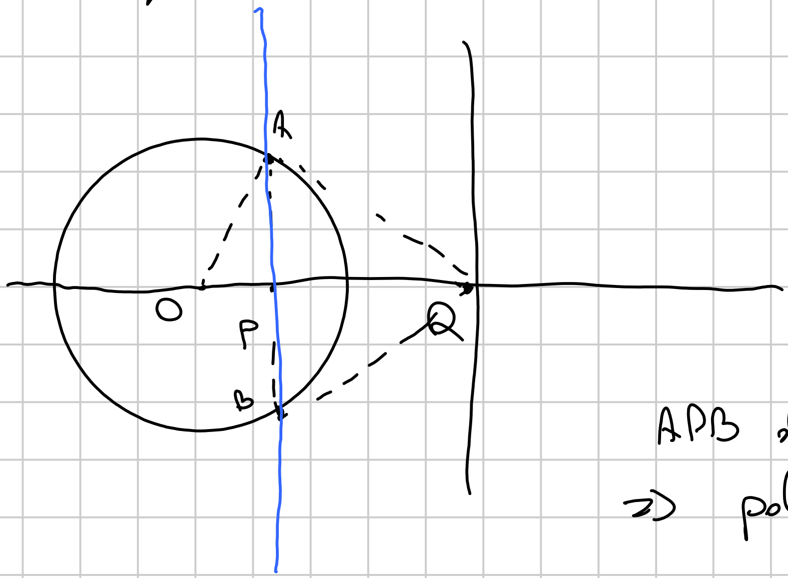
**Es** (con Pascal) Teorema di Brianchon




$\gamma$ inscritto in $ABCDEF$

$\Rightarrow$ $AD, BE, CF$ concorrono

• Poli, Polari, Dualità




polare di $P$ = pol($P$)

$Q$ inverso di $P$ rispetto $\Gamma$

pol($P$) $\perp$ $PQ$

e passa per $Q$

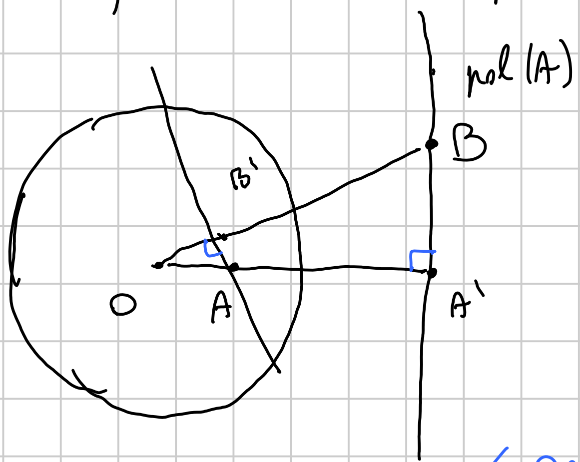
pol($Q$) = retta per $P$ $\perp$ $PQ$

$APB$ allineati, $AB \perp PQ$

$\Rightarrow$ pol($Q$) = $AB$

• Teorema di La Hire

Ho $\Gamma$, $A$ e $B$. $A \in$ pol($B$) $\Longleftrightarrow$ $B \in$ pol($A$)




$B \in$ pol($A$)

$A, A'$ e $B, B'$ inversi rispetto $\Gamma$

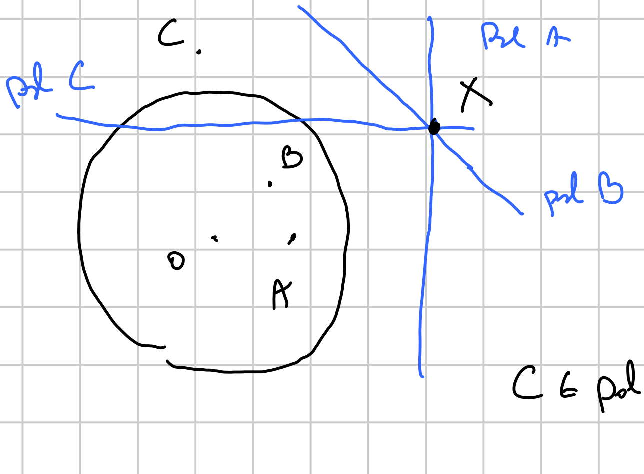
• L'inversione manda $\triangle OAB' \to \triangle OA'B$

$\triangle OAB' \sim \triangle OBA'$ $\left(OB' = \frac{R^2}{OB} \text{ etc}\right)$

$\angle OA'B = 90° \Rightarrow \angle OB'A = 90°$

$\Rightarrow B'A \perp OB \Rightarrow B'A = $ pol($B$)

Corollario: $A, B, C$ allineati $\Longleftrightarrow$ pol($A$), pol($B$), pol($C$) concorrono




$X$ = pol $A \cap$ pol $B$

$\boxed{X \in \text{pol } A}$ $\Rightarrow$ $A \in$ pol $X$

$\boxed{X \in \text{pol } B}$ $\Rightarrow$ $B \in$ pol $X$

$\Rightarrow$ pol($X$) = $AB$

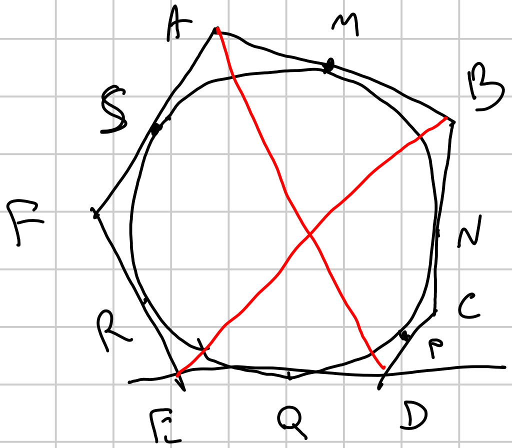
$C \in$ pol($X$) $\Rightarrow$ $\boxed{X \in \text{pol } C}$ $\rightarrow$ polari concorrono

• Prendo tutti i punti e tutte le rette

$P \to r$ $\quad r = $ pol($P$) $\quad$ e $P$ è il polo della retta $\quad P = $ Pol($r$)

- $P \in r \iff pol(P) \ni Pol(r)$
- $pol(P) \cap pol(Q) = \underline{Pol(PQ)}$
- $pol(r \cap s) = Pol(r)\,Pol(s)$

Es. Brianchon:




$M, N \dots S$ sono su $\Gamma$

$pol(M) = AB$

$A = AB \cap AF$

$pol(A) = MS \qquad pol(B) = MN$

$pol(C) = NP \qquad pol(D) = PQ$

$pol(E) = RQ \qquad pol(F) = RS$

$X = AD \cap BE \xrightarrow{pol} pol(X) = \overline{Pol(AD)\,Pol(BE)}$

CF

$Pol(AD) = Pol(A) \cap Pol(D)$

$= MS \cap QP$

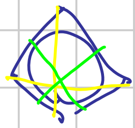
$Pol\,BE = MN \cap RQ$

$pol(CF) = NP \cap SR$

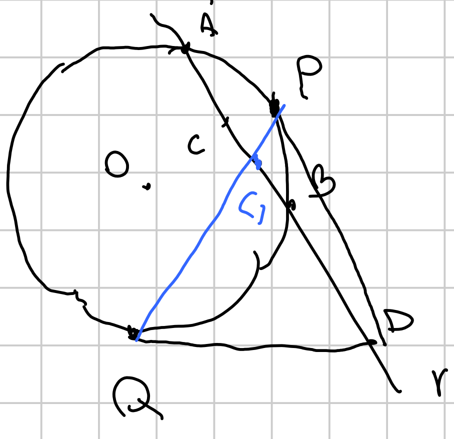
$X \in AD, BE, CF \rightsquigarrow pol(X) \ni MS \cap QP,\ MN \cap RQ,\ NP \cap SR$

Pascal su
$$\begin{matrix} M & P & R \\ Q & S & N \end{matrix}$$

Es: che teorema ottengo facendo il duale di Newton?



- Lemma della polare.




$r$ retta con $A, B, C, D$

$(A, B; C, D) = -1 \iff C \in pol(D)$

$(D \in pol\,C)$

$Pol(D) = PQ$

$C_1 = PQ \cap AB$

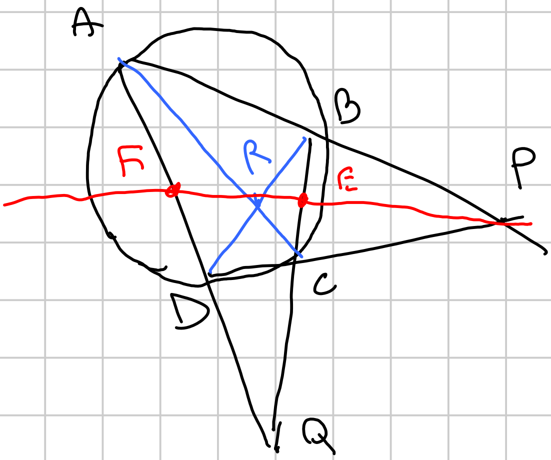
$AB$, $t_g P$, $t_g Q$ concorrono in $D \Rightarrow ABPQ$ è quadrilatero armonico

$(A, B; P, Q) = -1$ Proietto da $P$ $(A, B, D, C_1) = (A, B, C_1, D) = (A, B, C, D)$

$C = C_1$

$\Rightarrow C, P, Q$ allineati $\Rightarrow C \in \text{pol } D$

- Teorema di Brokard




$A, B, C, D$ su $\Gamma$

$AB \cap DC = P$

$AD \cap BC = Q$

$AC \cap BD = R$

$\boxed{\text{Tesi: pol } Q = PR \quad \text{pol } P = QR \quad \text{pol } R = PQ}$

$$(A, D; F, Q) \overset{P}{=} (B, C; E, Q) \quad x$$

$$R \quad ||$$

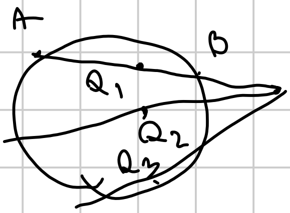
$$(C, B; E, Q) \quad \tfrac{1}{x}$$

$\Rightarrow x^2 = 1 \Rightarrow x = -1$

$\Rightarrow (B, C, E, Q) = -1$

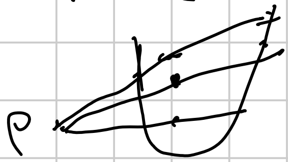
$BCEQ$, $ADFQ$ sono armoniche.

Per lemma polare, pol $(Q)$ passa per $F$, passa per $E$ $\Rightarrow$ pol $Q = EF = PR$ $\square$




$Q_1$ t.c. $(A, B, Q, P) = -1$

pol $(P)$ = luogo di $Q$ al variare di $AB$




$\Rightarrow$ pol $(P)$ per una conica generica