A1 BASIC

Note Title 05/05/2021

POLINOMI

termine noto

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ ANELLI: $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}$

coeff. di testa

$\deg(p) = \max\{n \in \mathbb{N} \mid a_n \neq 0\}$ $\deg 0 = -\infty$

$\deg(p+q) \leq \max\{\deg(p), \deg(q)\}$

$\deg(pq) = \deg(p) + \deg(q)$ $(\mathbb{Z}/6\mathbb{Z}: \ 2x \cdot 3x = 6x^2 = 0)$

$\deg(p \circ q) = \deg(p)\deg(q)$

P1: $P(Q(x)^2) = P(x) \cdot Q(x)^2$ in $\mathbb{Q}[x]$

- $P = 0 \Rightarrow 0 = 0$ $(0, \sim)$
- $Q = 0 \Rightarrow P(0) = 0$ ✓
- $\deg(P) = p$, $\deg(Q) = q$, $2pq = p + 2q$ $(2q-1)(p-1) = 1$

$q = 0, p = 0$ $a = ab^2, a \neq 0 \Rightarrow b = \pm 1$ ✓

- $q = 1, p = 2$ ~~ $ax^2 + bx + c = p(x)$, $dx + e = q(x)$

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p, q con $q \neq 0$ $\exists!\, a, b \in \mathbb{R}[x]$ tali che $\deg(b) < \deg(q)$, $p = aq + b$

$p = x^4 - 3x^2 + 2x - 1$, $q = x^2 + x - 1$

$$
\begin{array}{l|l}
x^4 + 0x^3 - 3x^2 + 2x - 1 & x^2 + x - 1 \\ \hline
-x^4 - x^3 + x^2 & x^2 - x - 1 = a \\
0x^4 - x^3 - 2x^2 + 2x - 1 & \\
+x^3 + x^2 - x & \\
0 \quad 0 - x^2 + x - 1 & \\
0 \quad \boxed{2x - 2} = b &
\end{array}
$$

$a \equiv b \ (q(x)) \Leftrightarrow q \mid a - b$

CONG. POLINOMIALI

$q(x) = x^2 + x + 1$, $p(x) = x^{35}$ $0 \equiv x^2 + x + 1 \ (x^2 + x + 1)$ $a \equiv c \ (q)$, $b \equiv d \ (q)$

$\Rightarrow ab \equiv cd \ (q)$

$x^3-1\equiv 0\ (x^2+x+1) \Rightarrow x^3\equiv 1\ (x^2+x+1)$

$x^{35}\equiv (x^3)^{11}\cdot x^2\equiv 1^{11}\cdot x^2\equiv x^2\equiv -x-1\ (x^2+x+1)$

TEOREMA DI RUFFINI

$p\in\mathbb{R}[x],\ \alpha\in\mathbb{R}$, allora $p(\alpha)=0 \Leftrightarrow x-\alpha \mid p$

$\Leftarrow$: $p(x)=(x-\alpha)q(x) \Rightarrow p(\alpha)=(\alpha-\alpha)q(\alpha)=0$. ✓

$\Rightarrow$: $p(x)=(x-\alpha)q(x)+k \Rightarrow p(\alpha)=0+k \Rightarrow k=0$

P2: $(x+1)P(x-1)-(x-1)P(x)=c$ con $P\in\mathbb{R}[x],\ c\in\mathbb{R}$

$x=1 \Rightarrow 2P(0)=c \Rightarrow P(0)=c/2$

$x=0 \Rightarrow P(-1)+P(0)=c \Rightarrow P(-1)=c/2$

$P(x)-c/2=x(x+1)q(x)$

$(x+1)(x-1)xq(x-1)+\frac{c}{2}\cancel{(x+1)}-\cancel{(x-1)}\frac{c}{2}-(x-1)x(x+1)q(x)=\cancel{c}$

y $x(x+1)(x-1)(q(x-1)-q(x))=0 \Rightarrow q(x-1)-q(x)=0$

$q\equiv k$ ✓ $\Rightarrow P(x)=kx(x+1)+c/2$

$p\in\mathbb{Z}[x],\ \forall a,b\in\mathbb{Z}\ \ a-b\mid p(a)-p(b)$

- $p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$, $p(a)-p(b)=\underbrace{a_n(b^n-a^n)}_{(b-a)}+\ldots+\underbrace{\cancel{a_0-a_0}}_{(b-a)}$ ✓

  $b^k-a^k=(b-a)(b^{k-1}+b^{k-2}a+\ldots a^{k-1})$

- $\alpha\in\mathbb{Q},\ p\in\mathbb{Q}[x],\ \alpha=\frac{a}{b}$ con $(a,b)=1$

  $p(\alpha)=0 \Rightarrow a_n\frac{a^n}{b^n}+\ldots+a_0=0 \Rightarrow a_na^n+a_{n-1}a^{n-1}b+\ldots+a_0b^n=0$

  $a_0b^n\equiv 0\ (a) \Rightarrow a_0\equiv 0\ (a) \quad a_na^n\equiv 0\ (b) \Rightarrow a_n\equiv 0\ (b)$

P3: $p\in\mathbb{Z}[x],\ n\in\mathbb{Z},\ k\geq 1 \mid p^k(n)=n \Rightarrow p^2(n)=n \qquad (p(n)\neq n)$

$\underbrace{\underset{d_1}{p(n)-n} \mid \underset{d_2}{p^2(n)-p(n)} \mid p^3(n)-p^2(n) \mid \ldots \mid \underbrace{p^k(n)}_{n}-p^{k-1}(n)}\mid \underbrace{p^{k+1}(n)}_{p(n)}-\underbrace{p^k(n)}_{n}$

d_{k+1}

$d_1+d_2+\ldots+d_k=0$ dovremmo avere $k/2$ + e $k/2$ − (se $d_i\neq 0$)

- se k è dispari $d_i=0\ \forall i$, assurdo.
- se k è pari

$\cancel{p^j(n)}-p^{j-1}(n)=-(p^{j+1}(n)-\cancel{p^j(n)}) \Rightarrow p^{j-1}(n)=p^{j+1}(n)$

$j \leq K-1 \Rightarrow \underset{n}{\underset{\shortparallel}{\rho^{K}(n)}} = \rho^{K-2}(n) = n \qquad T \mid (K-2, K) = (K-2, 2) \Rightarrow T \mid 2$

- Criterio di Eisenstein

$q \in \mathbb{Z}[x]$, p primo, se $p \nmid a_n$, $p \mid a_i$ per $i<n$ e $p^2 \nmid a_0 \Rightarrow q$ è irriducibile

- Lemma di Gauss

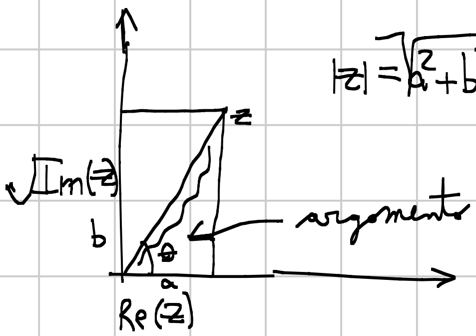
$p \in \mathbb{Z}[x]$ è riducibile in $\mathbb{Q}[x] \Rightarrow$ è riducibile in $\mathbb{Z}[x]$

$q \in \mathbb{Z}[x]$ si dice PRIMITIVO se $(a_0, a_1, \dots, a_n) = 1$

a, b prim. $\Rightarrow ab$ prim.

NUMERI COMPLESSI

$i^2 = -1 \qquad \mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\} \qquad (\bar{z} = a - ib)$



$|z| = \sqrt{a^2+b^2}$ e $|z|^2 = z \cdot \bar{z}$ NORMA

$z = |z|(\cos(\theta) + i\sin(\theta))$

$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad (\theta \in \mathbb{R})$

$(\cos(\alpha)+i\sin(\alpha))(\cos(\beta)+i\sin(\beta)) = (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\cos(\alpha)\sin(\beta)+\cos(\beta)\sin(\alpha))$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

Formula di De Moivre: $(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$

$p(x) = x^n - 1$, chiamiamo radici dell'unità le sue radici.

$z = r(\cos(\theta)+i\sin(\theta)) \quad p(z) = 0 \Rightarrow r^n(\cos(n\theta)+i\sin(n\theta)) = 1$

$(|a||b| = |ab|)$, $r^n = 1 \Rightarrow r = 1$, $\cos(n\theta) = 1$, $\sin(n\theta) = 0$

$n\theta = 2\pi K \Rightarrow \theta = \frac{2\pi K}{n}$ le RDU sono della forma $e^{\frac{2\pi K i}{n}}$ con $K = 0, \dots, n-1$

P4: $(\sqrt{3}+i)^{2022} = 2^{2022}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{2022} = 2^{2022} \cdot e^{i\frac{\pi}{6}\cdot 2022} = -2^{2022}$

TFDA: $p \in \mathbb{C}[x]$, $\deg(p) \geq 1 \Rightarrow p$ ha una radice

- In $\mathbb{R}[x]$ i fattori irriducibili hanno grado 1 o 2;

$\lambda = a+ib \mid p(\lambda)=0 \Rightarrow p(\bar\lambda) = \overline{p(\lambda)} = 0$

$p(\lambda) = 0 \Rightarrow (x-\lambda)(x-\bar\lambda) \mid p(x)$ se $\mathrm{Im}(\lambda) \neq 0$

$$(x-\lambda)(x-\bar\lambda) = x^2 - \underbrace{(\lambda+\bar\lambda)}_{2a \in \mathbb{R}}x + \lambda\bar\lambda, \quad \lambda\bar\lambda \geq 0$$

P5: $x^5+x^4+x^3+x^2+x+1$ scomporre in $\mathbb{R}$

$p(x) = a_n x^n + \dots + a_0 = (x-\lambda_1)\dots(x-\lambda_n)a_n \qquad S = \{1,2\dots,n\}$

$a_{n-1} = (-\lambda_1 - \lambda_2 \dots - \lambda_n)a_n = -e_1 a_n$

$\frac{a_{n-i}}{a_n} = \cancel{a_n}(-1)^i \sum_{\substack{F\subseteq S \\ |F|=i}} \prod_{k\in F} \lambda_k$ $\qquad e_i$ = somma simm. i-esima

(guardate Formule di Newton-Girard, $S_k = \sum \lambda_i^k$, $p(\lambda)=0 \Rightarrow \lambda^n = \frac{1}{a_n}(-a_{n-1}\lambda^{n-1} - \dots - a_0)$)

P6.1: $(x+1)p(x-1) - (x-1)p(x) = c \Rightarrow (x+1)p(x-1) = (x-1)p(x) + c$

caso $\deg \geq 2$: $\sum \lambda_i = S$, $\deg(p) = n \quad S+n-1 = S+1 = n=2$ ✓

P6.2: $x^{2020}+x+1 = p(x) \quad p(\lambda) = 0 \Rightarrow \lambda^{2020} = -\lambda - 1 \quad \sum \lambda_i^{2020} = -\underbrace{\sum \lambda_i}_{0} - 2020 = -2020$

ROOTS OF UNITY FILTER

$p \in \mathbb{R}[x] \quad p(x) = a_n x^n + \dots + a_0 \qquad \omega$ radice PRIMITIVA k-esima dell'unità

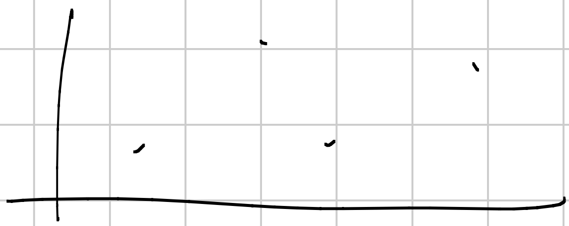
$$\sum_{k\mid l} a_l = \frac{\sum_{m=0}^{k-1} p(\omega^m)}{k} \quad : a_l x^l, \quad a_l \sum_{m=0}^{k-1} \omega^{lm} \begin{cases} k a_l & \text{se } k \mid l \\ 0 & \text{se } k \nmid l \end{cases}$$

ω^l è una radice $\underbrace{\frac{k}{(k,l)}}_{h}$-esima dell'unità PRIMITIVA

(x^n-1, la somma delle radici è 0)

- se $k \mid l$ $\quad a_l \sum_{m=0}^{k-1} 1 = k a_l$
- se $k \nmid l$ ho k/h pezzi da h termini con somma 0.

somma $a_{2k} = \frac{p(1)+p(-1)}{2}$



n+1 coppie (x_i, y_i)
con x_i distinti

$(\deg p, \deg q \le n)$

se p e q passano per le coppie $p-q$ ha $n+1$ radici distinte $\Rightarrow p-q=0$

$L_i = y_i \prod_{j\neq i} \frac{x-x_j}{x_i-x_j}$, Il p che cerco è $p(x)=\sum_{i=1}^{n+1} L_i$

- Metodo alternativo

$(1,2)$ $(3,7)$ $(4,3)$ $p_1(1)=2$ $p_1(x)=(x-1)p_2(x)+2$

$2p_2(3)+2=7 \Rightarrow p_2(3)=\frac{5}{2}$ $p_2(x)=(x-3)p_3(x)+\frac{5}{2}$

P7: $P\in\mathbb{Z}[x]$ | $P(0)=0, P(1)$ e $P(k)\equiv 0,1$ (q) per un certo q primo, allora
$\forall k\in\mathbb{Z}$

$\deg(P)\ge q-1$.

Per assurdo $\deg(P)\le q-2$ $P(x)=\sum_{k=0}^{q-1} P(k)\prod_{j\neq k}\frac{x-j}{k-j}$

coeff. di x^{q-1} a destra è $\sum_{k=0}^{q-1} P(k)\prod_{j\neq k}\frac{1}{k-j} = \sum_{k=0}^{q-1} P(k)\frac{(-1)^{q-k-1}}{k!\,(q-k-1)!} = \frac{1}{(q-1)!}\sum P(k)(-1)^{q-k-1}\binom{q-1}{k}$

$p\in\mathbb{R}[x]$, $\deg p = n\ge 1 \Rightarrow \deg(p(x+1)-p(x)) = n-1$

deg=3	2	5	11	7	x
~2		3	6	−4	x−7
~1			3	−10	x−3
~0				−13	x+7

p con $\deg(p)=3$ e $p(1)=2$

$p(2)=5$, $p(3)=11$, $p(4)=7$, $p(5)=?$

$-13=x+7 \Rightarrow x=-20 \Rightarrow p(5)=-20$

DERIVATE

$p(x) = \sum a_k x^k$, $p'(x) = \sum a_k k x^{k-1}$

$(p+q)' = p' + q'$, $\underline{(pq)' = pq' + p'q}$, $(x^n)' = nx^{n-1}$, $\underline{(p \circ q)' = p' \circ q \cdot q'}$ $(c)' = 0$

$f: \mathbb{R} \to \mathbb{R}$, $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

$p \in \mathbb{R}[x]$, $p(\alpha) = 0$, $\alpha \in \mathbb{C}$ $(x-\alpha)^m \| p(x) \Rightarrow (x-\alpha)^{m-1} \| p(x)$

$p(x) = (x-\alpha)^m q(x)$ con $x - \alpha \nmid q(x)$

$p'(x) = m(x-\alpha)^{m-1} q(x) + (x-\alpha)^m q'(x) = (x-\alpha)^{m-1}(mq(x) + (x-\alpha)q'(x)) \Rightarrow (x-\alpha)^{m-1} \| p'(x)$

CRITERIO DELLA DERIVATA

$p \in \mathbb{R}[x]$, $(x-\alpha) | p \wedge (x-\alpha) | p' \Rightarrow (x-\alpha)^2 | p$: $p(x) = (x-\alpha)q(x) \Rightarrow p'(x) = q(x) + (x-\alpha)q'(x)$

$\Rightarrow x - \alpha | q(x) \Rightarrow q(x) = (x-\alpha)g(x) \Rightarrow p(x) = (x-\alpha)^2 g(x)$.

$p(x) = \prod (x - \lambda_i)^{m_i}$, definisco $\mathrm{rad}(p(x)) = \prod (x - \lambda_i)$

- $\mathrm{rad}(p(x)) = \frac{p(x)}{(p(x), p'(x))}$ ✓

ES:

$p \in \mathbb{C}[x]$, $\deg p = n \Rightarrow p(x)(p(x)+1)$ ha almeno $n+1$ radici distinte.

$*$: $p(2n+1) = 2^{2n}$

CORREZIONE

1)

$(k+1)p(k) - k = 0$ per $k = 0, \dots, n$

$q(x) := (x+1)p(x) - x = c\,x(x-1)\dots(x-n)$, $c \in \mathbb{R}$

$x = -1 \Rightarrow 1 = c(-1)(-2)\dots(-n+1) = c(n+1)!\,(-1)^{n+1} \Rightarrow c = \frac{(-1)^{n+1}}{(n+1)!}$

$p(x) = \frac{c\,x(x-1)\dots(x-n) + x}{x+1}$

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2) $2^n = \sum_{i=0}^{n} \binom{n}{i}$, $\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!} = p_k(x)$

$P(x) = \sum_{k=0}^{n} p_k(x)$ $\quad P(z) = \sum_{k=0}^{n} p_k(z)$ $\begin{cases} \binom{z}{k} & \text{per } k \le z \\ 0 & \text{per } k > z \end{cases}$

$\overset{\shortparallel}{2^z}$ per $z = 0, \dots, n$

$\binom{n}{k} = \binom{n}{n-k}$ $\quad \sum_{k=0}^{n} p_k(2n+1) = \sum_{k=0}^{n} \binom{2n+1}{k} = \frac{1}{2}\left(\sum_{k=0}^{n} \binom{2n+1}{k} + \sum_{k=0}^{n} \binom{2n+1}{2n+1-k}\right) = \frac{1}{2} 2^{2n+1} = 2^{2n}$

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7) $\lambda_1, \lambda_2, \dots, \lambda_{2022}$, pongo $a_1 = \frac{\lambda_1 + \lambda_2}{2} \Rightarrow P_1(\lambda_1)^2 = P_2(\lambda_2)^2 = \frac{(\lambda_1 - \lambda_2)^2}{4}$

Vado da n a $n+1$: ho $n+1$ radici con la proprietà voluta e voglio aggiungere λ_{n+2}

$P(\lambda_1)^2 = P(\lambda_2)^2 = \dots = K \in \mathbb{C}$, $a_{n+1} = \frac{K + P_n(\lambda_{n+2})^2}{2}$ $\quad P_{n+1}(\lambda_i) = K^2 - a_{n+1}$ per $i \le n+1$

$P_{n+1}(\lambda_{n+2}) = P_n(\lambda_{n+2})^2 - a_{n+1}$

$P_{n+1}(\lambda_{n+2}) = -P_{n+1}(\lambda_1)$

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8) $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} - 1 = p(x)$ $\quad \deg p \le 2$

$p(a) = 0 + \overset{1}{\frac{\cancel{(a-b)}\cancel{(a-c)}}{\cancel{(a-b)}\cancel{(a-c)}}} + 0 - 1 = 0$

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11) $(\underbrace{\lambda_1^2+1}_{(\lambda_1+i)(\lambda_1-i)})(\lambda_2^2+1)(\lambda_3^2+1)$, $p(i) = a(i-\lambda_1)(i-\lambda_2)(i-\lambda_3)$

$= \frac{1}{a^2} p(i) p(-i)$

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12) $\sum_{n=0}^{668} (-1)^n \binom{2004}{3n}$ sono i coeff. in posizione $3k$ di $(x-1)^{2004}$

$$=\left((\omega-1)^{2004}+(\omega^2-1)^{2004}+0\right)\frac{1}{3} \qquad \omega=e^{\frac{2\pi i}{3}}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$=\frac{1}{3}\left(\left(-\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)^{2004}+\left(-\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)^{2004}\right) \qquad \omega^2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$=3^{1001}\left(\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{2004}+\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{2004}\right)$$

$$150^\circ = \frac{5}{6}\pi \qquad 210^\circ = \frac{7}{6}\pi$$

$$\frac{5\pi}{6}\cdot 2004=\underbrace{334\cdot 5\pi}_{0} \qquad \frac{7\pi}{6}\cdot 2004=\underbrace{334\cdot 7\pi}_{0}$$

$$=3^{1001}(1+1)=2\cdot 3^{1001}.$$