A2 BASIC

Note Title 07/05/2021

NOTAZIONE

$f(a,b,c)$

$$\sum_{cyc} f(a,b,c) = f(a,b,c) + f(c,a,b) + f(b,c,a)$$

$$f(a,b,c) = a^2 b \Rightarrow \sum_{cyc} a^2 b = a^2 b + b^2 c + c^2 a$$

$$\sum_{sym} f(a,b,c) = f(a,b,c) + f(a,c,b) + f(b,a,c) + \dots$$

$x_1, x_2, \dots, x_n \geq 0$ MEDIA P-ESIMA $\rightarrow$ $M_p(x_1, \dots, x_n) = \sqrt[p]{\dfrac{x_1^p + x_2^p + \dots + x_n^p}{n}}$

$M_{-\infty} = \min_{1 \leq i \leq n} (x_i)$

$M_{+\infty} = \max_{1 \leq i \leq n} (x_i)$

$M_1 = AM$

$M_{-1} = HM = \dfrac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots \frac{1}{x_n}}$

$M_2 = QM$

$M_0 = GM = \sqrt[n]{x_1 x_2 \dots x_n}$

SOS

$x \in \mathbb{R} \Rightarrow x^2 \geq 0 \qquad x^2 + 6x + 10 = (x+3)^2 + 1^2 > 0$

P1.1: $\sum_{cyc} a^2 \geq \sum_{cyc} ab \quad \forall a,b,c \in \mathbb{R}$: $\quad x,y \in \mathbb{R},\ \underbrace{(x-y)^2 \geq 0} \Rightarrow \dfrac{x^2+y^2}{2} \geq xy$

$\underbrace{\dfrac{a^2+b^2}{2}} \geq ab$

$\dfrac{b^2+c^2}{2} \geq bc$

$\dfrac{c^2+a^2}{2} \geq ca$

$\underbrace{\left(\tfrac{1}{2} + \tfrac{1}{2}\right)}_{1} a^2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\right) b^2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\right) c^2 \geq ab + bc + ca$

caso uguaglianza: $a = b = c$

$$\sum_{cyc} ab \leq \sum_{cyc} \frac{(a^2+b^2)}{2} = \sum_{cyc} a^2$$

P1.2: $a,b,c > 0 \quad \sum_{cyc} \dfrac{2a^3}{a^2+b^2} \geq \sum_{cyc} a$;

$\dfrac{2a^3}{a^2+b^2} \geq 2a - b \Leftrightarrow \cancel{2a^3} \geq (2a-b)(a^2+b^2) = \cancel{2a^3} + 2ab^2 - ba^2 - b^3$

$\Leftrightarrow b^3 + ba^2 \geq 2ab^2 \Leftrightarrow b^2 + a^2 \geq 2ab$

$\sum_{cyc} \dfrac{2a^3}{a^2+b^2} \geq \sum_{cyc} 2a - b = \sum_{cyc} a \quad \checkmark$

P1.3: $x,y,z\geq 0$ $\quad x^3+y^3+z^3\geq 3xyz$:

$$x^3+y^3+z^3-3xyz=\underbrace{(x+y+z)}_{\geq 0}\underbrace{(x^2+y^2+z^2-(xy+yz+zx))}_{\geq 0}\geq 0$$

2) MEDIE P-ESIME

$p,q\in\mathbb{R}\cup\{\pm\infty\}$ tali che $p<q \Rightarrow M_p(x_1,\dots,x_n)\leq M_q(x_1,\dots,x_n)$

caso uguaglianza: $x_1=x_2\dots=x_n$

$HM\leq GM\leq AM\leq QM$

P2.1: $AM\geq GM \Rightarrow GM\geq HM$ $\left(\sqrt[n]{x_1x_2\dots x_n}\geq \dfrac{n}{\frac{1}{x_1}+\dots+\frac{1}{x_n}}\right)$

$$\frac{\frac{1}{x_1}+\dots+\frac{1}{x_n}}{n}\geq\frac{1}{\sqrt[n]{x_1x_2\dots x_n}}\Rightarrow\frac{n}{\frac{1}{x_1}+\dots+\frac{1}{x_n}}\leq\sqrt[n]{x_1\dots x_n}$$

✓

P2.2: $x^2+y^5+1\geq kxy$, determinare k più grande t.c. è vera $\forall x,y\geq 0$:

$$\frac{5\left(\frac{1}{5}x^2\right)+2\left(\frac{1}{2}y^5\right)+3\left(\frac{1}{3}\right)}{\cancel{10}}\geq 10\sqrt[10]{\left(\tfrac{1}{5}x^2\right)^5\left(\tfrac{1}{2}y^5\right)^2\left(\tfrac{1}{3}\right)^3}=xy\underbrace{\frac{10}{\sqrt[10]{5^5 2^2 3^3}}}_{k}$$

$\frac{1}{5}x^2=\frac{1}{2}y^5=\frac{1}{3}\longrightarrow$ diventa un $=$ ✓

P2.3: $a,b,c\geq 0$, $\sum_{cyc}a^4\geq\sum_{cyc}a^2bc$: speranza: $xa^4+yb^4+zc^4\geq a^2bc$

(con $x+y+z=1$

$$\frac{2a^4+b^4+c^4}{4}\geq a^2bc\iff\frac{1}{2}a^4+\frac{1}{4}b^4+\frac{1}{4}c^4$$

$$\sum_{cyc}a^2bc\leq\sum_{cyc}\left(\tfrac{1}{2}a^4+\tfrac{1}{4}b^4+\tfrac{1}{4}c^4\right)=\sum_{cyc}a^4$$ ✓

3) Cauchy-Schwarz: $(x_1,\dots,x_n)$, $(y_1,\dots,y_n)$ in $\mathbb{R}$, allora

$$\left(\sum_{k=1}^n x_k^2\right)\left(\sum_{k=1}^n y_k^2\right)\geq\left(\sum_{k=1}^n x_ky_k\right)^2$$

PROOF: $p_k(t)=(x_kt+y_k)^2$ per $k=1,\dots,n$ $\quad P(t):=\sum_{k=1}^n p_k(t)\geq 0\quad\forall t\in\mathbb{R}$

$$P(t)=\left(\sum_{k=1}^n x_k^2\right)t^2+2\left(\sum_{k=1}^n x_ky_k\right)t+\left(\sum_{k=1}^n y_k^2\right)\geq 0$$

caso uguaglianza: $\exists t\in\mathbb{R}$

$p_k(t)=0\ \forall k\Rightarrow x_kt+y_k=0\ \forall k$

$$\cancel{4}\left(\sum x_k y_k\right)^2 - \cancel{4}\left(\sum x_k^2\right)\left(\sum y_k^2\right) \le 0 \iff \left(\sum x_k^2\right)\left(\sum y_k^2\right) \ge \left(\sum x_k y_k\right)^2 \quad \checkmark$$

LEMMA DI TITU (Corollario):

$x_1,\dots,x_n \ge 0$ e $y_1,y_2,\dots,y_n > 0$

$$\sum_{k=1}^{n} \frac{x_k^2}{y_k} \ge \frac{\left(\sum x_k\right)^2}{\sum y_k}$$

AM-QM: $(x_1,\dots,x_n)$ e $(1,\dots,1)$ in CS $\Rightarrow \frac{\left(\sum x_k^2\right)\cancel{n}}{n} \ge \frac{\left(\sum x_k\right)^2}{n^2}$

$$\Rightarrow \sqrt{\frac{\sum x_k^2}{n}} \ge \frac{\sum x_k}{n}$$

Nesbitt: $a,b,c \ge 0$ $\quad \sum_{cyc} \frac{a}{b+c} \ge \frac{3}{2}$: scriviamo come $\sum_{cyc} \frac{a^2}{ab+ca} \ge \frac{3}{2}$

Per Titu $\sum_{cyc} \frac{a^2}{ab+ca} \ge \frac{\left(\sum a\right)^2}{2\left(\sum ab\right)} \overset{?}{\ge} \frac{3}{2} \iff \sum a^2 + \cancel{2\sum ab} \ge \overset{1}{\cancel{3}}\sum ab \quad \checkmark$

PT.2: $c_1 \le \sum \frac{a}{a+b} \le c_2$ migliori costanti c_1, c_2 al variare di a,b,c in $\mathbb{R}_{>0}$

$\sum \frac{a}{a+b} \ge \sum \frac{a}{a+b+c} = 1$ $\quad (a,b,c) = (x^2,1,x)$ $\quad \frac{\cancel{x^2}}{\cancel{x^2}+1/x^2} + \frac{1}{1+x} + \frac{x}{x+x^2}$ per $x \to +\infty$

$\to 1$, $\to 0$, $\to 0$, $x(x+1)$

$\Rightarrow c_1 = 1$, $\sum \frac{a}{a+b} =: f(a,b,c)$

$f(b,a,c) = \sum \frac{b}{a+b} \Rightarrow f(a,b,c) + f(b,a,c) = 3 \Rightarrow f(a,b,c) = 3 - f(b,a,c)$

$c_2 = 3 - c_1 = 2 \quad \checkmark$

4) Riarrangiamento: $x_1,\dots,x_n \in \mathbb{R}$ e $y_1,\dots,y_n \in \mathbb{R} \mid x_1 \ge \dots \ge x_n$ e $y_1 \ge \dots \ge y_n$
allora $\forall \sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ permutazione abbiamo

$$\sum_{k=1}^{n} x_k y_k \ge \sum_{k=1}^{n} x_k y_{\sigma(k)} \ge \sum_{k=1}^{n} x_k y_{n+1-k}$$

(x,y) e (x,y)

$x^2 + y^2 \ge xy + yx = 2xy$

Chebyshev: $x_1,\dots,x_n \in \mathbb{R}$ e $y_1,\dots,y_n \in \mathbb{R} \mid x_1 \ge \dots \ge x_n$ e $y_1 \ge \dots \ge y_n$

$$n\left(\sum_{k=1}^{n} x_k y_k\right) \ge \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) \ge n\left(\sum_{k=1}^{n} x_k y_{n+1-k}\right)$$

$\ge x_1y_1 + \dots + x_ny_n$

$x_1y_2 + \dots + x_ny_1$

$\vdots$

$x_1y_n + \dots + x_ny_{n-1}$

P4; a,b,c>0, allora $\sum_{cyc}\frac{ab}{a+b}\le\frac{3\sum ab}{2\sum a}$: $a\ge b\ge c$ w/log

$\frac{1}{a}\le\frac{1}{b}\le\frac{1}{c}$

$\sum_{cyc}\frac{1}{\frac{1}{a}+\frac{1}{b}}\le\frac{3\sum ab}{2\sum a}$ $\left(\frac{1}{\frac{1}{a}+\frac{1}{b}}\ge\frac{1}{\frac{1}{a}+\frac{1}{c}}\ge\frac{1}{\frac{1}{b}+\frac{1}{c}}\right)$

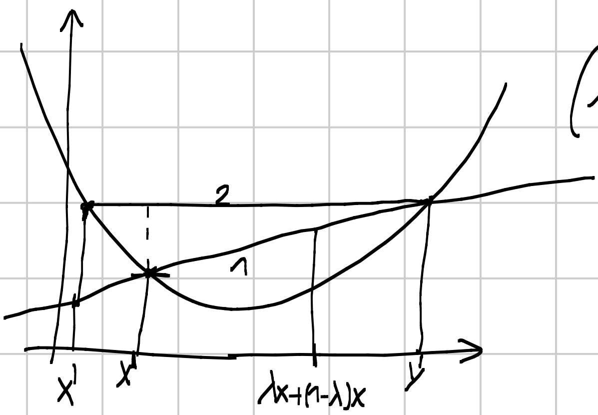
$(a+b,\ a+c,\ b+c)$

$$3\left(\sum_{cyc}\frac{ab}{a+b}\cancel{(a+b)}\right)\ge\left(\sum_{cyc}\frac{ab}{a+b}\right)\left(2\sum_{cyc}a\right)\Rightarrow\sum\frac{ab}{a+b}\le\frac{3\sum ab}{2\sum a}$$

5) CONVESSITÁ

$f:\mathbb{R}\to\mathbb{R}$ si dice convessa se $\forall x,y\in\mathbb{R}$ $\forall\lambda\in[0,1]$ abbiamo

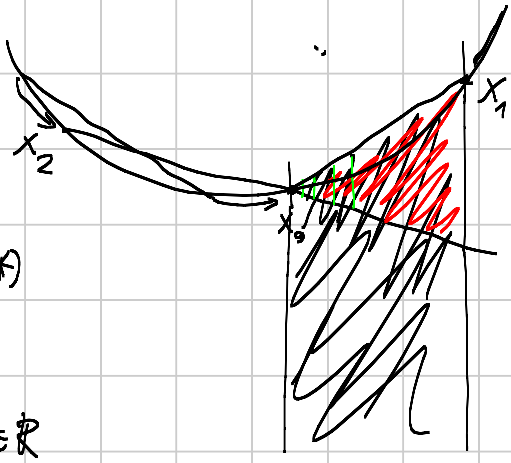
$$\lambda f(x)+(1-\lambda)f(y)\ge f(\lambda x+(1-\lambda)y)$$



(concave: disuguaglianza opposta)

f convessa ⇒ continua:

⇒ $\frac{f(x)-f(y)}{x-y}$ crescente in x e y (*)



se $\exists f'':\mathbb{R}\to\mathbb{R}$, allora f convessa $\Leftrightarrow$ $f''(x)\ge 0\ \forall x\in\mathbb{R}$

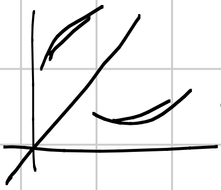
x^n è convessa

in $\mathbb{R}$ per n pari — in $[0,+\infty)$ per n dispari

se $f:[0,1]\to\mathbb{R}$ può essere fatta così:

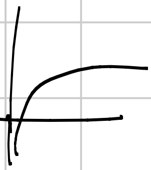
e^x è convessa, a^x è convessa

inversa di convessa è concava



sopra → sotto

log è concava



Jensen: f convessa $x_1,\dots,x_n \in \mathbb{R}$, $w_1,\dots,w_n \in [0,1]$ tali che $\sum_{k=1}^{n} w_k = 1$ allora

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \le \sum_{k=1}^{n} w_k f(x_k)$$

PS.1: AM-GM con Jensen: $\frac{x_1+\dots+x_n}{n} \ge \sqrt[n]{x_1\cdots x_n}$ prendiamo $x_1,\dots,x_n > 0$

$\Longleftrightarrow$ $\log\left(\frac{x_1+\dots+x_n}{n}\right) \ge \sum_{k=1}^{n} \frac{1}{n}\log(x_k)$ $\quad w_1=\dots=w_n=\frac{1}{n}$ vera per concavità

PS.2: $a,b \ge 0$, $p,q>1 \mid \frac{1}{p}+\frac{1}{q}=1$, allora $\frac{a^p}{p}+\frac{b^q}{q} \ge ab$:

$\Longleftrightarrow \log\left(\frac{1}{p}a^p+\frac{1}{q}b^q\right) \ge \log(a)+\log(b) = \frac{1}{p}\log(a^p)+\frac{1}{q}(b^q)$ vera per concavità ✓

6) Schur: $x,y,z \ge 0$ e $t \ge 0$, allora (wlog $x \ge y \ge z$)

$$\sum_{cyc} x^t(x-y)(x-z) \ge 0 \Longleftrightarrow \underbrace{(x-y)}_{\ge 0}\underbrace{\left(x^t(x-z)-y^t(y-z)\right)}_{\ge 0}+\underbrace{\underbrace{z^t}_{\ge 0}\underbrace{(z-x)}_{\le 0}\underbrace{(z-y)}_{\le 0}}_{\ge 0} \ge 0$$ ✓

$t=1 \Rightarrow x^3+y^3+z^3+3xyz \ge \sum_{sym} x^2y$

7) Esempi vinceli: $\sum_{i=1}^{n} \frac{x_i}{\sqrt{1-x_i}} \ge \sqrt{\frac{n}{n-1}}$ $\quad x_1,\dots,x_n \ge 0$ e $\sum_{i=1}^{n} x_i = 1$

con Jensen: $f(x) = \frac{x}{\sqrt{1-x}} = x \cdot \frac{1}{\sqrt{1-x}}$ $\quad \sum_{i=1}^{n} \frac{x_i}{\sqrt{1-x_i}} \ge f\left(\frac{1}{n}\right) = \sqrt{\frac{n}{n-1}}$

(Provate con Chebyschev)

f è omogenea di esponente λ e $\forall a,b,c \in \mathbb{R}$ vale che

$$f(ta,tb,tc) = t^\lambda f(a,b,c)$$

$a+b+c=3$ $\quad a,b,c \ge 0$ $\quad a^2+b^2+c^2 \ge 3\left(\frac{a+b+c}{3}\right)^2 \Longleftrightarrow \sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3}$

CORREZIONE

2) $1+a_k \ge 2\sqrt{a_k} \Rightarrow \prod_{k=1}^{n}(1+a_k) \ge \prod_{k=1}^{n}(2\sqrt{a_k}) = 2^n \cdot 1$ ✓

$$\frac{1+a_k = (k-1)\frac{1}{k-1}+a_k}{k} \ge \sqrt[k]{\frac{1}{(k-1)^{k-1}}a_k} \Rightarrow (1+a_k)^k \ge \frac{k^k}{(k-1)^{k-1}}a_k$$

$$\prod_{k=2}^{n}(1+a_k)^k \geq \prod_{k=2}^{n}\frac{k^k}{(k-1)^{k-1}}\cdot 1 = n^n,$$ caso uguaglianza ipotetico: $a_k = \frac{1}{k-1}$

$$\Rightarrow \prod_{k=2}^{n} a_k = \frac{1}{(n-1)!} \neq 1,$$ impossibile

4) $$\sum \frac{a^2}{a^2+2/a} \geq \frac{\cancel{(a+b+c)^2}}{\cancel{a^2+b^2+c^2+2\sum ab}} = 1$$

5) W/log $a \geq b \geq c$ $\quad (a^n, b^n, c^n)$ $\left(\frac{1}{b+c}, \frac{1}{a+c}, \frac{1}{a+b}\right)$

$$\sum \frac{a^n}{b+c} \geq \frac{1}{3}(a^n+b^n+c^n)\left(\sum \frac{1}{b+c}\right) *$$

$$\boxed{\sqrt[n]{\frac{a^n+b^n+c^n}{3}} \geq \frac{a+b+c}{3} \Leftrightarrow a^n+b^n+c^n \geq \frac{(a+b+c)^n}{3^{n-1}}}$$

$\left(\frac{1}{b+c}, \frac{1}{a+c}, \frac{1}{a+b}\right)$ $(b+c, a+c, a+b)$

$$\left(\sum \frac{1}{b+c}\right)\left(\sum b+c\right) \geq \left(\sum 1\right)^2 = 9 \Rightarrow \sum \frac{1}{b+c} \geq \frac{9}{2(a+b+c)}$$

$$* \geq \frac{1}{\cancel{3}} \frac{(a+b+c)^{\cancel{n}\ n-1}}{3^{n-1}} \cdot \frac{\cancel{9}\ 3}{2\cancel{(a+b+c)}} = \frac{(a+b+c)^{n-1}}{2\cdot 3^{n-2}}$$

7) $$\sum \frac{1}{1+xy} \overset{?}{\leq} \frac{3}{4} :$$ $\quad$ ~~$x+y+z = xyz$~~, $x,y,z>0$

$x+y+z=1$

$$\sum \frac{z}{z+xyz} = \sum \frac{z}{x+y+2z} = \sum_{cyc} \frac{z}{1+z} \leq \frac{3}{4} \Leftrightarrow \sum_{cyc}\left(1-\frac{1}{1+z}\right) \leq \frac{3}{4} \Leftrightarrow \sum \frac{1}{1+z} \geq \frac{9}{4}$$

$$f(x) = \frac{1}{1+x} \Rightarrow LHS \geq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{4}$$