Algebra 2 - Disuguaglianze

- $x^2 \geq 0 \quad x \in \mathbb{R}$ SOS (sum of squares)

$p(x_1, \dots x_n) \geq 0 \quad \forall (x_1, \dots, x_n) \quad x_i \in \mathbb{R}$

$\Rightarrow p = \sum_{i=1}^{N} \frac{a_i^2}{b_i^2}$. (N dipende da n)

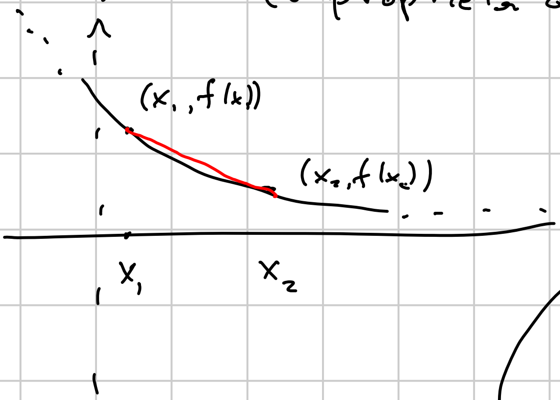
- Cauchy-Schwartz: $v = (x_1 \dots x_n)$, $w = (y_1 \dots y_n)$

$v \cdot w = \langle v, w \rangle$

$\sum x_i y_i \leq \sqrt{\sum x_i^2} \cdot \sqrt{\sum y_i^2}$

prodotto scalare tra v e w

- Convessità (o proprietà di funzioni, analoghe)



$f: \mathbb{R} \to \mathbb{R}$

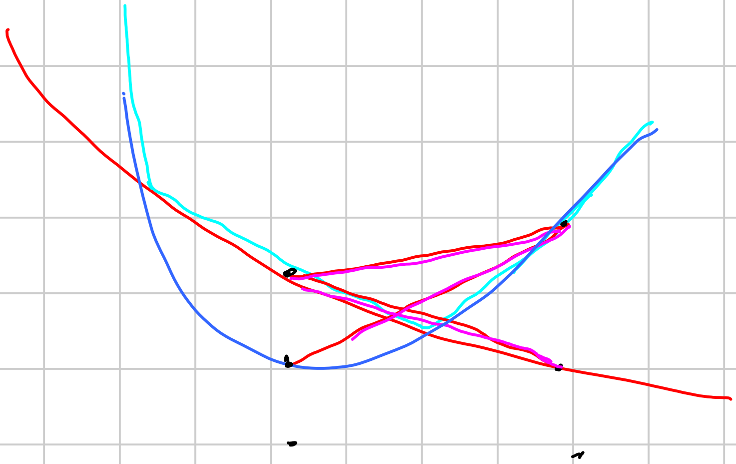
f si dice convessa se $\forall x_1, x_2 \in \mathbb{R}$, il segmento fra $(x_1, f(x_1))$ e $(x_2, f(x_2))$ è tutto contenuto nella parte di piano "sopra" al grafico di f

e $\forall \lambda \in [0,1]$

$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2)$

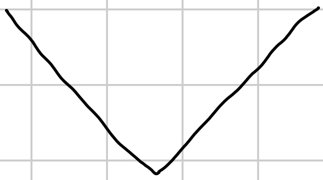
Nota: se $\forall \lambda \in (0,1)$ vale il $>$, f si dice strettamente convessa

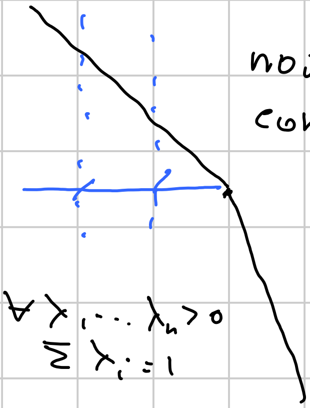
- $f(x) = a + bx$ è convessa (non strettamente)
- f convessa $\Rightarrow f(x+k)$ è convessa.
- f convessa $g: \mathbb{R} \to \mathbb{R}$ lineare $\Rightarrow f(g(x))$ convessa.
- f, g convessa $\Rightarrow f+g$ convessa, $\max\{f, g\}$ è convessa.



Se f è derivabile 2 volte e $f''(x) \geq 0$ $\forall x \in \mathbb{R} \Rightarrow$ f è convessa

$$f(x) = |x| =$$
$$= \max\{-x, x\}$$





non è convessa.

In generale, se f è convessa $\forall x_1 \dots x_n$ $\forall \lambda_1 \dots \lambda_n > 0$, $\sum \lambda_i = 1$

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq f\left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

IMO 2001/2

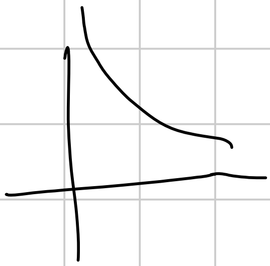
$a,b,c>0$

$$\sum_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1$$

f

$x_1, \dots x_n$

$\lambda_1, \dots \lambda_n$



$$f(x) = \frac{1}{\sqrt{x}}$$

x^α è convessa per $x>0$ $\alpha \geq 1$ o $\alpha \leq 0$

$$\sum_c \frac{a}{\sqrt{a^2+8bc}} = \sum_{i=1}^{3} \lambda_i f(x_i) \geq f\left(\sum \lambda_i x_i\right)$$

x^{2k} $k \in \mathbb{N}$ è convessa su tutto $\mathbb{R}$

$x_i = a^2+8bc$

$\lambda_i = \frac{a}{a+b+c}$

[f concava se $-f$ è convessa]

$$\frac{1}{a+b+c} \sum_c \frac{a}{\sqrt{a^2+8bc}} \geq \frac{1}{\sqrt{\sum_c \frac{a}{a+b+c} \cdot (a^2+8bc)}} = \frac{(a+b+c)^{1/2}}{\sqrt{\sum_c a^3 + 8abc}}$$

$$\sum_c \frac{a}{\sqrt{a^2+8bc}} \geq \frac{(a+b+c)^{3/2}}{\sqrt{\sum_c a^3+8abc}} \overset{?}{\geq} 1 \iff \frac{(a+b+c)^3}{\sum_c a^3+8abc} \geq 1$$

$$(a+b+c)^3 \geq \sum_c a^3 + 8abc \quad = \left(\sum a^3\right) + 24abc$$

$$a^3+b^3+c^3 + \underset{\text{grado 3}}{\text{24 termini di}\ \dots} \geq a^3+b^3+c^3+24abc$$

$$[3,0,0] + 3[2,1,0] + 6[1,1,1] \geq [3,0,0] + 24[1,1,1]$$

$$[3,0,0]+[1,1,1] \geq 2[2,1,0]$$

Schur $\quad \sum_c a^r(a-b)(a-c) \geq 0$

$g(a)$ crescente

Uguaglianza:

$a=b=c$ $\quad$ $a=b$ $\ c=0$; $\ a=c$ $\ b=0$; $\ b=c$ $\ a=0$

Lemma "di Titu"

$$a_i, b_i > 0 \qquad \sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$$

$f = x^2 \quad x_i = a_i$

$$\lambda_i = \cancel{\frac{b_i}{\sum b_i}} \quad \frac{\frac{1}{b_i}}{\sum \frac{1}{b_i}}$$

$$\frac{\frac{1}{b_i}}{\sum \frac{1}{b_i}} \sum a_i^2 \geq \left(\frac{\sum \frac{a_i}{b_i}}{\sum \frac{1}{b_i}}\right)^2$$

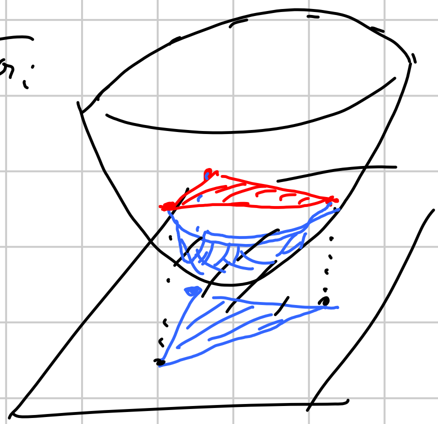
$$f(x) = \frac{1}{x}$$

$$\frac{1}{\sum \lambda_i x_i} = \frac{(\sum a_i)^2}{\sum b_i} \qquad \sum \lambda_i x_i = \frac{\sum b_i}{(\sum a_i)^2}$$

$$x_i = \frac{b_i}{a_i} \qquad \lambda_i = \frac{a_i}{\sum a_i} \qquad \sum \frac{a_i}{b_i} \cdot \frac{a_i}{\sum a_i} \geq \frac{1}{\sum \frac{b_i}{\sum a_i}}$$

$$\sum \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$$

- $f(x,y) = x^2+y^2 \quad \mathbb{R}^2 \to \mathbb{R}$ è convessa.



Karamata $\qquad (a_1 \dots a_n) \succ (b_1 \dots b_n)$

$a_1 \geq b_1$

$a_1+a_2 \geq b_1+b_2$

$\vdots$

$a_1 + \dots + a_{n-1} \geq b_1+b_2+\dots b_{n-1}$

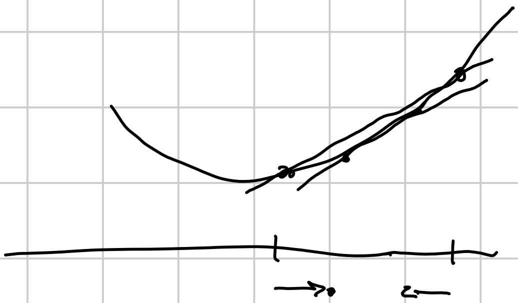
$a_1 + \dots + a_n = b_1 + \dots + b_n$

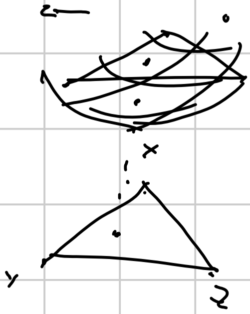
$a_i \in \mathbb{R}$ $\quad b_i \in \mathbb{R}$ $\quad a_1 \geq a_2 \geq a_3 \dots \geq a_n$

$b_1 \geq b_2 \geq \dots \geq b_n$ $\qquad (a_1 \dots a_n) \succ (b_1 \dots b_n)$

f convessa

$\Rightarrow \sum f(a_i) \geq \sum f(b_i)$.





$x = (x_1 \dots x_n)$ $\qquad y = (y_1 \dots y_n)$ $\qquad z = (z_1 \dots z_n)$

$t_1 x + t_2 y + t_3 z$ $\qquad 0 \leq t_1, t_2, t_3 \leq 1$ $\quad \sum t_i = 1$

$$t_1 f(x) + t_2 f(y) + t_3 f(z) \geq f(t_1 x + t_2 y + t_3 z)$$

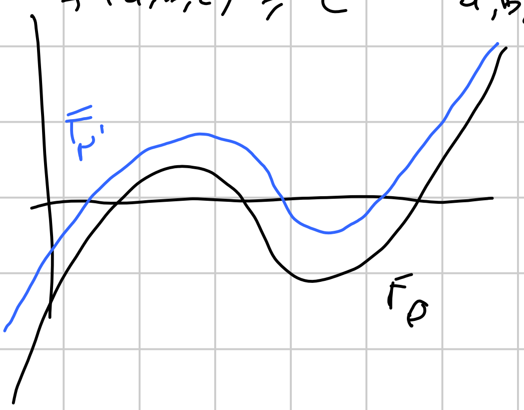
Lemma ABC

f(a,b,c) è simm. in a,b,c (pol!) di grado ≤ 5

$F_p = (x-a)(x-b)(x-c) = x^3 - s x^2 + q x - p$

$f(a,b,c) = T(s,q,p)$

$f(a,b,c) \geq C$ $\qquad a,b,c \in \mathbb{R}$ $(\in \mathbb{R}^+)$



Teorema di Newton:

$f(x_1 \dots x_n)$ polinomio simmetrico di grado k $\Rightarrow$ f è funzione (polinomiale) delle funzioni simmetriche elementari di grado $\leq k$

$$x^3 + y^3 + z^3 = (x+y+z)^3 - 3(xy+yz+zx)(x+y+z) + 3xyz$$

$p' > p$

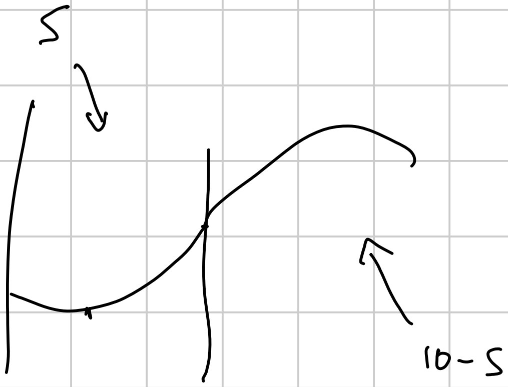
p ha un valore massimo e uno minimo, altrimenti non ci sarebbero 3 radici reali; ma max e min di p sono casi in cui 2 radici coincidono.

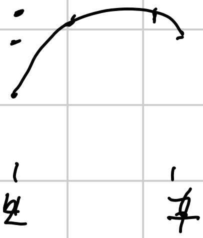
Ma se f ha grado ≤ 5, in T p ha grado ≤ 1, e allora f ha max o min (in p) quando p è massimo o minimo $\Rightarrow$ quando 2 tra a,b,c sono uguali.

Concave? Min $\longleftrightarrow$ max convesse.

Il min di una f concava (max convessa) è sul bordo (se esiste)

$f(a)+f(b)+f(c)$





$a+b+c=10$

$\sum x_i = 10$

Studio termine a termine

$a,b,c>0$

$$\sum_c \frac{2a^3}{a^2+b^2} \geq \sum_c a \qquad \frac{2a^3}{a^2+b^2} \geq 2a-b$$

$$\sum_c 2a-b \underset{=}{\geq} \sum_c a$$

$$2a^3 \geq (2a-b)(a^2+b^2) = 2a^3 + \overbrace{2ab^2-a^2b-b^3}^{\leq 0} =$$
$$= 2a^3 - b(a+b)^2$$

Bernoulli: $(1+x)^n \geq 1+nx$ $\quad x>0,\ n\geq 1$ $\qquad e^x \geq 1+x$ $\quad x\in\mathbb{R}$

IMO 2012/2 $\quad a_2, a_3 \dots a_n > 0 \quad a_2 a_3 \cdots a_n = 1 \quad n \geq 3$

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n$$

$$x^k = (1+(x-1))^k \geq 1 + k(x-1) \qquad x = \frac{k-1}{k}(a_k+1)$$

$$\left(\frac{k-1}{k}(a_k+1)\right)^k \geq 1 + k\left(\frac{k-1}{k}(a_k+1) - 1\right) = 1 + (k-1)(a_k+1) - k =$$

$$= (k-1)a_k + k - \cancel{1} + \cancel{1} - \cancel{k} = (k-1)a_k$$

$$(1+a_k)^k \geq \frac{k^k}{(k-1)^k}(k-1)a_k = \frac{k^k}{(k-1)^{k-1}}a_k .$$

$$(1+a_2)^2 \cdot \ldots \cdot (1+a_n^n) \geq a_2 a_3 \cdots a_n \cdot n^n = n^n$$