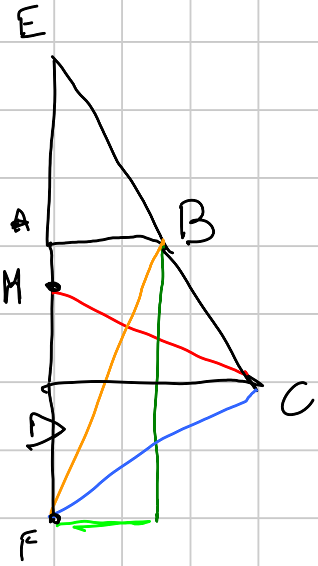
SENIOR 2023 - GEOMETRIA 1

Titolo nota 04/09/2023

ANALITICA
VETTORI
COMPLESSI
TRIGONOMETRIA

$\boxed{y = mx + q}$

TF - 2016 P11



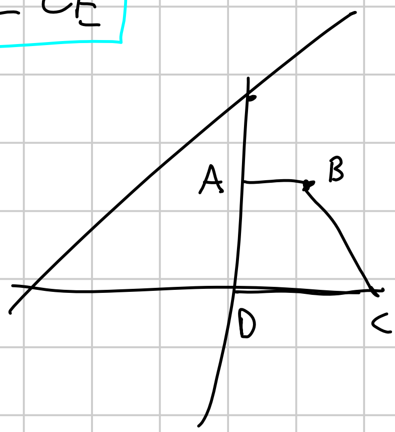
E = AD ∩ BC
F = simm. di A risp D
M = pto medio di EF

$CM \perp BF$

$\Downarrow$

$\boxed{CF \perp CE}$

$A = (0, a)$
$B = (b, a)$
$C = (c, 0)$
$D = (0, 0)$



$x = a$

$y = mx + q$

AD ha eq. $\boxed{x = 0}$

BC ha eq $y = mx + q$ $\quad m = \frac{y_B - y_C}{x_B - x_C} = \frac{a}{b-c}$

ha eq. $\boxed{y = \boxed{\frac{a}{b-c}}(x - c)}$ pend. giusta e passa per $(c, 0)$

$E = AD \cap BC \quad x = 0 \quad y = \frac{a}{b-c} \cdot -c = \frac{ac}{c-b}$

$E = \left(0, \frac{ac}{c-b}\right)$

$F = (0, -a)$

$M = \left(0, \frac{\frac{ac}{c-b} - a}{2}\right) = \frac{ac - ac + ab}{2(c-b)} = \boxed{\frac{ab}{2(c-b)}}$

Se ho due rette $y = mx + q$ $\quad y = m'x + q'$
quando sono ortogonali? $\quad m = -\frac{1}{m'} \quad m \cdot m' = -1$

coeff BF $\frac{y_B - y_F}{x_B - x_F} = \frac{2a}{b} =$

coeff CM $\frac{y_M - y_C}{x_M - x_C} = \frac{\frac{ab}{2(c-b)} - 0}{0 - c}$

$BF \perp CM = \frac{2a}{b} = -1 \cdot \frac{-c}{\frac{ab}{2(c-b)}} = \frac{2c(c-b)}{ab}$

$\cancel{2} a^2 \cancel{b} = \cancel{2} \cancel{b} c (c-b)$ $\boxed{a^2 = c(c-b) = c^2 - bc}$

coeff CF $\frac{y_C - y_F}{x_C - x_F} = \frac{a}{c}$

coeff CE = coeff BC $= \frac{a}{b-c}$

$CF \perp CE$ $\frac{a}{c} = -\frac{b-c}{a}$ $\boxed{a^2 = c(c-b)}$

GEO ANALITICA

BUONO

rette parallel

intersez. perpend.

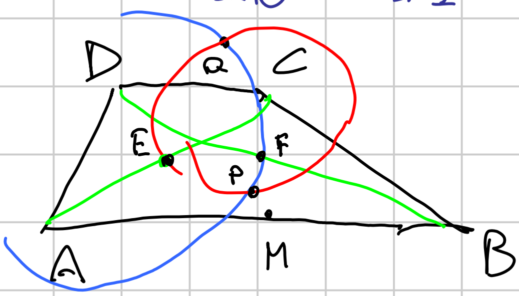
p.ti medi

MALE

TANTE CIRC.

UGUAGLIANZE DI ANGOLI

TF 2010 P11



$AB = 3 \cdot CD$

E = p.to medio di AC

F = p.to medio di DB

ω_1 circ. centro C passante per E

ω_2 circ. centro D passante per F

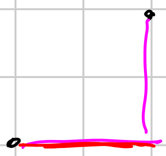
M = p.to medio di AB

Tesi: M, P, Q allineati.

$M=(0,0)$ $\quad C=(2a+1,\ 2h)$ $\quad |CD| = (2a+1)-(2a-1)=2$

$A=(-3,\ 0)$ $\quad D=(2a-1,\ 2h)$

$B=(\ 3,\ 0)$



$$E=\left(\frac{2a+1-3}{2},\ \frac{2h}{2}\right)=(a-1,\ h)$$

$$F=(\ .\ \ -\ -\ -\ -\ \ )=(a+1,h)$$

$$d(C,X)^2=R^2$$

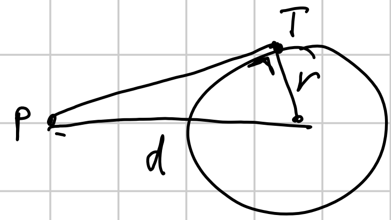
ω_1

$$\left(x-(2a+1)\right)^2+(y-2h)^2-\overbrace{(a+2)^2-h^2}^{R^2}=0$$

ω_2

$$\left(x-(2a-1)\right)^2+(y-2h)^2-(a-2)^2-h^2=0$$

$$\mathrm{pow}_P(\Gamma)=d^2-r^2$$



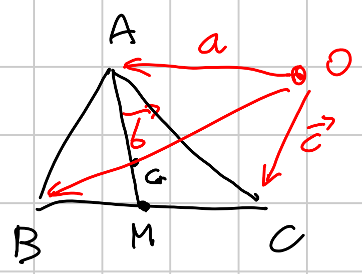
Basta verificare

$$\underbrace{(2a+1)^2+(2h)^2-(a+2)^2-h^2}=\underbrace{(2a-1)^2+(2h)^2-(a-2)^2-h^2}$$

Si verifica

Vettori $\quad$ Scegliamo origine $\vec{O}$ e poi punti = Vettori $O\to$ p.to.

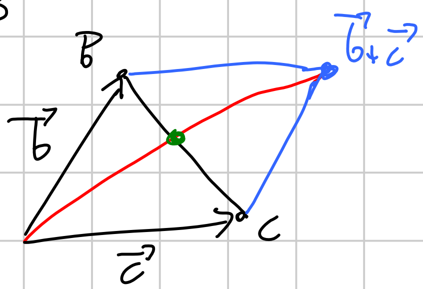
$\vec{P}=\vec{OP}$



$$\vec{G}=\frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

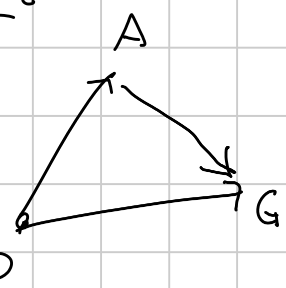
$$\vec{AG}=2\,\vec{GM}$$

$$M=\frac{\vec{b}+\vec{c}}{2}$$

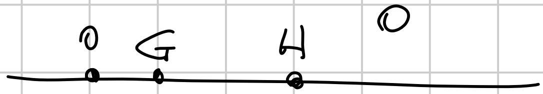


$\vec{AG} = -(\vec{A}-\vec{G}) \qquad \vec{GM} = -(\vec{G}-\vec{M})$

$\vec{A}-\vec{G} = 2\vec{G} - 2\vec{M} = \qquad 3\vec{G} = \vec{A}+2\vec{M} = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$



$\vec{H}$ se origine è circocentro $H = \vec{a}+\vec{b}+\vec{c}$



Dimostriamolo:

$AH \perp BC \qquad \vec{AH} = (\vec{H}-\vec{A}) \qquad BC = (\vec{C}-\vec{B})$

$(\vec{H}-\vec{a})(\vec{c}-\vec{b}) = (\vec{b}+\vec{c})(\vec{c}-\vec{b}) = \vec{c}\cdot\vec{c} - \vec{b}\cdot\vec{b} + \cancel{\vec{b}\cdot c} - \cancel{\vec{b}\cdot c}$

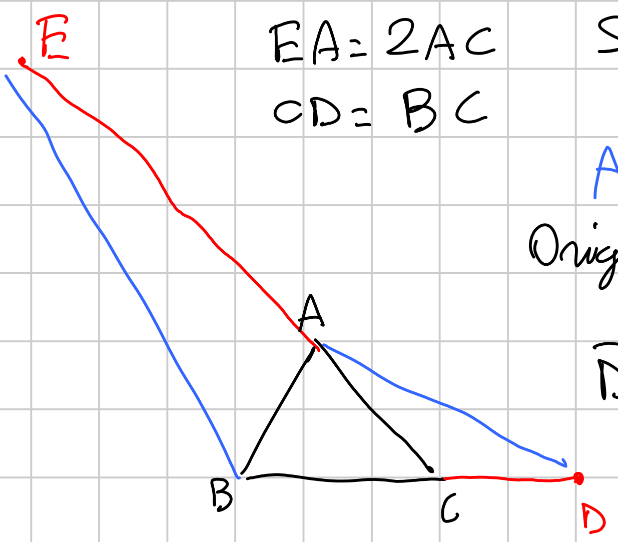
$= \vec{c}^{\,2} - \vec{b}^{\,2}$

$= |\vec{c}|^2 - |\vec{B}|^2 = R^2 - R^2 = 0.$

H sta su altezza di A.

$|OH| = \sqrt{\vec{H}\cdot\vec{H}}$

EGMO 2013-P1



$EA = 2AC$ Supponiamo $EB = AD$

$CD = BC$

Allora $\hat{A}$ è retto.

Origine in C $\quad C = 0 \quad \vec{CA} = \vec{a} \quad \vec{CB} = \vec{b}$

$\vec{D} = -\vec{b} \qquad \vec{E} = 3\vec{a}$

$|AD|^2 = |BE|^2$

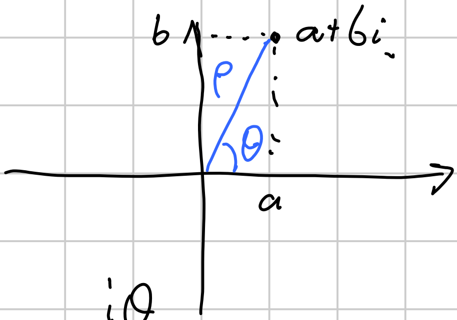
$|AD|^2 = \vec{AD}\cdot\vec{AD} = (\vec{D}-\vec{A})(\vec{D}-\vec{A}) = \vec{D}^2 + \vec{A}^2 - 2\vec{D}\vec{A} = \vec{b}^2 + \vec{a}^2 + 2\vec{b}\vec{a}$

$|BE|^2 = (\vec{E}-\vec{B})(\vec{E}-\vec{B}) = 9\vec{a}^2 + \vec{b}^2 - 6\vec{a}\vec{b} \qquad 8\vec{a}^2 = 8\vec{a}\cdot\vec{b}$

$8\vec{a}\cdot(\vec{a}-\vec{b}) = 0 =$
- $\vec{a} = 0 \quad A = C$
- $\vec{a} = \vec{b} \quad A = B$
- Tesi $AC \perp AB$

Complessi

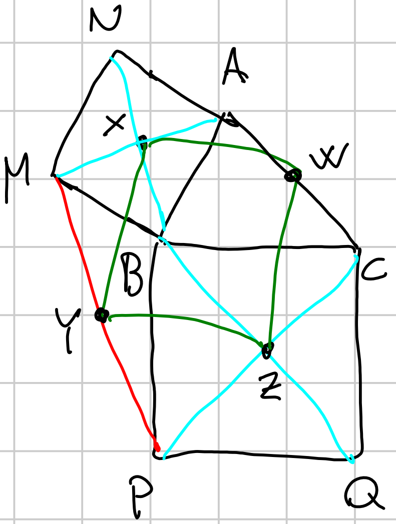
$z = (a+bi) = e^{i\theta}$



$$\rho e^{i\theta} \cdot r e^{i\varphi} = \rho \cdot r\, e^{i(\theta+\varphi)}$$

ruotare di θ = moltiplicare per $e^{i\theta}$.

G2-10



X, Z centri quadrati
Y, W p.ti medi.

Tesi XYZW quadrato.

In complessi

$B=0 \qquad C=1 \qquad A=a \quad (\operatorname{Im} a > 0)$

$P=-i \qquad Q = 1-i$

$Z = \frac{Q}{2} = \frac{1-i}{2}. \qquad W = \frac{a+1}{2}$

(p.to medio BQ)

M = ruotato di A di 90° rispetto al pto B (origine) $e^{i\frac{\pi}{2}} = i$

$M = a \cdot i \qquad X = \frac{M+A}{2} = \frac{ai+a}{2} = \frac{1}{2}a(1+i)$

$Y = \frac{M+P}{2} = \frac{ai-i}{2} = \frac{1}{2}i(a-1)$

$|XW| = |WZ| \quad X\hat{W}Z = 90° \Leftarrow$ WZ = ruotato di XW di 90° antiorario rispetto a W.

CAVEAT Attenti al verso!

Se fosse $W=0$ basta $iX = Z$.

Devo vedere $i(X-W) = (Z-W)$

$$\left.\begin{aligned} i(X-W) &= \tfrac{1}{2} i(\cancel{a}+ai-\cancel{a}-1) = \tfrac{1}{2}i(ai-1) = \tfrac{1}{2}(-a-i) \\ (Z-W) &= \tfrac{1}{2}(\cancel{1}-i-a-\cancel{1}) = \tfrac{1}{2}(-a-i) \end{aligned}\right\} \text{sono uguali.}$$

Poi lo faccio per altri vertici.

complessi
allineamenti
perp. parall.
rotazioni!
angoli - simmetrie

circ. di centro 0 raggio 1

$|z| = 1$. $\quad z\bar{z} = |z|^2 = 1$

Dodecagono regolare ⇒ vertici sono radici di 1.

a, b, c allineati. Come si dice?

0, b, c allineati. $b = \lambda c$ se $\frac{b}{c}$ è reale, $\frac{\frac{b}{c}}{\frac{\bar{b}}{\bar{c}}} = 1$,

a, b, c allineati se $b-a$, $c-a$, 0 allineati

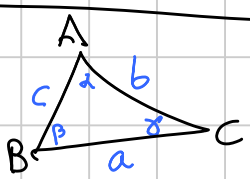
$$\frac{\frac{b-a}{c-a}}{\frac{\bar{b}-\bar{a}}{\bar{c}-\bar{a}}} = 1 \Leftrightarrow a, b, c \text{ allineati.}$$

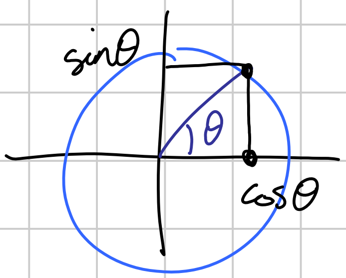
$a \overset{?}{=} ib$

a b sono perpendicolari $\frac{a}{b}$ = immaginario puro.

z è imm. puro ⇔ $\bar{z} = -z$ $\qquad \boxed{\frac{a}{b} = -\frac{\bar{a}}{\bar{b}}}$ $\perp$

Trigonometria



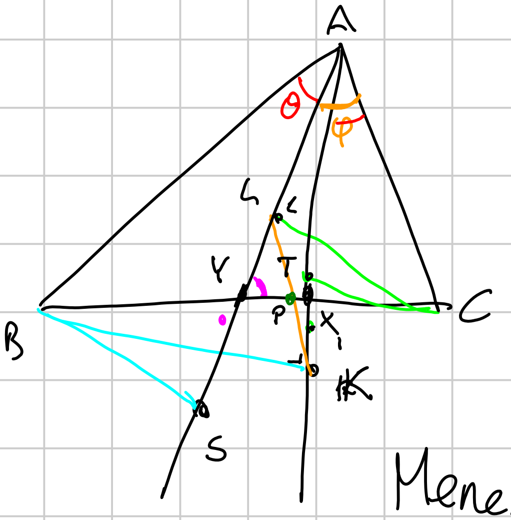


Teo seni $\quad \frac{a}{\sin\alpha} = 2R = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$.

Teo coseno (Carnot) $\quad a^2 = b^2 + c^2 - 2bc\cos\alpha$

Dati i dati di uno dei teo di congruenza, calcolare gli altri dati del triangolo.

BMO 2019 - 3



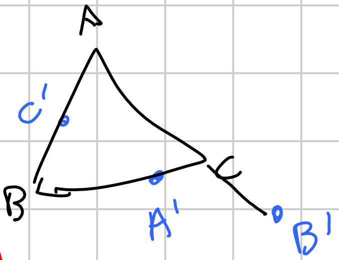
$C\hat{A}X = Y\hat{A}B$

KS TL piedi di perp.

Th: $KL \cap ST$ sta su BC.

$P = KL \cap BC$ dimostro S T P allineati

Menelao:



$$\boxed{A', B', C' \text{ allineati} \iff \frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = -1}$$

Usiamolo sul triangolo AXY $\underbrace{\frac{AL}{LY} \cdot \frac{YP}{PX} \cdot \frac{XK}{KA} = -1}$;

Vogliamo (Menelao) $\underbrace{\frac{AS}{SY} \cdot \frac{YP}{PX} \cdot \frac{XT}{TA} = -1}$.

Voglio. $\frac{AL}{LY} \cdot \frac{XK}{KA} = \frac{AS}{SY} \cdot \frac{XT}{TA}$.

$$\frac{AC \cos\varphi}{CY \cos C\hat{Y}A} \cdot \frac{BX \cos B\hat{X}K}{AB \cos\varphi} = \frac{AB \cos\theta}{BY \cos B\hat{Y}S} \cdot \frac{CX \cos C\hat{X}T}{AC \cos\theta}$$

Si semplifica tutto e $\frac{BX \cdot BY}{CX \cdot CY} = \frac{AB^2}{AC^2}$.

Lemma Se ABC, X, Y come sopra allora $\frac{BX \cdot BY}{CX \cdot CY} = \frac{AB^2}{AC^2}$

Dim esercizio.

Esercizi: Lemma.

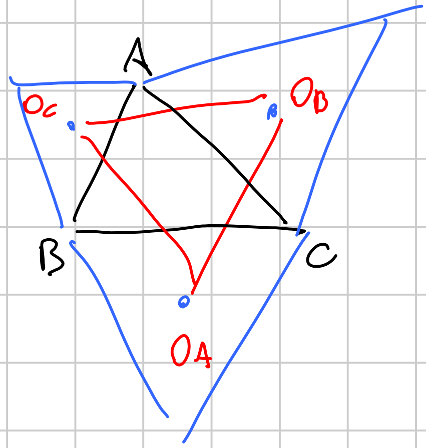
es: 7, 9, 11, 12, 24, 25, 26.

G1: 8. 12

G2: 7 16

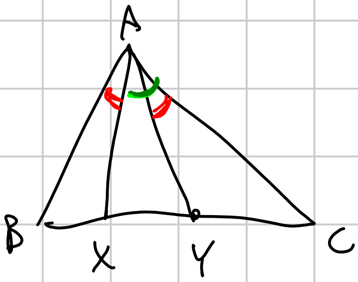
Bonus. Napoleone

$O_A O_B O_C$ equilatero.



Lemma Se ABC, X, Y come sopra allora $\boxed{\dfrac{BX \cdot BY}{CX \cdot CY} = \dfrac{AB^2}{AC^2}}$

Dim



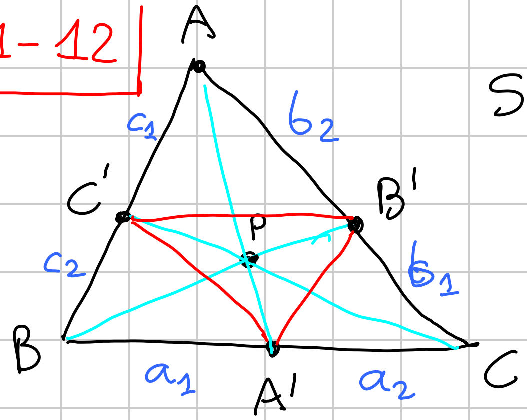
$$\left.\begin{aligned} \frac{BX}{\sin(\textcolor{red}{\bullet})} &= \frac{AX}{\sin\beta} \\ \frac{CX}{\sin(\textcolor{green}{\bullet})} &= \frac{AX}{\sin(\gamma)} \end{aligned}\right\} \quad \frac{BX}{CX} = \frac{\sin(\textcolor{red}{\bullet})}{\sin(\textcolor{green}{\bullet})} \; \frac{\sin\gamma}{\sin\beta}$$

$$\frac{BY}{CY} = \frac{\sin(\textcolor{green}{\bullet})}{\sin(\textcolor{red}{\bullet})} \cdot \frac{\sin(\gamma)}{\sin(\beta)}$$

$$\boxed{\frac{BX}{CX} \cdot \frac{BY}{CY} = \frac{(\sin\gamma)^2}{(\sin\beta)^2} =} \qquad \frac{AC}{\sin\beta} = \frac{AB}{\sin\gamma} \qquad \boxed{\frac{AB}{AC} = \frac{\sin\gamma}{\sin\beta}}$$

$$= \frac{AB^2}{AC^2}$$

G1-12



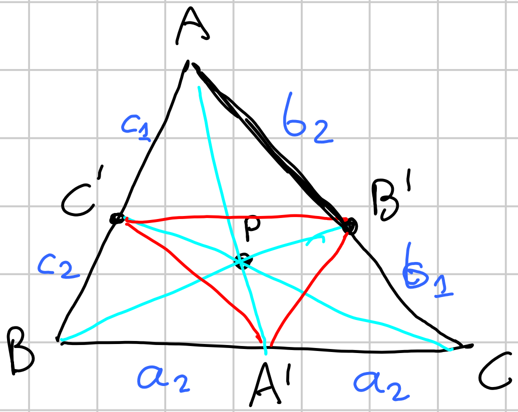
$S' = [A'B'C']$ $\quad d = $ diam circoscr. ABC

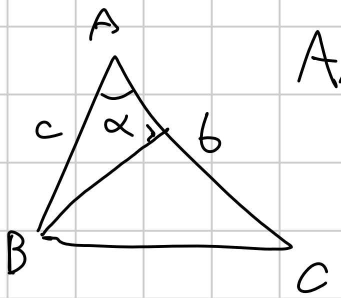
$$\underbrace{d S'} = AB' \cdot BC' \cdot CA' = \underbrace{b_2 \cdot c_2 \cdot a_2}$$

Ceva AA', BB', CC' concorrono $\Leftrightarrow \dfrac{AC'}{C'B} \cdot \dfrac{BA'}{A'C} \cdot \dfrac{CB'}{B'A} = 1$

$$\frac{c_1}{c_2} \cdot \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} = 1$$

$$\boxed{[A'B'C'] = [ABC] - [AB'C'] - [BC'A'] - [CA'B']}$$





$\text{Area} = \frac{1}{2} bc \sin\alpha$

$$[ABC] = \frac{1}{2} bc \sin\alpha \qquad [AB'C'] = \frac{1}{2} b_2 c_1 \sin\alpha$$

$$= \frac{b_2 c_1}{bc} [ABC]$$

$$[A'B'C'] = [ABC] \left(1 - \frac{b_2 c_1}{bc} - \frac{c_2 a_1}{ac} - \frac{a_2 b_1}{ab} \right)$$

$$[ABC] = \frac{1}{2} bc \sin\alpha = \frac{1}{2} abc \cdot \frac{1}{d}$$

$\frac{a}{\sin\alpha} = d \qquad \sin\alpha = \frac{a}{d}$

$$dS' = \frac{1}{2} abc \left(1 - \frac{b_2 c_1}{bc} - \frac{c_2 a_1}{ac} - \frac{a_2 b_1}{ab} \right) \overset{?}{=} a_2 \cdot b_2 \cdot c_2$$

$$a = a_1 + a_2 \qquad b = b_1 + b_2 \qquad c = c_1 + c_2$$

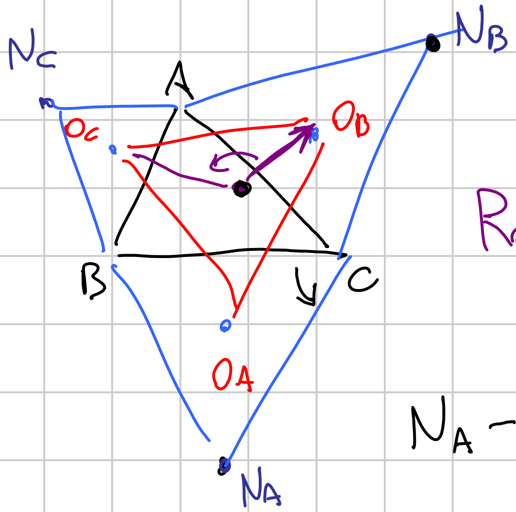
$$\underline{ds'} = \frac{1}{2}\left(abc - a\,b_2c_1 - b\,c_2a_1 - c\,a_2b_1\right)$$

$$= \frac{1}{2}\Big(\underbrace{(a_1+a_2)(b_1+b_2)(c_1+c_2)} - \underbrace{(b_1+b_2)c_2a_1} - \underbrace{(a_1+a_2)b_2c_1} - \underbrace{(c_1+c_2)a_2b_1}\Big)$$

$$= \frac{1}{2}(a_1b_1c_1 + a_2b_2c_2) = \frac{1}{2}(2\,a_2b_2c_2) = \underline{a_2b_2c_2}$$

Ceva $a_1b_1c_1 = a_2b_2c_2$

Napoleone



Non fisso un origine, decido dopo.

$A = a \quad B = b \quad C = c.$

Rotaz. di $60° =$ molt. per $e^{i\frac{\pi}{3}} = \omega$

$$\boxed{\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

$$N_A - c = (b-c)\,\omega$$

$$n_A = c + (b-c)\,\omega$$
$$n_B = a + (c-a)\,\omega$$
$$n_C = b + (a-b)\,\omega$$

O_A = baricentro di $BCN_A \longrightarrow \sigma_A = \dfrac{b+c+n_A}{3}$

$$\sigma_A = \frac{1}{3}\left(b + 2c + \omega(b-c)\right)$$
$$\sigma_B = \frac{1}{3}\left(c + 2a + \omega(c-a)\right)$$
$$\sigma_C = \frac{1}{3}\left(a + 2b + \omega(a-b)\right)$$

Devo mostrare $O_AO_BO_C$ equil.

$$\frac{\sigma_A + \sigma_B + \sigma_C}{3} = \frac{1}{3}(a+b+c) = 0.$$

Vedere: ruotato di O_A di $120°$ è O_B (risp. al centro $= 0$)

$$\omega^2\sigma_a = \frac{1}{3}\omega^2(b + 2c + \omega(b-c)) = \frac{1}{3}\left(\underline{\omega^2}(b+2c) + \underline{\omega^3}(b-c)\right)$$

$\omega^2 = \omega - 1$

$$= \frac{1}{3}\left(\omega b + 2\omega c - b - 2c - b + c\right) = \frac{1}{3}\left(\omega b + 2\omega c - 2b - c\right)$$

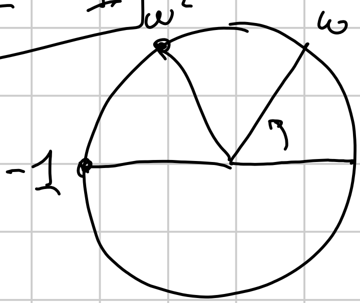
$=$

$b = -c-a$ (perché origine in baricentro)

$$\omega^2 \sigma_a = \frac{1}{3}\left(-\omega a - \omega c + 2\omega c + 2a + 2c - c\right) = \frac{1}{3}\left(\omega(c-a) + 2a + c\right) = \sigma_B$$

e quindi è finita.

$\boxed{\omega^3 = -1}$



$\boxed{\omega^2 = \omega - 1}$

$\boxed{\omega^6 - 1} = 0$