G1 M

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Note Title 04/09/2022

LEMMA

Dati 4 punti A, B, C, D, $\theta = \angle(AB, CD)$

$$e^{2i\theta} = \frac{a-b}{\bar{a}-\bar{b}} : \frac{c-d}{\bar{c}-\bar{d}}$$

DIM

La traslazione conserva l'angolo

$a - b = |a-b| e^{i\theta_1}$

$c - d = |c-d| e^{i\theta_2}$

$\theta = \theta_1 - \theta_2$

$\frac{a-b}{c-d} = \frac{|a-b|}{|c-d|} e^{i\theta}$

$|z|^2 = z \cdot \bar{z}$

$$e^{2i\theta} = \frac{(a-b)^2}{|a-b|^2} : \frac{(c-d)^2}{|c-d|^2} = \frac{a-b}{\bar{a}-\bar{b}} : \frac{c-d}{\bar{c}-\bar{d}}$$

$\square$

PARALLELISMO $AB /\!/ CD$

$$\frac{a-b}{c-d} = \frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}}$$

PERPENDICOLARITÀ $AB \perp CD$

$$\frac{a-b}{c-d} = -\frac{\bar{a}-\bar{b}}{\bar{c}-\bar{d}}$$

ALLINEAMENTO A-B-C

$$\frac{a-c}{b-c} = \frac{\bar{a}-\bar{c}}{\bar{b}-\bar{c}}$$

SIMILITUDINE $A\hat{B}C \sim X\hat{Y}Z$

$$\frac{a-b}{a-c} = \frac{x-y}{x-z}$$

DIM

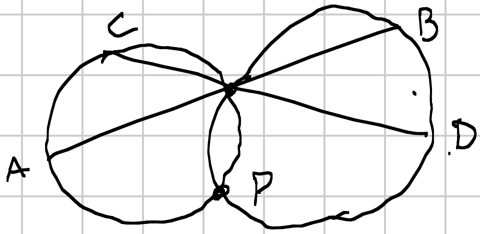
$\frac{AB}{AC} = \frac{XY}{XZ}$ $\quad \angle BAC = \angle YXZ$

$$\frac{|a-b|}{|a-c|} = \frac{|x-y|}{|x-z|} \qquad \frac{a-b}{\bar{a}-\bar{b}} : \frac{a-c}{\bar{a}-\bar{c}} = \frac{x-y}{\bar{x}-\bar{y}} : \frac{x-z}{\bar{x}-\bar{z}}$$

$$\frac{a-b}{a-c}\cdot\frac{|\cancel{a}-b|}{|\cancel{a}-c|}=\frac{x-y}{x-z}\cdot\frac{|\cancel{x}-y|}{|\cancel{x}-z|}$$

$\square$

ROTOMOTETIA $AB \to CD$



P centro rotomotetia

$$\frac{a-p}{b-p}=\frac{c-p}{d-p} \longrightarrow p=\frac{ad-bc}{a+d-b-c}$$

CICLICITÀ ABCD ciclico

$\frac{b-a}{c-a} : \frac{b-d}{c-d}$ reale

AREA

$$[ABC]=\frac{i}{4}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}$$

INTERSEZIONE $P=AB\cap CD$

$$p=\frac{(\bar{a}b-a\bar{b})(c-d)-(a-b)(\bar{c}d-c\bar{d})}{(\bar{a}-\bar{b})(c-d)-(a-b)(\bar{c}-\bar{d})}$$

SIMMETRIA. Q simm. di P risp. AB

$$q=\frac{(a-b)\bar{p}+\bar{a}b-a\bar{b}}{\bar{a}-\bar{b}}$$

DIM

$$z \longmapsto \frac{z-a}{b-a} \qquad a\mapsto 0 \quad b\mapsto 1$$

$$\frac{q-a}{b-a}=\overline{\left(\frac{p-a}{b-a}\right)}$$

$\square$

CIRCOCENTRO $\overset{\triangle}{ABC}$

$$O = \frac{\begin{vmatrix} a & a\bar{a} & 1 \\ b & b\bar{b} & 1 \\ c & c\bar{c} & 1 \end{vmatrix}}{\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}}$$

DIM

$|O-a|^2 = R^2 \quad |O-b|^2 = R^2 \quad |O-c|^2 = R^2$

Sistema $O \quad \bar{O} \quad R^2 - O\bar{O}$ $\square$

CIRCONFERENZA UNITARIA $z\bar{z}=1 \quad \bar{z} = \frac{1}{z}$

$AB \parallel CD \quad ab = cd$

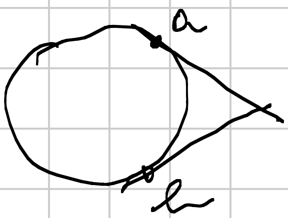
$AB \perp CD \quad ab = -cd$

$P \in AB$

$$\frac{p-a}{b-a} = \frac{\bar{p} - \frac{1}{a}}{\frac{1}{b} - \frac{1}{a}} \longrightarrow a+b = p + ab\bar{p}$$

$P = AB \cap CD$

$$P = \frac{ab(c+d) - cd(a+b)}{ab - cd}$$



$\frac{2ab}{a+b} \quad AA \cap BB$

SIMMETRIA

$q = a + b - ab\bar{p}$

RETTA DI EULERO

$o = 0 \quad g = \frac{a+b+c}{3} \qquad p = \lambda(a+b+c), \ \lambda \in \mathbb{R}$

$h = a+b+c$

$n = \frac{a+b+c}{2}$

INCENTRO

Esistono u, v, w sulla circonferenza unitaria tali che

$a = u^2 \quad b = v^2 \quad c = w^2$.

$m_A = -vw \quad m_B = -wu \quad m_C = -uv$

$j = -(uv + vw + wu)$

EGMO 2017-6

$\overset{\triangle}{ABC}$ triangolo, G_1, G_2, G_3 simm. G rispetto BC, CA, AB. O_1, O_2, O_3 simm. O rispetto BC, CA, AB.

TESI.

(ABC) (G_2G_3A), (G_3G_1B), (G_1G_2C)

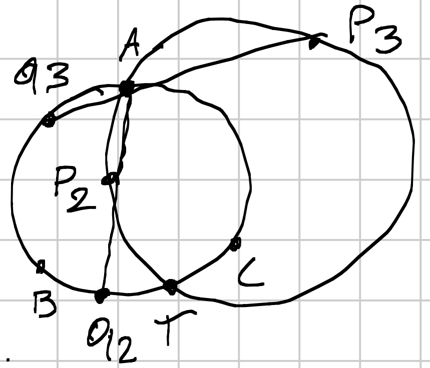
(O_2O_3A), $(O_3O_1B$ (O_1O_2C)

concorrono

DIM.

$p = \lambda(a+b+c)$

calcoliamo $(P_2P_3A) \cap (ABC)$



$p_2 = a + c - ac\bar{p}$

$p_3 = a + b - ab\bar{p}$

q_2 sec. int. (ABC) e AP_2

q_3 // // AP_3

$$a + q_2 = p_2 + aq_2\bar{p}_2$$

$$q_2 = \frac{p_2 - a}{1 - a\bar{p}_2} = \frac{c(1-a\bar{p})}{1 - a\left(\frac{1}{a} + \frac{1}{c} - \frac{p}{ac}\right)} = \frac{c^2(a\bar{p}-1)}{a - p}$$

$$q_3 = \frac{b^2(a\bar{p}-1)}{a-p}$$

$$t = \frac{p_2q_3 - p_3q_2}{p_2 - p_3 + q_3 - q_2}$$

$$N = (a + c - ac\bar{p})\frac{b^2(a\bar{p}-1)}{a-p} - (a + b - ab\bar{p})\frac{c^2(a\bar{p}-1)}{a-p}$$

$$\stackrel{!}{=} \frac{a\bar{p}-1}{a-p}\Big((c-b)(ab+bc+ca-abc\bar{p})\Big)$$

$$D = (c-b)(a\bar{p}-1) + \frac{a\bar{p}-1}{a-p}(c-b)(c+b)$$

$$t = \frac{ab+bc+ca-abc\bar{p}}{a+b+c-p} = \frac{ab+bc+ca}{a+b+c}$$

SIMMETRICA
NON DIPENDE DA λ $\square$

IMO SL 2018 G7

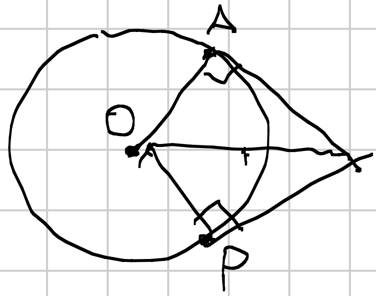
$\hat{ABC}$ triangolo, $P \in (ABC)$, O_A, O_B, O_C circ. $\hat{AOP}$, $\hat{BOP}$ $\hat{COP}$. ℓ_A, ℓ_B, ℓ_C perp BC, CA, AB per O_A, O_B, O_C.

TESI circoscritta al triangolo formato da ℓ_A, ℓ_B, ℓ_C tange OP

DIM.

$p = 1$

$x = \ell_B \cap \ell_C \quad y = \ell_C \cap \ell_A \quad z = \ell_A \cap \ell_B$



$$o_A = \frac{a}{a+1}$$

$z o_A \perp BC \quad z o_B \perp CA$

$$\frac{z-o_A}{\bar{z}-\bar{o}_A} = -\frac{b-c}{\bar{b}-\bar{c}} = bc$$

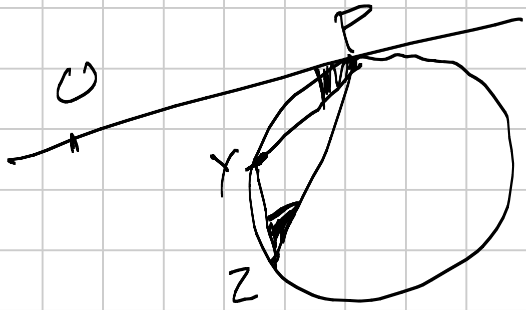
$$z - \frac{a}{a+1} = bc\left(\bar{z} - \frac{1}{a+1}\right)$$

$$z - \frac{b}{b+1} = ca\left(\bar{z} - \frac{1}{b+1}\right)$$

$$\left(\frac{1}{b} - \frac{1}{a}\right) z = c\left(\frac{1}{b+1} - \frac{1}{a+1}\right) + \frac{a}{b(a+1)} - \frac{b}{a(b+1)}$$

$$= \frac{a-b}{ab} \cdot \frac{ab+a+b+abc}{(a+1)(b+1)}$$

$$Z-1=\frac{ab+a+b+abc}{(a+1)(b+1)}-1=\frac{abc-1}{(a+1)(b+1)}$$



$$Y-1=\frac{abc-1}{(a+1)(c+1)}$$

$$e^{2i\angle YPO}=\frac{Y-1}{\bar{Y}-1}:1=\frac{\dfrac{abc-1}{(a+1)(c+1)}}{\dfrac{\frac{1}{abc}-1}{\left(\frac{1}{a}+1\right)\left(\frac{1}{c}+1\right)}}=-b$$

$$e^{2i\angle YZP}=\frac{Y-Z}{\bar{Y}-\bar{Z}}:\overset{-c}{\frac{1-Z}{1-\bar{Z}}}=\frac{\dfrac{(abc-1)(b-c)}{(a+1)(b+1)(c+1)}}{\dfrac{\left(\frac{1}{abc}-1\right)\left(\frac{1}{b}-\frac{1}{c}\right)}{\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right)}}=bc:(-c)=-b$$

OP tange (YZP) e cicliche

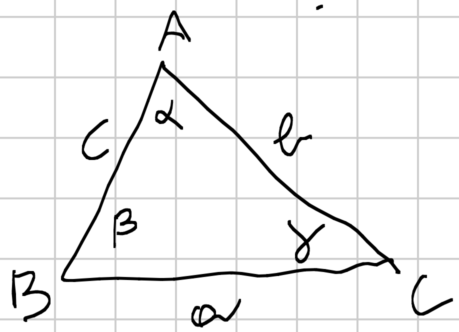
$\square$

IMO 2000-6

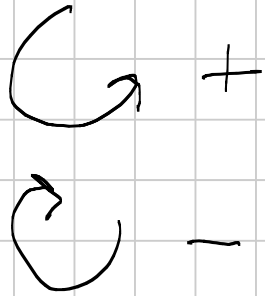
IMO SL 2016 G4

IMO SL 2016 G5

COORDINATE BARICENTRICHE



[XYZ] AREA ORIENTATA DI $\overset{\triangle}{XYZ}$



LEMMA

origine qualsiasi, per ogni punto P

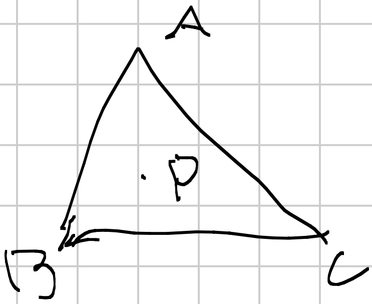
$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \qquad x+y+z=1$

x, y, z sono unici

$\vec{P} - \vec{O} = x(\vec{A} - \vec{O}) + y(\vec{B} - \vec{O}) + z(\vec{C} - \vec{O})$

non dipende dall'origine

$P = (x:y:z)$ terna omogenea $(x:y:z) = (kx:ky:kz) \quad \forall k \neq 0 \in \mathbb{R}$



$P = ([PBC] : [PCA] : [PAB])$

$A = (1:0:0) \quad B = (0:1:0) \quad C = (0:0:1)$

$G = (1:1:1)$

$I = (a:b:c) \quad I_A = (-a:b:c)$

$H = \left(\frac{1}{-a^2+b^2+c^2} : \frac{1}{a^2-b^2+c^2} : \frac{1}{a^2+b^2-c^2} \right)$

$S_A = \frac{-a^2+b^2+c^2}{2}$

$S_B = a^2-b^2+c^2$

$S_C = \frac{a^2+b^2-c^2}{2}$

$O = (a^2(-a^2+b^2+c^2) : cyc : cyc)$

$P = (x:y:z) \quad P' = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z} \right)$ con. isogonale

AREA $\quad P = (p_1 : p_2 : p_3) \quad p_1+p_2+p_3 = 1$ lo stesso per Q e R

$[PQR] = [ABC] \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$

RETTA PER $P = (p_1 : p_2 : p_3) \quad Q = (q_1 : q_2 : q_3)$

$$\begin{vmatrix} x & y & z \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = 0$$

DETERMINANTE

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} = AEI + BFG + CDH - GEC - AHF - IBD$$

$$ux + vy + wz = 0$$

CEVIANA ~~per~~ A e $P = (p_1 : p_2 : p_3)$

$$\frac{y}{z} = \frac{p_2}{p_3} \qquad \begin{vmatrix} 1 & 0 & 0 \\ p_1 & p_2 & p_3 \\ x & y & z \end{vmatrix} = 0 \longrightarrow p_2 z - p_3 y = 0$$

$(k : p_2 ; p_3)$

CONCORRENZA DI RETTE $\quad u_i x + v_i y + w_i z = 0$

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$$

RETTA ALL'INFINITO

$x + y + z = 0$

Punto all'infinito di BC $\quad (0 : 1 : -1)$

DISTANZA TRA DUE PUNTI

$P = (p_1 : p_2 : p_3) \quad Q = (q_1 : q_2 : q_3) \qquad \sum p_i = \sum q_i = 1$

$x = p_1 - q_1 \qquad y = p_2 - q_2 \qquad z = p_3 - q_3$

$\vec{PQ} = \vec{P} - \vec{Q} = x\vec{A} + y\vec{B} + z\vec{C} \qquad \vec{A}\cdot\vec{A} = R^2$

origine in O, R raggio (ABC) $\qquad \vec{B}\cdot\vec{C} = R^2 - \frac{a^2}{2}$

$$PQ^2 = PQ \cdot PQ = R^2 \sum x^2 + \sum 2yz\left(R^2 - \frac{a^2}{2}\right)$$

$$= R^2 \sum x^2 + 2R^2 \sum yz - \sum a^2 yz$$

$$\overset{!}{=} R^2\underbrace{(x+y+z)^2}_{=0} - \underbrace{\sum a^2 yz}$$

PERPENDICOLARITÀ $\quad P_1P_2 \perp Q_1Q_2$

$\overrightarrow{P_1P_2} = x_1\vec{A} + y_1\vec{B} + z_1\vec{C}$ $\quad$ origine in O

$\overrightarrow{Q_1Q_2} = x_2\vec{A} + y_2\vec{B} + z_2\vec{C}$?

$\left.\begin{array}{l} x_1+y_1+z_1=0 \\ x_2+y_2+z_2=0 \end{array}\right]$ ne basta uno

$$0 = \overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}$$

$$\overset{!}{=} \sum x_1x_2R^2 + \sum(z_1y_2+y_1z_2)\left(R^2-\frac{a^2}{2}\right)$$

$$\overset{!}{=} R^2\left(\sum x_1x_2 + \sum(z_1y_2+y_1z_2)\right) - \frac{1}{2}\sum a^2(z_1y_2+y_1z_2)$$

$$\overset{!}{=} R^2\underbrace{(x_1+y_1+z_1)(x_2+y_2+z_2)}_{=0} - \frac{1}{2}\sum a^2(z_1y_2+y_1z_2)$$

CIRCONFERENZE

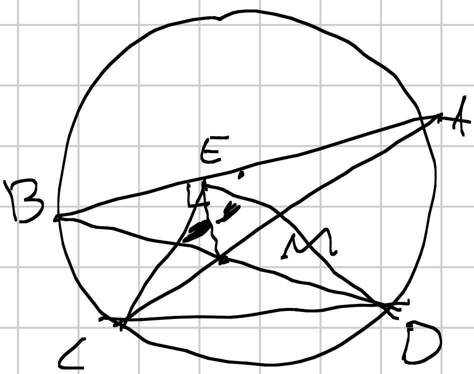
$$\sum a^2yz = (x+y+z)(ux+vy+wz)$$

(ABC) $\quad a^2yz + b^2zx + c^2xy = 0$

BMO 2018-1

$(ABCD)$ $\quad AB \nparallel CD$ $\quad M = AC \cap BD$

E proiez. di M su AB. Se EM biseca $\angle CED$ allora AB è diametro



$E=(1:0:0)$, $C=(0:1:0)$, $D=(0:0:1)$

AB bis. esterna

$A=(k:-b:c)$ $B=(h:-b:c)$

AC $\frac{x}{z}=\frac{k}{c}$

BD $\frac{x}{y}=-\frac{h}{b}$

$M=(\cancel{x}:-\frac{b}{h}\cancel{x}:\frac{c}{k}\cancel{x})$

$=(kh:-bk:ch)$

$\frac{b}{c}=-\frac{bk}{ch} \longrightarrow h=-k$ $B=(k:b:-c)$

(ABCD) $\sum a^2yz=(x+y+z)(ux+vy+wz)$

$v=w=0$ per il passaggio per C e D

$-a^2bc+b^2ck-c^2bk=(k-b+c)ku$

$-a^2bc-b^2ck+c^2bk=(k+b-c)ku$

$\frac{bc}{k}(-a^2+(b-c)k)=(k-b+c)u$

$\frac{bc}{k}(-a^2-(b-c)k)=(k+b-c)u$

$(-a^2+(b-c)k)\cdot(k+b-c)=(-a^2-(b-c)k)(k-b+c)$

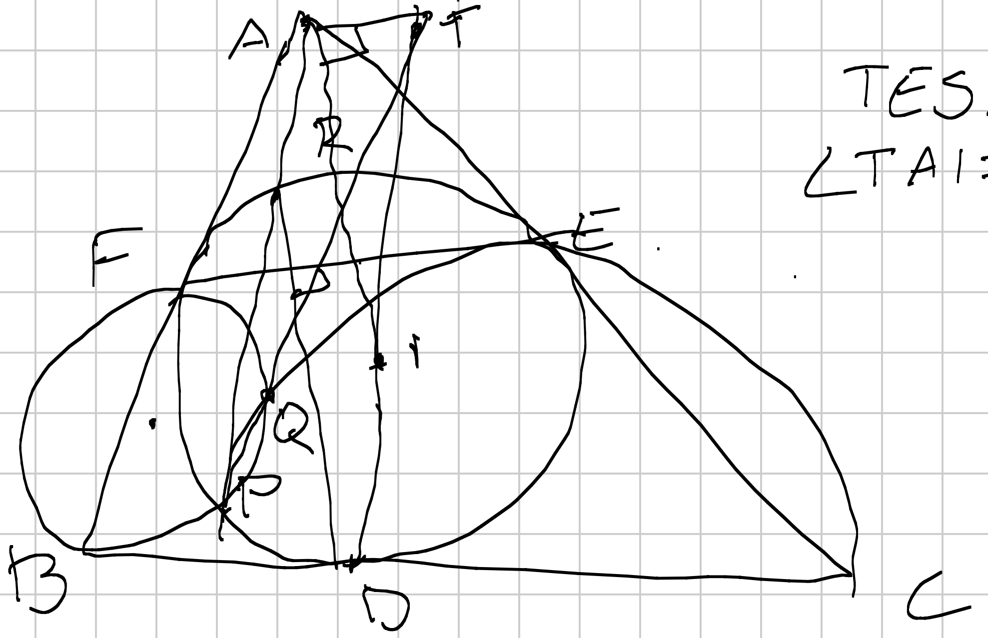
$(b-c)(k-a)(k+a)=0$

$b\neq c$ per ipotesi

WLOG $k=a$ $A=(a:-b:c)$ $B=(a:b:-c)$

SEGUE LA TESI

$\square$



TESI
$\angle TAI = 90^\circ$

COMPLESSI

$$a = \frac{2yz}{y+z} \qquad r = -\frac{yz}{x}$$

$$p + r = a + pr\bar{a} \longrightarrow p = \frac{a-r}{1-r\bar{a}} = \frac{yz(2x+y+z)}{2yz + x(y+z)}$$

$T \in ID$ e $\angle TAI = 90^\circ$

$$\frac{t}{x} = \frac{\bar{t}}{\bar{x}} \qquad \bar{t} = \frac{t}{x^2} \qquad \frac{t-a}{-a} = -\left(\quad\right)$$

$$t = \frac{x^2}{x^2+yz} \cdot \frac{4yz}{y+z}$$

o_B centro $\widehat{BFP}$

$$o_B = \begin{vmatrix} p & 1 & 1 \\ z & 1 & 1 \\ b & b\bar{b} & 1 \end{vmatrix} : \begin{vmatrix} p & \bar{p} & 1 \\ z & \bar{z} & 1 \\ b & \bar{b} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} p & 1 & 1 \\ z & 1 & 1 \\ \frac{2xz}{x+z} & \frac{4xz}{(x+z)^2} & 1 \end{vmatrix} : \begin{vmatrix} \quad \end{vmatrix}$$

$$= \frac{1}{(x+z)^2} \begin{vmatrix} p & 1 & 1 \\ z & 1 & 1 \\ 2xz(x+z) & 4xz & (x+z)^2 \end{vmatrix} : \begin{vmatrix} \quad \end{vmatrix}$$

$$= \frac{1}{x+z} \begin{vmatrix} p & 0 & 1 \\ z & 0 & 1 \\ 2xz(x+z) & -(x-z)^2 & (x+z)^2 \end{vmatrix} : \begin{vmatrix} p & 1/p & 1 \\ z & 1/z & 1 \\ 2xz & 2 & x+z \end{vmatrix}$$

$$= \frac{x-y}{x+z} \frac{yz(2x+y+z)}{(y-z)(x+y)}$$

$$O_B - O_C = \frac{yz(2x+y+z)}{(x+y)(x+z)} \left(\frac{x-y}{y-z} - \frac{x-z}{z-y} \right)$$

$$= \frac{yz(2x+y+z)(2x-y-z)}{(x+y)(x+z)}$$

$$P - t = -yz \frac{(2x-y-z)(x^2y+yz^2+4xyz+y^2z+yz^2)}{(y+z)(x^2+yz)(2yz+xy+xz)}$$

$$\frac{O_B - O_C}{P-t} = \frac{(2x+y+z)(x^2+yz)(2yz+xy+xz)(\cancel{y+z})}{(x+y)(x+z)(x^2x+yz^2+4xyz+\dots)}$$

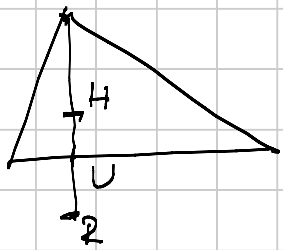
BARICENTRICHE

$D = (1:0:0)$ ...

$I = (a^2S_A : b^2S_B : c^2S_C)$

$A = (-a^2 : b^2 : c^2)$

$H = (S_BS_C : S_CS_A : S_AS_B)$



$$\vec{U} = \frac{\vec{H} + \vec{R}}{2}$$

$\vec{R} = 2\vec{U} - \vec{H}$ $\qquad U = (0 : S_C : S_B)$

$R = (-a^2 S_B S_C : S_C(b^2 S_B + c^2 S_C) : —)$

$$AR \quad \begin{vmatrix} -a^2 & b^2 & c^2 \\ -a^2 S_B S_C & S_C(b^2 S_B + c^2 S_C) & \sim \\ x & y & z \end{vmatrix} = 0$$

$$a^2 yz + b^2 xz + c^2 xy = 0$$

$$z^2 S_C^2 = y^2 S_B^2$$

$$z S_C = y S_B \longrightarrow R$$

$$z S_C = -y S_B$$

$$P = (a^2 S_B S_C : S_C(c^2 S_C - b^2 S_B) : \text{—})$$

$$\sum a^2 yz = (ux + vy + wz)(x+y+z) \qquad (PBF)$$

$F \to w = 0$

$P \to a^2 S_B u = -S_B(c^2 S_C - b^2 S_B) v$

$$u = t S_B (b^2 S_B - c^2 S_C)$$

$$v = t a^2 S_B$$

$$a^2 b^2 c^2 - a^2 b^2 c^2 - a^2 b^2 c^2 = \overset{2S_A}{(-a^2 + b^2 + c^2)}(-ua^2 + vb^2)$$

$$t = \frac{b^2}{2 S_B S_C}$$

FINIRE PER CASA

IMO SL 2019 G4
IMO SL 2013 G4
IMO SL 2005 G5
SHARYGIN 2020 CR
USA TSTST 2016-6