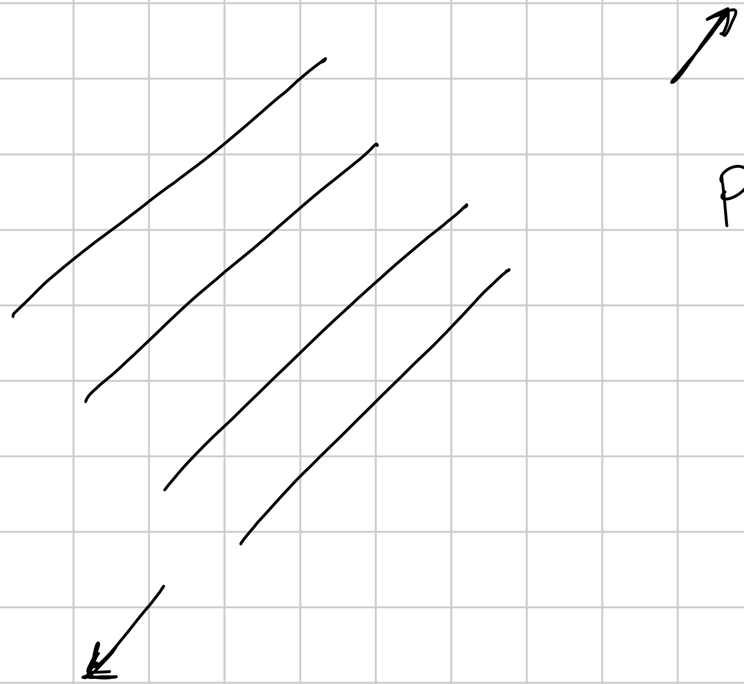
G2 MEDIUM

Titolo nota

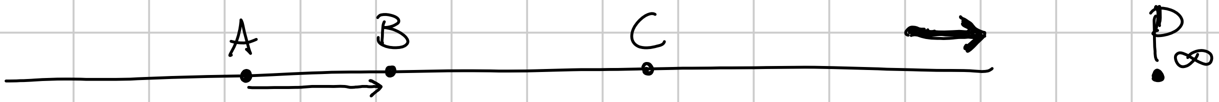


Piano proiettivo:

Punti "finiti" $\cup$ punti all'∞

Rette "finite" $\cup$ Retta all'∞

Segmenti orientati



$\overline{AB}$ è positivo — $\overline{BA}$ è negativo

$\overline{AC}$ — $\overline{CA}$

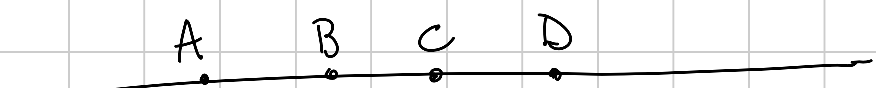
$\overline{BC}$ — $\overline{CB}$

$\dfrac{\overline{AB}}{\overline{AC}}$ pos , $\dfrac{\overline{BA}}{\overline{BC}}$ neg ..

$$\frac{\overline{AP_\infty}}{\overline{BP_\infty}} = 1$$

$$\frac{\overline{AC}}{\overline{BP_\infty}} = 0 \qquad \frac{\overline{BP_\infty}}{\overline{AC}} = \infty$$

Birapporto



$$(A,B;C,D) := \frac{\overline{AC}}{\overline{CB}} \Big/ \frac{\overline{AD}}{\overline{DB}} = \lambda$$

$(B, A; C, D) = \frac{1}{\lambda}$ $\qquad$ $(A, B; D, C) = \frac{1}{\lambda}$

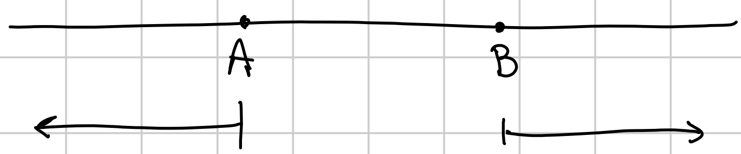
Fissiamo $A, B, C \in r$

$$P \in r \xleftrightarrow{\text{bigez.}} \lambda \in \mathbb{R} \cup \{\infty\}$$
$$\longrightarrow \quad \lambda = (A, B; C, P)$$

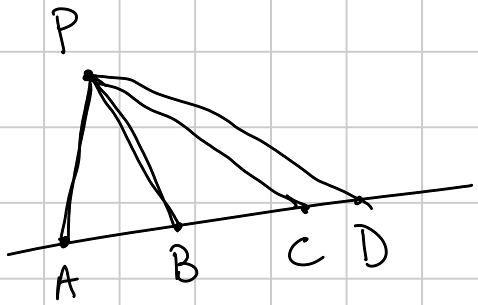
$P_1, P_2 \in r \quad (A,B;C,P_1) = (A,B;C,P_2) \rightarrow \frac{AC}{CB} \Big/ \frac{AP_1}{P_1B} = \frac{AC}{CB} \Big/ \frac{AP_2}{P_2B}$

$$\rightarrow \frac{AP_1}{P_1B} = \frac{AP_2}{P_2B}$$

$\frac{AP}{PB} = ?$ $\qquad$ $P = A \rightarrow 0$, $\quad P = B \rightarrow \infty$



$0 > \frac{AP}{PB} > -1$ $\qquad$ $-1 > \frac{AP}{PB}$



$(A,B;C,D)$ si scrive in termini degli angoli in P

$$\frac{AC}{CB} \Big/ \frac{AD}{DB}$$

$\Delta PAC \rightarrow \frac{AC}{\sin\angle APC} = \frac{AP}{\sin\angle PCA}$

$\Delta PBC \rightarrow \frac{BC}{\sin\angle BPC} = \frac{BP}{\sin\angle PCA}$

$\Delta PAD \rightarrow \frac{AD}{\sin\angle APD} = \frac{AP}{\sin\angle PDA}$

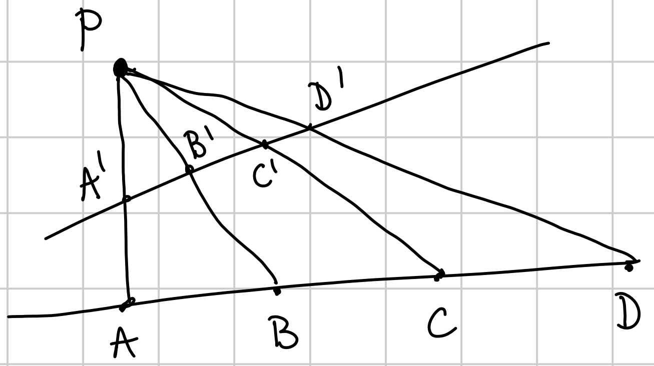
$\Delta PBD \rightarrow \frac{BD}{\sin\angle BPD} = \frac{BP}{\sin\angle PDA}$

$$\frac{AC}{BC} \cdot \frac{\sin \angle BPC}{\sin \angle APC} \Big/ \left(\frac{AD}{BD} \cdot \frac{\sin \angle BPD}{\sin \angle APD} \right) =$$

$$= \frac{AP}{BP} \Big/ \frac{AP}{BP} = 1$$

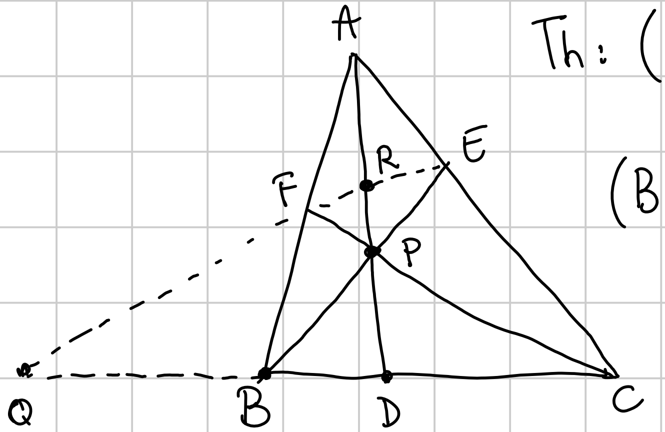
$$\frac{AC}{BC} \Big/ \frac{AD}{BD} = \frac{\sin \angle APC}{\sin \angle BPC} \Big/ \frac{\sin \angle APD}{\sin \angle BPD}$$

$$\frac{AC}{CB} \Big/ \frac{AD}{DB} = \frac{\sin \angle APC}{\sin \angle CPB} \Big/ \frac{\sin \angle APD}{\sin \angle DPB}$$



$$(A,B;C,D) \overset{P}{=} (A',B';C',D')$$

$$= (PA, PB; PC, PD) =$$

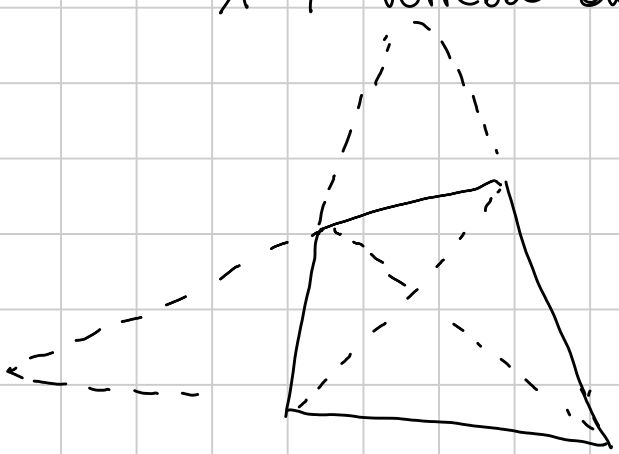


Th: $(B,C;D,Q) = -1$

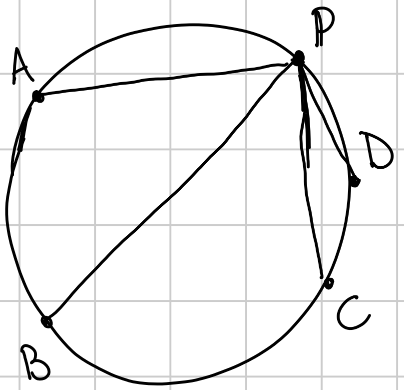
$$(B,C;D,Q) \overset{P}{=} (E,F;R,Q) \overset{A}{=} (C,B;D,Q)$$

$$\lambda = \frac{1}{\lambda} \rightarrow \lambda = \pm 1$$

$\lambda = 1$ vorrebbe dire $B = C$, oppure $D = Q$. $\longrightarrow \lambda = -1$

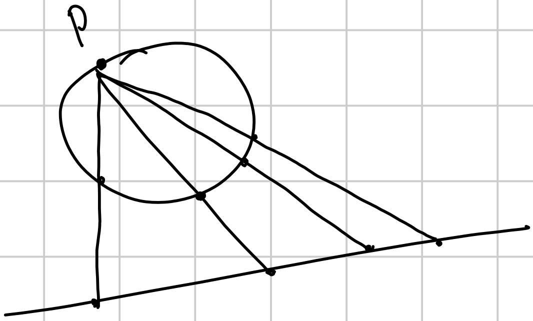


Stessa configurazione con un quadrilatero.



$(A, B; C, D) := (PA, PB; PC, PD)$

$P = A$? $(A, B; C, D) = (AA, AB; AC, AD)$



Teorema (Desargues)

Due triangoli sono prospettici rispetto a un punto P

$A_1B_1C_1$, $A_2B_2C_2$

se A_1A_2, B_1B_2, C_1C_2 concorrono in P.

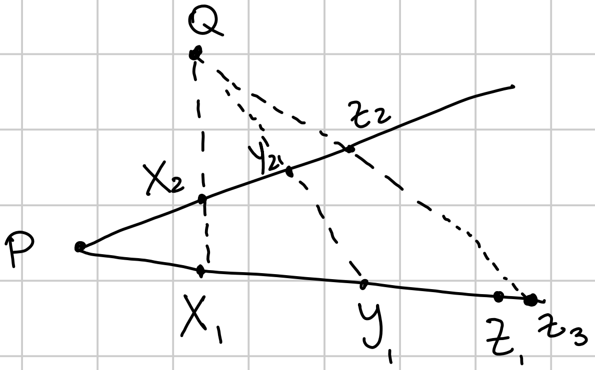
- - - - prospettici rispetto a una retta r

se $A_1B_1 \cap A_2B_2$, $A_1C_1 \cap A_2C_2$, $B_1C_1 \cap B_2C_2$ $\in r$.

Due triangoli sono prospettici rispetto a un punto

$\iff$ sono prospettici rispetto a una retta.

Lemma preliminare:



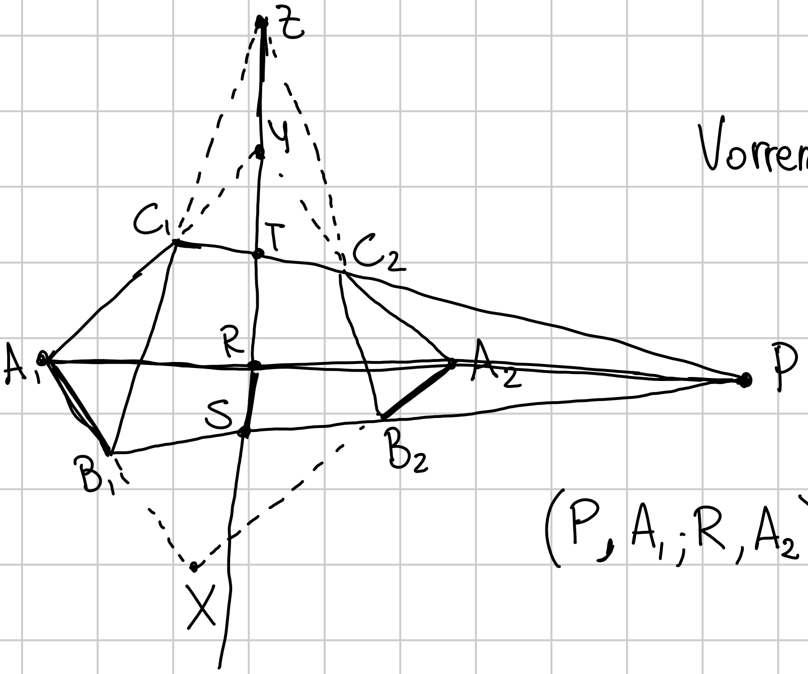
$X_1X_2,\ Y_1Y_2,\ Z_1Z_2$ concorrono

$\iff (P,X_1;Y_1,Z_1) = (P,X_2;Y_2,Z_2)$

Dim: $Q := X_1X_2 \cap Y_1Y_2$

$\to (P,X_1;Y_1,Z_1) = (P,X_1,Y_1,Z_3)$

$\to Z_1 = Z_3$ $\square$

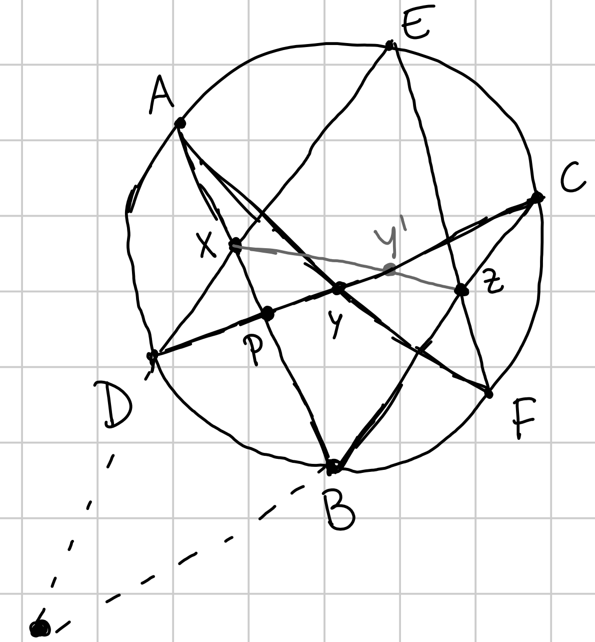


Vorremmo $(P,A_1;R,A_2) = (P,B_1;S,B_2)$

$(P,A_1;R,A_2) \stackrel{Y}{=} (P,C_1;T,C_2)$

$\stackrel{Z}{=} (P,B_1,S,B_2)$

Teorema (Pascal)



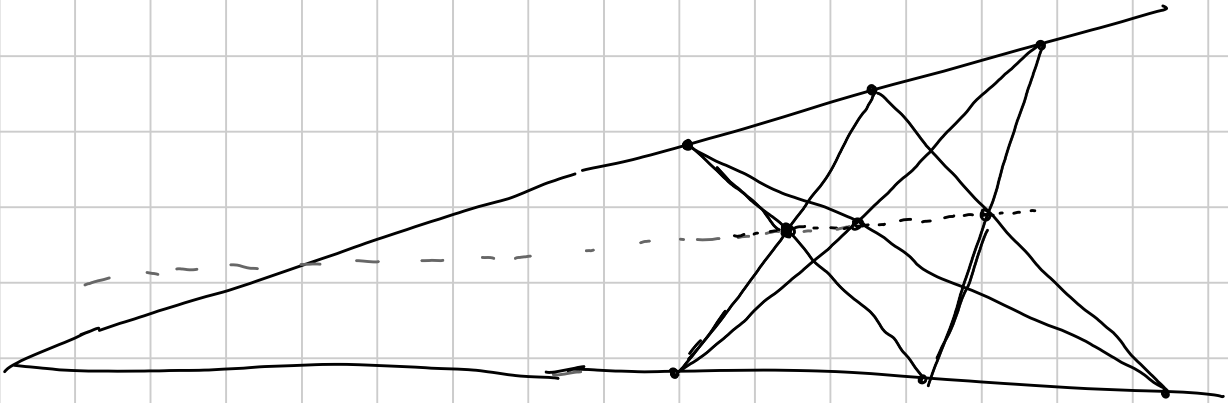
A, B, C, D, E, F conciclici.
Allora $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ sono allineati.

$(D,P;Y,C) \stackrel{?}{=} (D,P;Y',C)$

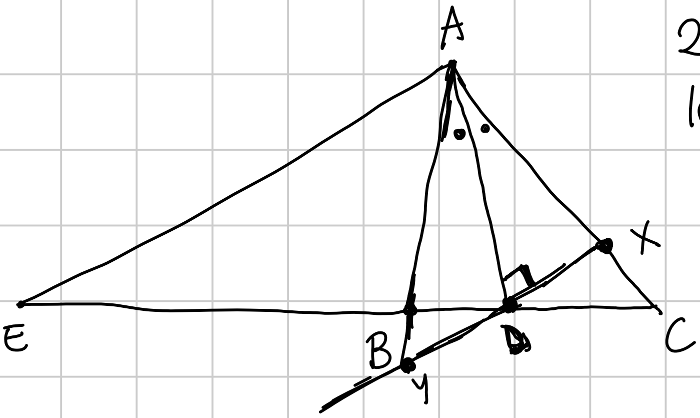
$$(D,P;Y',C) \overset{X}{=} (DE\cap BC\,,B\,;\,Z,C) \overset{E}{=} (D,B;F,C) \overset{A}{=} (D,P;Y,C)$$

$$\to Y = Y' \qquad \square$$

<u>Teorema</u> (Pappo)



Lemma:

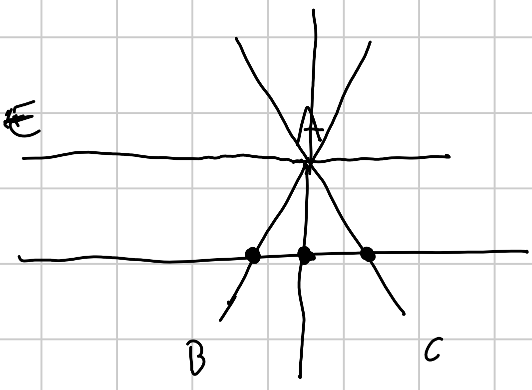


2 condizioni qualsiasi implicano la terza:

<u>(i)</u> $(B,C;D,E) = -1$

<u>(ii)</u> AD bisettrice

<u>(iii)</u> $AD \perp AE$



$\to P_\infty$

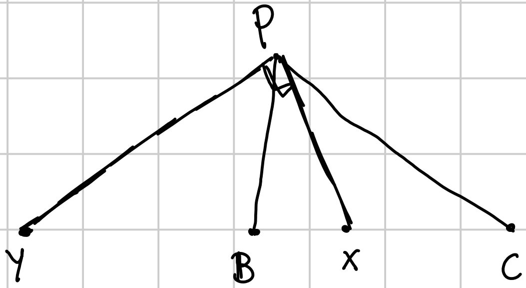
$$\left.\begin{aligned} \frac{BD}{DC} &= \frac{AB}{AC} \\ \frac{BE}{EC} &= -\frac{AB}{AC} \end{aligned}\right\} (B,C;D,E) = -1$$

(In generale, $(X,Y;M,P_\infty) = -1 \iff M$ è pt. medio di XY)

$$\underbrace{\frac{XM}{MY}}_{1} \Big/ \underbrace{\frac{XP_\infty}{P_\infty Y}}_{-1}$$

Circonferenza di Apollonio.

Qual è il luogo dei punti P tale che $\frac{BP}{PC} = k$?

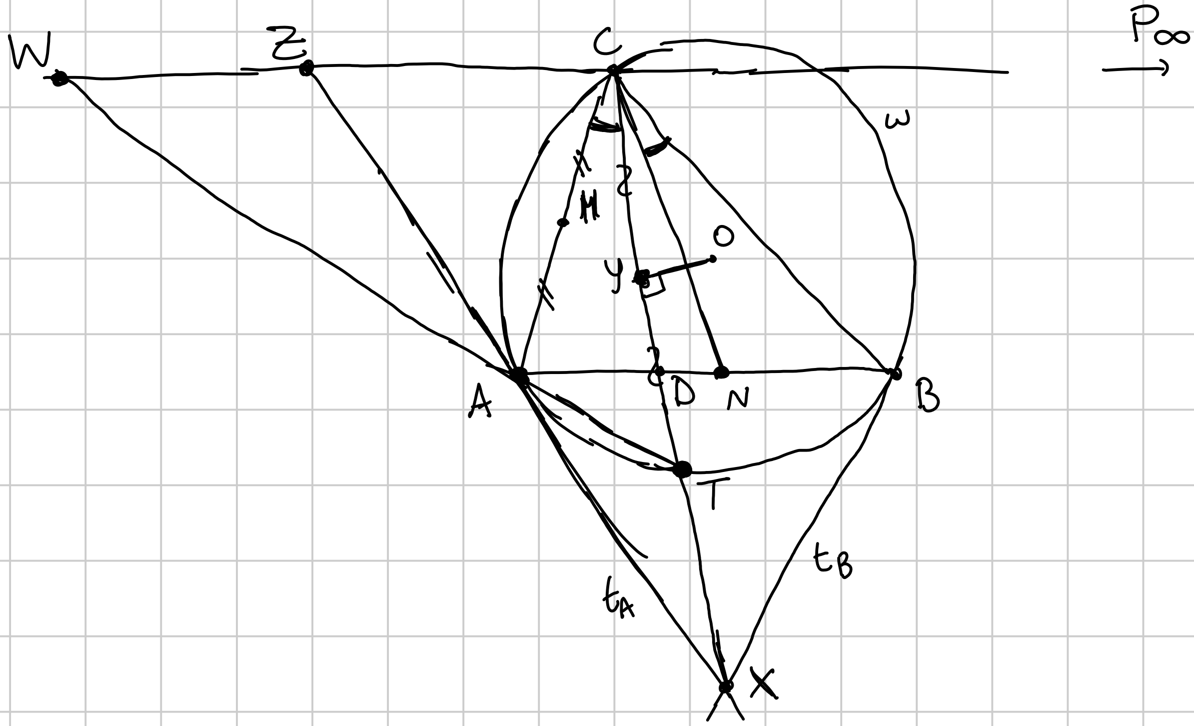


$$(B,C;X,Y) = -1$$

$$\frac{BX}{XC} \Big/ \frac{BY}{YC} = k/{-k} = -1$$

$\longrightarrow P \in (XY)$

BMO 2022.1



Tesi: Z, M, Y allineati

$(A,B;C,T) = -1$

$\overset{A}{=} (X,D;C,T)$

$\overset{B}{=} (D,X;C,T)$

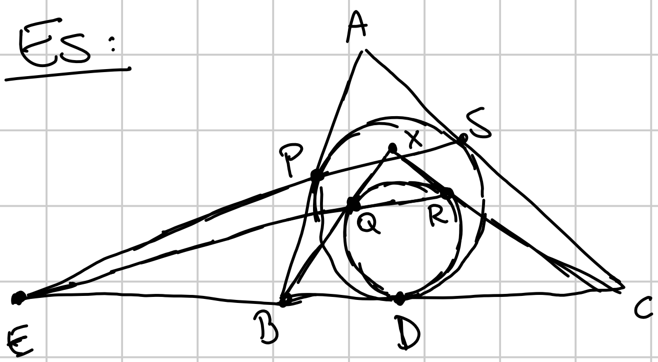
uguali & inversi

$\rightarrow (A,B;C,T) = -1$

Tesi $\Leftrightarrow$ $(W, C; Z, P_\infty) = -1$

Proiettando da A sulla circonferenza. $\square$

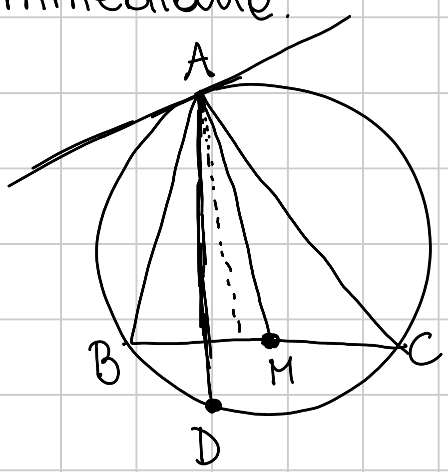
Es:



Th: PQRS

$EP \cdot ES = ED^2 = EQ \cdot ER$

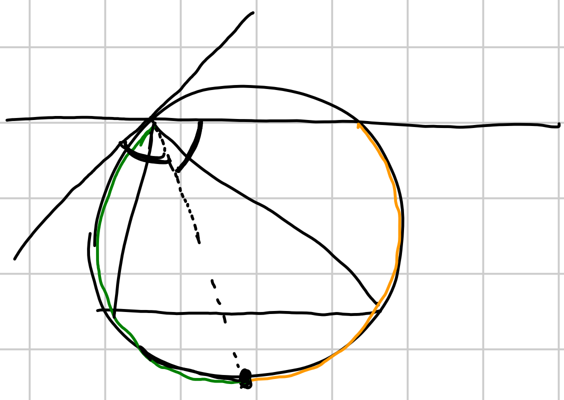
Simmediane.

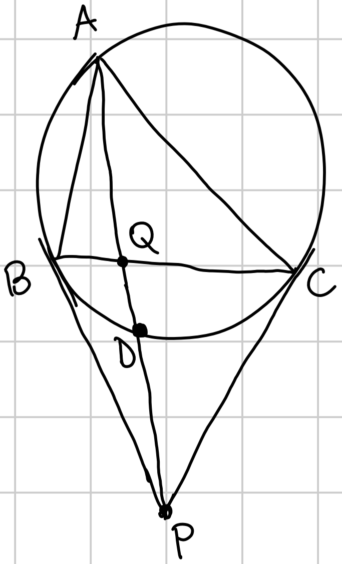


$(A,D;B,C) = -1$

$(AA, AD; AB, AC) = -1$

$(\;??\;, AM; AC, AB) = -1$

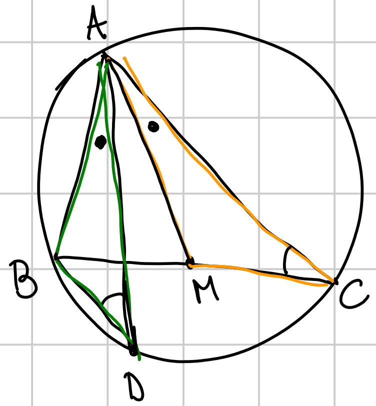




$$(A,D;B,C) \overset{B}{=} (A,D;P,Q) \overset{C}{=} (A,D;Q,P)$$

$$\rightarrow \underline{\underline{-1}}$$

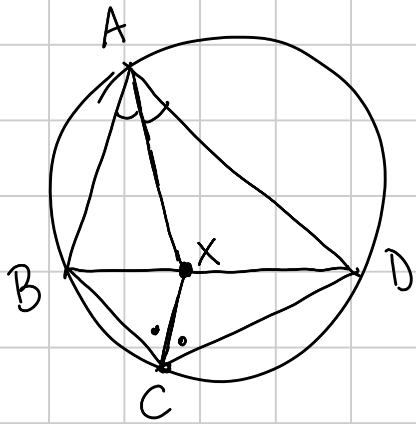
- $\underline{AB \cdot DC = AC \cdot BD}$



$\Delta ABD \sim \Delta AMC$

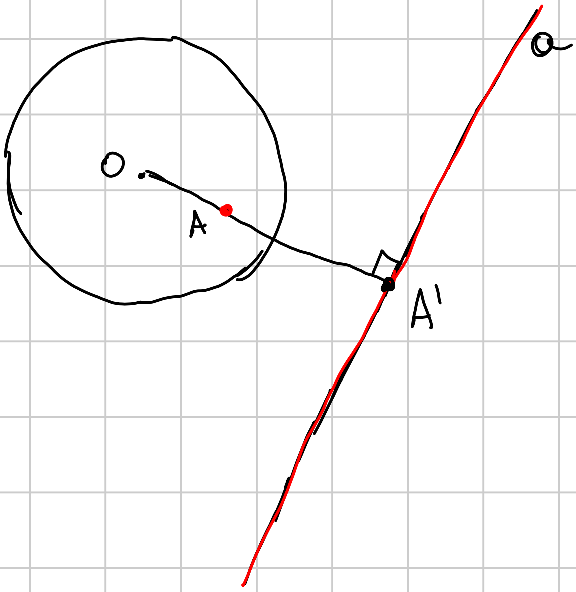
$\Delta ACD \sim \Delta AMB$

$$\left.\begin{aligned} \rightarrow \frac{AB}{BD} &= \frac{AM}{MC} \\ \rightarrow \frac{AC}{CD} &= \frac{AM}{MB} \end{aligned}\right\} \rightarrow \boxed{\frac{AB}{BD} = \frac{AC}{CD}}$$



$$\frac{BX}{XD} = \frac{BC}{CD} = \frac{AB}{AD}$$

Poli e polari

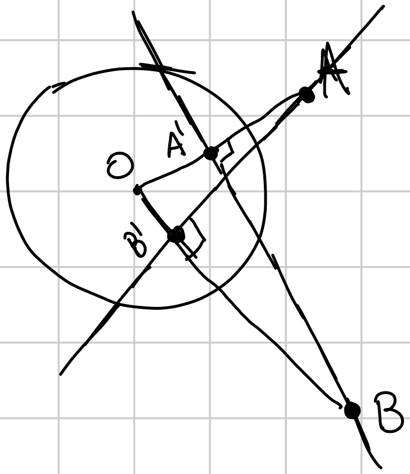


a si dice POLARE di A.

A si dice POLO di a.

Teorema (La Hire)

Dati A, B con polari a, b, $A \in b \Leftrightarrow B \in a$



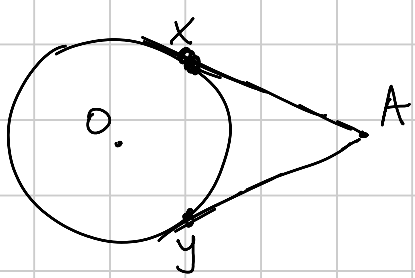
$A \in b \to \angle BB'A = \frac{\pi}{2}$

$OA \cdot OA' = r^2 = OB \cdot OB'$

$\to AA'BB'$ cyc

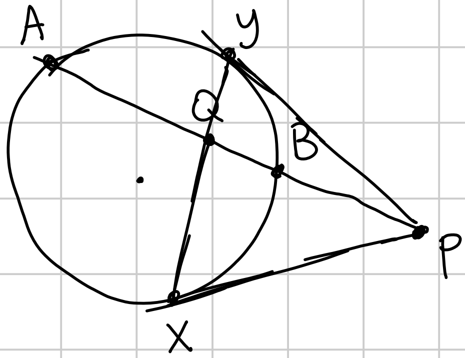
$\to \angle BA'A = \frac{\pi}{2} \to B \in a$

Costruzione polare:

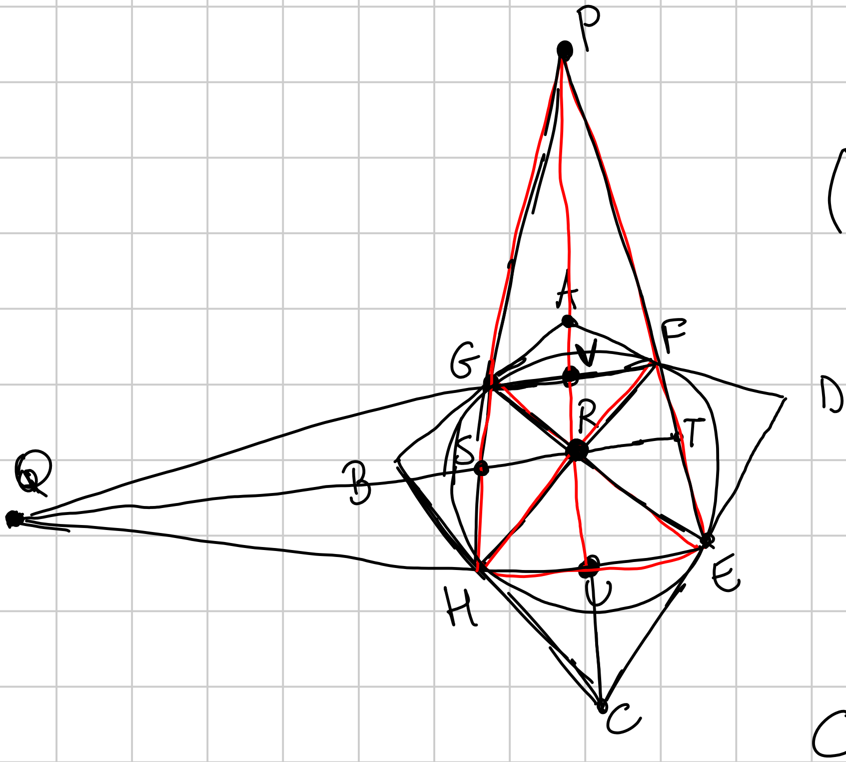


$A \in x$
$A \in y$ $\to$ $X \in a$
$Y \in a$ $\to$ $a = XY$

Lemma della polare



$(A,B;P,Q) = -1 \iff Q \in p$



$(Q,U;H,E) = -1$

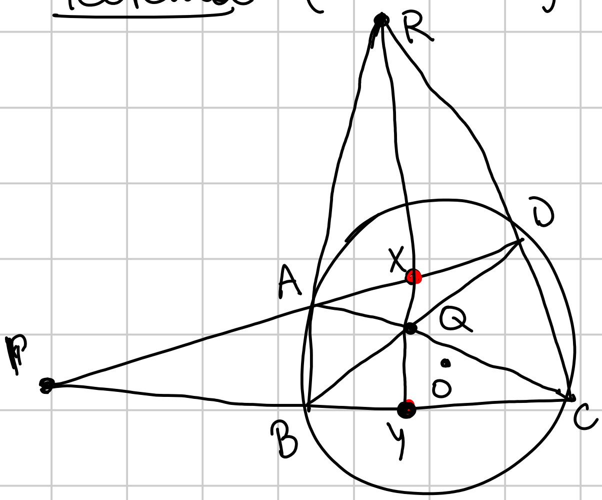
$(Q,R;S,T) = -1$

$R \in q$

$R \in p$

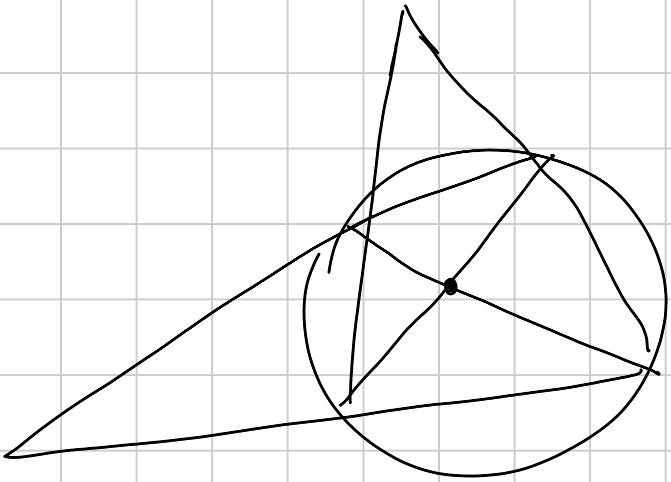
C, U, R, V, A, P allineati.

Teorema (Brocard)



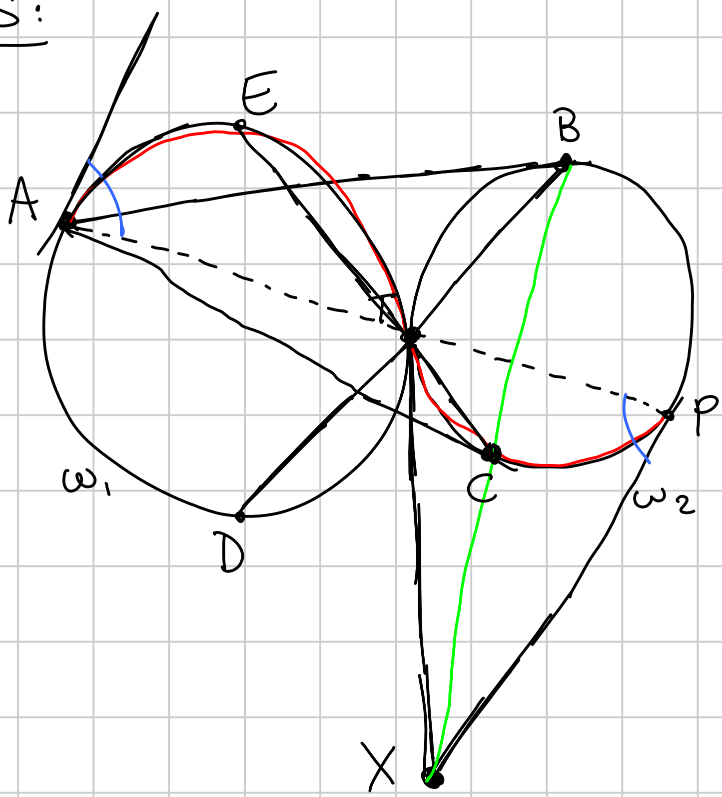
PQR è "autopolare"

$X, Y \in p \rightarrow RQ$ polare di P.



Brocard dà un modo di costruire la polare solamente con nozioni proiettive.

Es:



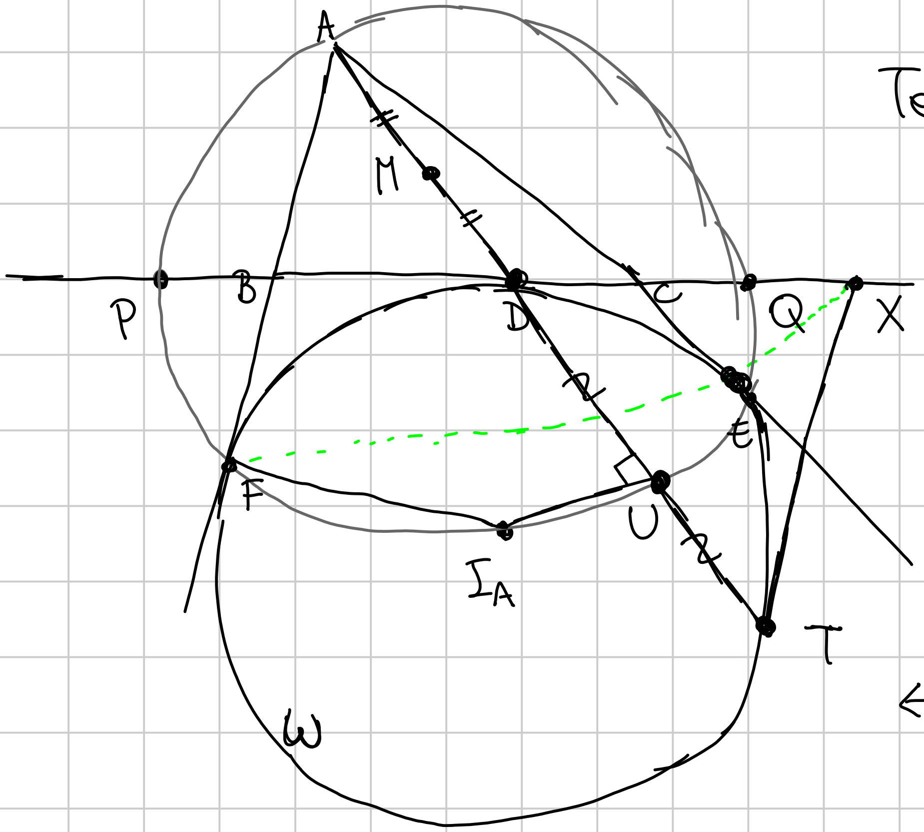
Tesi: $\underline{AA \cap DE}$ sta su una retta al variare di A su ω_1.

$$\underline{(B, C; T, P) = -1}$$

Tesi $\Longleftrightarrow$ $BC \cap PP$ sta su una retta

$\square$

IMO SL G4 2017



Tesi: (PQM) e ω si tangono.

PQMT cyc

$\longleftrightarrow DM \cdot DT = DP \cdot DQ$

$= AD \cdot DU$

$\longleftrightarrow 2\cancel{DM} \cdot DU = \cancel{DM} \cdot DT$

$(D, T; E, F) = -1$

XT tange ω

$XP \cdot XQ = XT^2$

$= XE \cdot XF \underset{\omega}{=}$

□