# G2M - PROIETTIVA

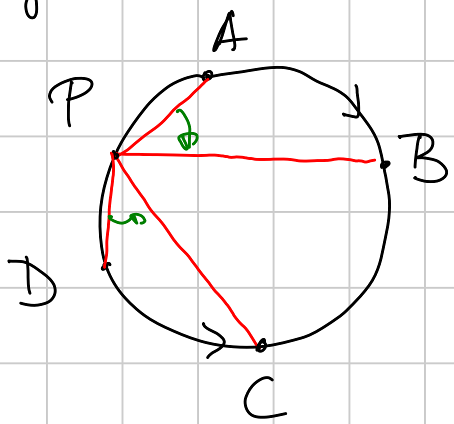
Note Title 04/02/2023

1) BIRAPPORTI
2) QUATERNE ARMONICHE
3) POLI & POLARI
4) ESERCIZI

## 1- BIRAPPORTI

Rapporto con segno di segmenti allineati ✓

Uguale per archi (o corde) di circonferenza



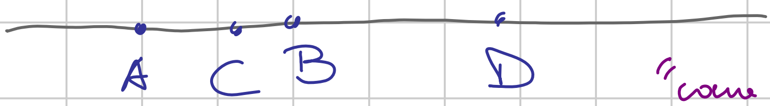

$\frac{\overset{\frown}{AB}}{\overset{\frown}{DC}} < 0 \qquad \frac{DB}{BC} > 0$

$\widehat{APB}$

$\widehat{DPC}$

Birapporto di 4 punti allineati

A, B, C, D punti allineati $\quad (A,B;C,D) = \frac{AC}{CB} \Big/ \frac{AD}{DB}$




$$\frac{\text{"come C divide AB"}}{\text{"come D divide AB"}}$$

Convenzioni: $\quad AA = 0 \quad , \quad \frac{* \neq 0}{0} = \infty$

Oss: $\frac{AC}{CB} = -1$ dove sta C? non c'è.

Aggiungiamo, alla retta $r$ che passa per A e B un punto $r_\infty$ t.c. $\frac{Ar_\infty}{r_\infty B} = -1$. Si scopre che, per mantenere un minimo di coerenza, per $r_\infty$ passano anche tutte le parallele a $r$.

$\Rightarrow$ Se $M$ è il pt. medio di AB

$$(A,B;M,r_\infty) = \frac{AM}{MB} \Big/ \frac{Ar_\infty}{r_\infty B} = -1$$

Es: $(A,B;C,D) = 1$ possibile solo se $C \equiv D$ o $A \equiv B$

Es: Se $(A,B;C,D) = \lambda$, quanto vale $(B,A;C,D)$?

$$(B,A;C,D) = \frac{1}{\lambda}$$

$$(A,B;D,C) = \frac{1}{\lambda}$$

$$(B,A;D,C) = \lambda$$

$$(C,D;A,B) = \lambda$$

$$(D,C;B,A) = \lambda$$

Es: $(C,A;B,D) = \frac{CB}{BA} \Big/ \frac{CD}{DA} =$

$$= \frac{CB}{-(AC+CB)} \cdot \frac{DA}{-(DB+BC)} = \frac{1}{\left(\frac{AC}{CB}+1\right)\left(\frac{DB}{DA}+\frac{BC}{DA}\right)}$$

$$= \frac{1}{\underbrace{\left(\frac{AC}{CB} \Big/ \frac{DA}{DB}\right)}_{-\lambda} + \frac{DB}{DA} - \frac{AC}{DA} + \frac{BC}{DA}} = \frac{1}{1-\lambda}$$

Le 24 permutazioni di A, B, C, D producono 6 valori, ognuno 4 volte

$$\lambda,\ \frac{1}{\lambda},\ \frac{1}{1-\lambda},\ 1-\lambda,\ \frac{\lambda}{1-\lambda},\ \frac{1-\lambda}{\lambda}$$

Oss: $r \cup \{r_\infty\} \longleftrightarrow \mathbb{R} \cup \{\infty\}$ fissi A, B, C distinti su $r$

$$D \longleftrightarrow (A,B;C,D)$$

biezione

$A \longrightarrow (A,B;C,A) = \frac{AC}{CB} \Big/ \frac{AA}{AB} = \frac{*\neq 0}{0} = \infty$

$B \longrightarrow 0$

$C \longrightarrow 1$

$r_\infty \longrightarrow -\frac{AC}{CB}$

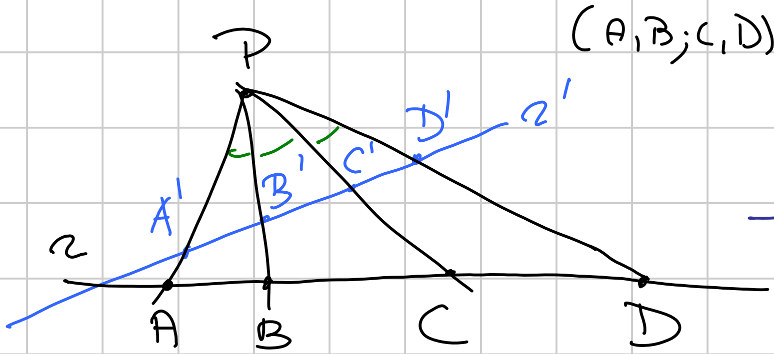
**Lemma 1** (Invarianza per proiezione)

$A,B,C,D \in r$, $P \notin r$

(a) il birapporto $(A,B;C,D)$ dipende solo dagli angoli in $P$

(b) se $r'$ è un'altra retta, detti $A' = AP \cap r'$, $B' = BP \cap r'$, $C' = CP \cap r'$, $D' = DP \cap r'$, si ha

$$(A,B;C,D) = (A',B';C',D')$$




dim: (a) $A\hat{P}C = \alpha$ $\quad A\hat{P}D = \gamma$
$C\hat{P}B = \beta$ $\quad D\hat{P}B = \delta$

Teo dei seni su $\overset{\triangle}{APC}$: $\frac{AC}{\sin\alpha} = \frac{AP}{\boxed{\sin A\hat{C}P}}$

su $\overset{\triangle}{BPC}$: $\frac{CB}{\sin\beta} = \frac{BP}{\boxed{\sin B\hat{C}P}}$

su $\overset{\triangle}{APD}$: $\frac{AD}{\sin\gamma} = \frac{AP}{\boxed{\sin A\hat{D}P}}$

su $\overset{\triangle}{BPD}$: $\frac{DB}{\sin\delta} = \frac{BP}{\boxed{\sin B\hat{D}P}}$

$$\Longrightarrow \frac{AC}{CB} \Big/ \frac{AD}{DB} = \frac{\sin\alpha}{\sin\beta} \Big/ \frac{\sin\gamma}{\sin\delta}.$$

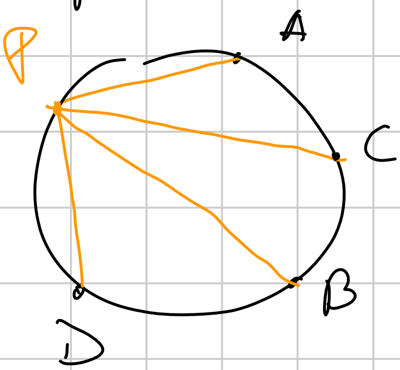
(b) ovvio.

**Def**: Date 4 rette $r_1, r_2, r_3, r_4$ concorrenti in un punto $P$, si definisce

$$(r_1, r_2; r_3, r_4) = (A,B;C,D)$$

con $A = r_1 \cap r$ $\quad B = r_2 \cap r$ $\quad C = r_3 \cap r$ $\quad D = r_4 \cap r$ per una qualsiasi retta $r$ non passante per $P$.

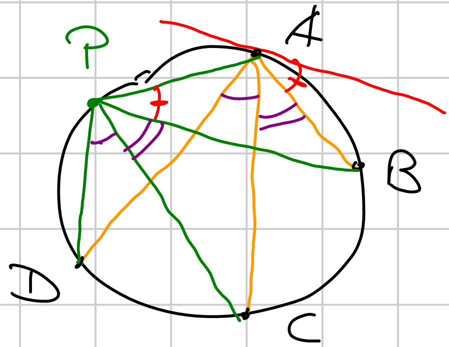
**Corollario:** Se $A, B, C, D$ stanno su una circonferenza $\Gamma$, allora $(PA, PB; PC, PD)$ non dipende dalla scelta di $P$ finché $P \in \Gamma$.




dim: $\dfrac{\sin A\hat{P}C}{\sin C\hat{P}B} \Big/ \dfrac{\sin A\hat{P}D}{\sin D\hat{P}B} = \dfrac{AC}{CB} \Big/ \dfrac{AD}{DB}$. $\square$

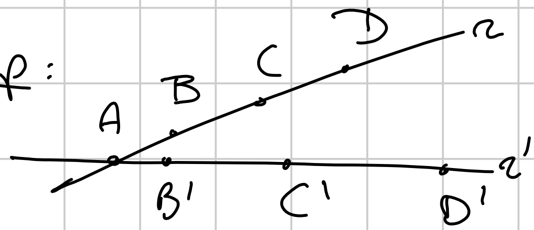
teo delle corde

**Def:** Se $A, B, C, D$ stanno su $\Gamma$ circonf., il loro birapporto è
$$(A,B;C,D) = (PA, PB; PC, PD)$$
con $P \in \Gamma$.




**OSS:** $P \equiv A$ che succede?
In questo caso $AA$ è la tangente in $A$ a $\Gamma$.

**Prop:**




$BB', CC', DD'$ concorrono se e solo se $(A,B;C,D) = (A,B';C',D')$.

dim: ($\Longrightarrow$) è ovvio proiettando dal pt di concorrenza.

($\Longleftarrow$) $P = BB' \cap CC'$

interseco con $r$

$$(A,B;C,D) = (A,B';C',D') = (PA, PB'; PC', PD') = (A,B;C,D'')$$

$D'' = PD' \cap r$

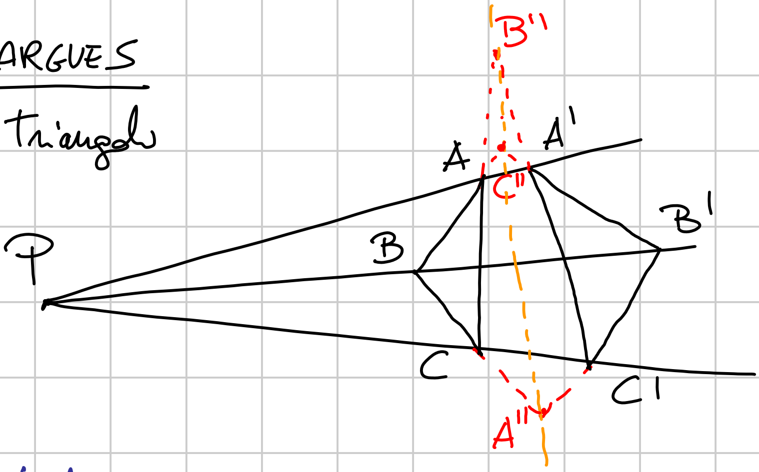
per l'unicità del birapporto, $D'' = D$. $\Rightarrow$ concorrono. $\square$

$(A,B';C',D') \underset{P}{=} (A,B;C,D'')$ ← proiettando da $P$

## Teo di DESARGUES

$ABC$, $A'B'C'$ triangoli

$A'' = BC \cap B'C'$
$B'' = AC \cap A'C'$
$C'' = AB \cap A'B'$

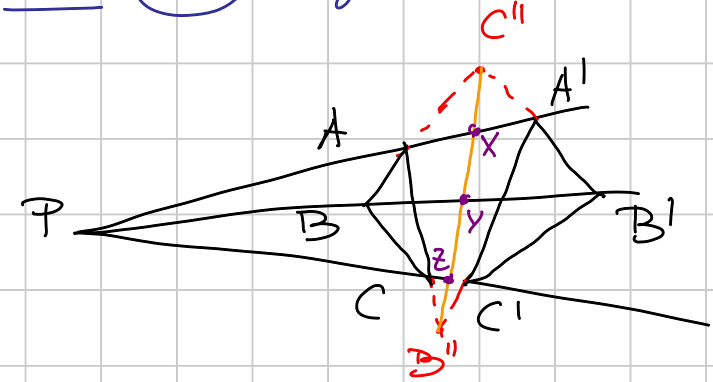



$AA'$, $BB'$, $CC'$ concorrono

$\Updownarrow$

$A''$, $B''$, $C''$ allineati

dim $\Leftarrow$ Voglio dim che $BC$, $B'C'$, $B''C''$ concorrono.



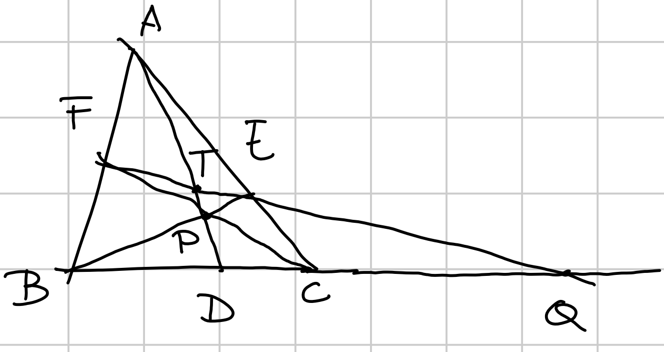

$X = AA' \cap B''C''$
$Y = BB' \cap B''C''$
$Z = CC' \cap B''C''$

$$(P,B;Y,B') \underset{C''}{=} (P,A;X,A') \underset{B''}{=} (P,C;Z,C')$$

$\Rightarrow$ per la prop. precedente $BC$, $B'C'$, $B''C''$ concorrono. $\square$

## Esempi di birapporti

(a)



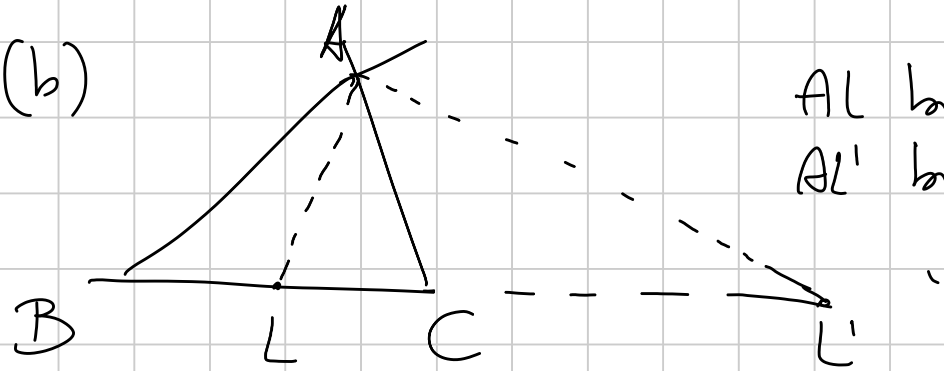

$$(B,C;D,Q) \underset{A}{=} (F,E;T,Q) \underset{P}{=} (C,B;D,Q) = \frac{1}{(B,C;D,Q)}$$

$\Rightarrow$ $(B,C;D,Q) = \pm 1$ ma se i pt. sono distinti, l'unico val possibile è $-1$.

$\Rightarrow$ $(B,C;D,Q) = -1$.

Es: Dimostrarlo con CEVA & MENELAO

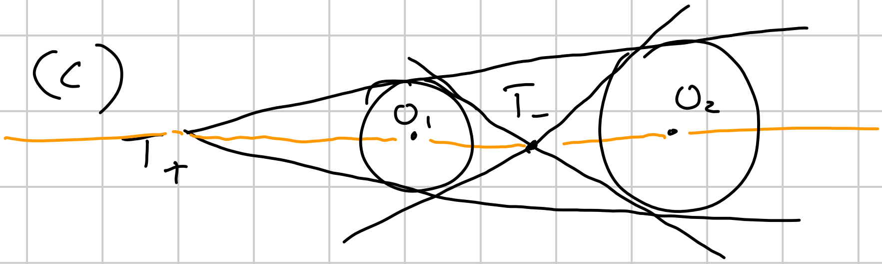
(b)




AL bisett. interna
AL' bisett. esterna

$$(B,C;L,L') = \frac{BL}{LC} \Big/ \frac{BL'}{L'C} = \frac{AB}{AC} \Big/ -\frac{AB}{AC} = -1.$$

(c)



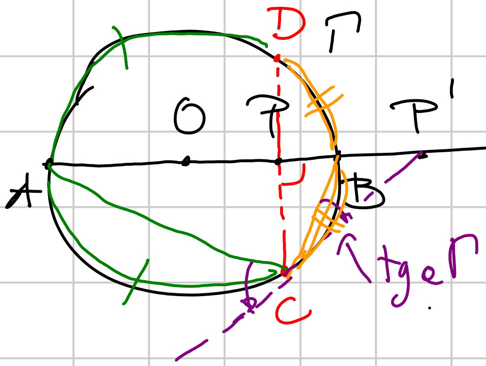

$$(O_1,O_2;T_+,T_-) = \frac{O_1T_+}{T_+O_2} \Big/ \frac{O_1T_-}{T_-O_2}$$

$T_+, T_-$ centri di similitudine tra le due cf. $\Rightarrow$ $\frac{O_1T_+}{T_+O_2} = -\frac{R_1}{R_2}$

$$\frac{O_1T_-}{T_-O_2} = +\frac{R_1}{R_2}$$

$\Rightarrow (O_1,O_2;T_+,T_-) = -1$

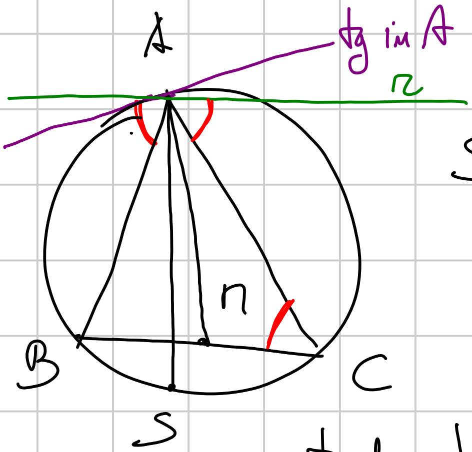
(d)




P' = inverso di P in Γ

$$(A,B;P,P') \underset{C}{=} (A,B;D,C) = -\frac{AD}{DB} \Big/ \frac{AC}{CB} = -1$$

(e)




S su Γ t.c. $B\hat{A}S = C\hat{A}H$.
AS simmediana

$(A,S;B,C)$

<u>metodo brutto</u>: teo dei seni

$$(A,S;B,C) = (AA, AS; AB, AC) \underset{\substack{\text{inv. rigs.} \\ \text{alla bisett.}}}{=} (r, AD; AC, AB) =$$

$r$ è la parallela a $BC$ per $A$

$$\underset{BC}{=} (r_\infty, D; C, B) = (C, B; r_\infty, D) = (B, C; D, r_\infty) = -1.$$

**Lemma** (invarianza sotto inversione)

I birapporti si conservano sotto inversione

dim: (1) l'inversione conserva l'ordine dei punti $\Rightarrow$ segno ok

(2) $A, B, C, D \longrightarrow A', B', C', D'$

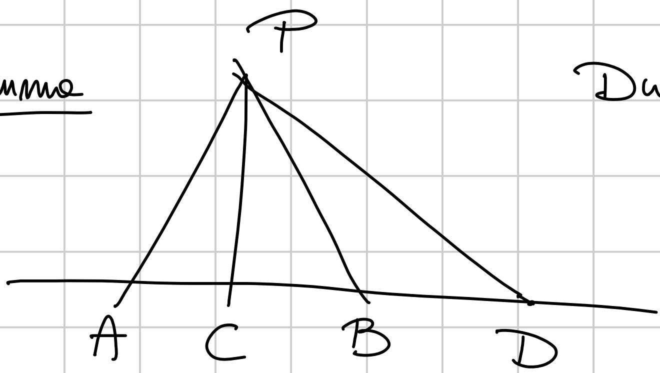
$$\left|\frac{A'C'}{C'B'}\right| = \left| R^2 \frac{AC}{OA \cdot OC} \cdot \frac{OC \cdot OB}{CB} \frac{1}{R^2}\right| = \left|\frac{OB}{OA}\right| \cdot \left|\frac{AC}{CB}\right|$$

$$\left|\frac{A'D'}{D'B'}\right| = \left| R^2 \frac{AD}{OA \cdot OD} \cdot \frac{OB \cdot OD}{DB} \frac{1}{R^2}\right| = \left|\frac{OB}{OA}\right| \cdot \left|\frac{AD}{DB}\right|$$

$$\Rightarrow \left| \frac{A'C'}{C'B'} \Big/ \frac{A'D'}{D'B'} \right| = \left| \frac{AC}{CB} \Big/ \frac{AD}{DB} \right|. \quad \square$$

## 2- QUATERNE ARMONICHE

**Lemma**




Due delle seguenti implicano l'altra:

(i) $(A,B;C,D) = -1$

(ii) $PC$ biseca $A\hat{P}B$

(iii) $PC \perp PD$

dim: (ii) + (iii) $\Rightarrow$ $C, D$ piedi bisett. interna/est $\Rightarrow$ $(A,B;C,D) = -1$

(i) + (ii) $\Rightarrow$ per l'unicità del birapporto $PD$ bisett esterna $\Rightarrow PD \perp PC$

(i) + (iii) $\Rightarrow$ $A\hat{P}D = A\hat{P}C + \frac{\pi}{2}$ $\Big\|$ $\Leftarrow$ (iii)

$B\hat{P}D = B\hat{P}C + \frac{\pi}{2}$

(i) $\searrow$

$$\frac{\sin(A\hat{P}C)}{\sin(C\hat{P}B)} = -\frac{\sin(A\hat{P}D)}{\sin(D\hat{P}B)} = \frac{\cos(A\hat{P}C)}{\cos(C\hat{P}B)}$$

$\Rightarrow \operatorname{tg}(A\hat{P}C) = \operatorname{tg}(C\hat{P}B) \Rightarrow A\hat{P}C = C\hat{P}B \Rightarrow PC$ bisettrice. □

**Def**: $\delta(A,B;C,D) = -1$, $A,B,C,D$ si dicono <u>quaterna armonica</u>

$D$ si dice <u>quarto armonico</u> di $A,B,C$

## Cond. equiv. sulle rette [TFAE]

Sia $O$ il pt. medio di $AB$, allora le seguenti sono equivalenti:

(i) $(A,B;C,D) = -1$

(ii) $\dfrac{2}{AB} = \dfrac{1}{AC} + \dfrac{1}{AD}$

(iii) $CA \cdot CB = CO \cdot CD$

(iv) $OC \cdot OD = OA^2$

(v) $\dfrac{OC}{OD} = \left(\dfrac{AC}{AD}\right)^2 = \left(\dfrac{BC}{BD}\right)^2$

<u>dim</u>: <u>x esercizio</u>

**Def**: $A,B,C,D$ su $\Gamma$ con $(A,B;C,D) = -1$ si dicono <u>quadrilatero armonico</u>.

## Cond equivalenti sulla circonferenza

$A,B,C,D$ su $\Gamma$, allora TFAE

(i) $AB \cdot CD = BC \cdot AD$

(ii) $BD$ simmediana di $\overset{\triangle}{ABC}$

(iii) le tg in $A$ e $C$ a $\Gamma$ si incontrano su $BD$

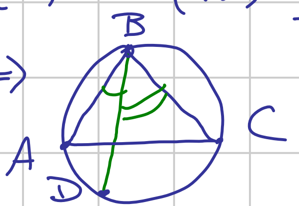
(iv) $(A,C;B,D) = -1$

(v) $\exists M$ pt. medio di $AC$, $MD, MB$ sono simmetriche risp. a $AC$.

(vi) la bisettrice di $A\hat{B}C$ e quella di $A\hat{D}C$ si intersecano su $AC$

(vii) $\dfrac{AB^2}{AD^2} = \dfrac{MB}{MD}$

<u>dim</u>: (ii) $\Longleftrightarrow$ (iii) <u>ok</u>

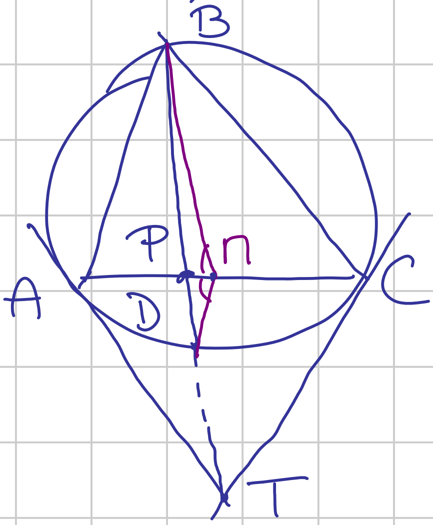
(i) $\Longleftrightarrow$ 


$\left|\dfrac{AB}{BC}\right| = \left|\dfrac{AD}{DC}\right| = \left|\dfrac{\sin A\hat{B}D}{\sin C\hat{B}D}\right| = \left|\dfrac{\operatorname{dist}(D,AB)}{\operatorname{dist}(D,BC)}\right|$

$\iff D \in$ simmediana.

$(i) \iff (iv)$

(v)




AC bisett. di $B\hat{M}D$

$T =$ tg in $A$ $\cap$ tg in $C$

$(A, C; B, D) = -1 \iff T \in BD.$

$T\hat{M}A = \frac{\pi}{2}$

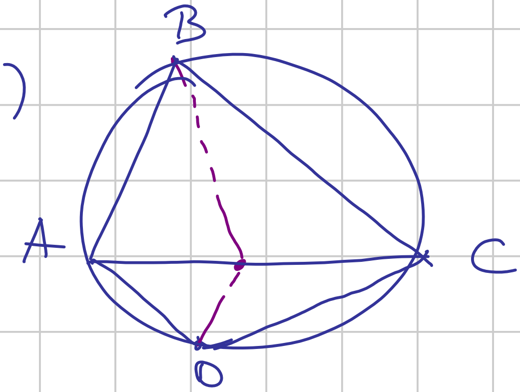
AC bisett. di $B\hat{M}D \iff (B, D; P, T) = -1$

$\Updownarrow$

$(A, C; B, D) = -1$

$$(B, D; P, T) \underset{C}{=} (B, D; A, C) = (A, C; B, D)$$

(vi)




Siano $L, L'$ su $AC$ t.c.

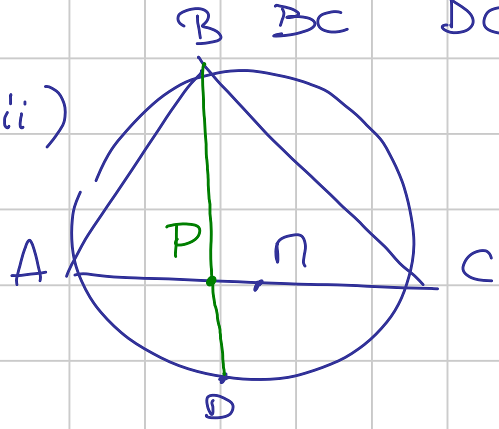
$BL$ bisett. di $A\hat{B}C$

$DL'$ bisett di $A\hat{D}C$

$$\Rightarrow \frac{AL}{LC} = \frac{AB}{BC} \qquad \frac{AL'}{L'C} = \frac{AD}{DC}$$

$$L = L' \iff \frac{AB}{BC} = \frac{AD}{DC} \underset{(i)}{\iff} (A, C; B, D) = -1$$

(vii)




$$\frac{AB^2}{AD^2} = \frac{MB}{MD}$$

$$\frac{BP}{PD} = \frac{BM}{MD}$$

$\Updownarrow$

$MP$ bisett. $D\hat{M}B$

$\Updownarrow$ (v)

$$\frac{AB}{BC} \Big/ \frac{AD}{DC} = \frac{AB \cdot DC}{BC \cdot AD} = \frac{AB}{AD} \cdot \frac{DC}{BC} = -1$$

$\Updownarrow$

$$\left|\frac{BC}{DC}\right| = \left|\frac{AB}{AD}\right|$$

$\Updownarrow$

$$\frac{BP}{PD} = \left(\frac{AB}{AD}\right)^2.$$

$$BP = \frac{AP}{\sin(A\hat{B}P)} \cdot \sin P\hat{A}B \qquad PD = \frac{AP}{\sin(A\hat{D}P)} \cdot \sin(P\hat{A}D)$$

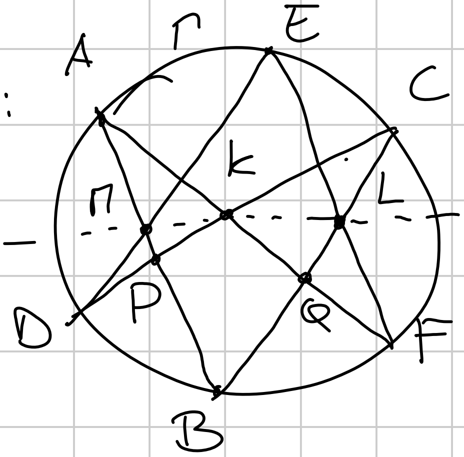
$$\frac{BP}{PD} = \frac{\sin(\overset{C\hat{A}B}{\overbrace{P\hat{A}B}})}{\sin(\underset{A\hat{B}D}{\underbrace{A\hat{B}P}})} \cdot \frac{\sin(\overset{A\hat{D}B}{\overbrace{A\hat{D}P}})}{\sin(\underset{C\hat{A}D}{\underbrace{P\hat{A}D}})} = \frac{BC}{AD} \cdot \frac{AB}{CD}$$

## Teo (PASCAL)

$\Gamma$ cp $A, B, C, D, E, F$ in $\Gamma$

$\Rightarrow$ $AB \cap DE = M$, $BC \cap EF = L$, $CD \cap FA = K$ sono allineati

dim:



$(C, L; Q; B) \underset{F}{=} (C, E; A, B) \underset{D}{=}$

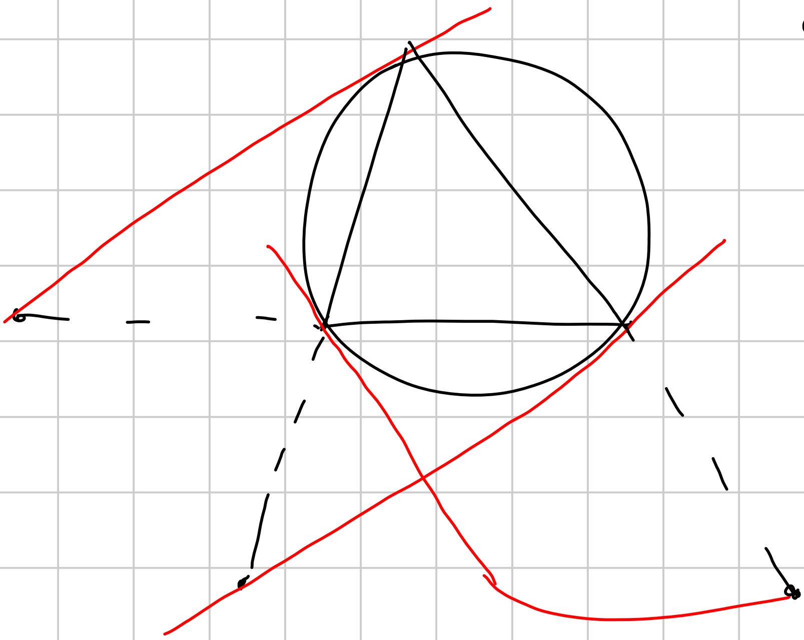
$\underset{D}{=} (P, M; A, B) \underset{K}{=} (C, MK \cap BC; Q, B)$

per l'unicità del birapporto

$L = MK \cap BC \Rightarrow M, K, L$ allineati. □

Oss: Se due pt. coincidono, considero la tg.

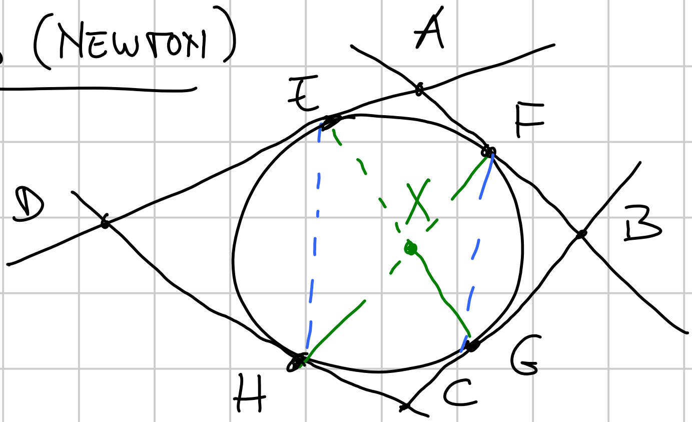
Es: $AABBCC \Rightarrow AA \cap BC$, $AB \cap CC$, $BB \cap CA$ sono allineati sulla retta di Lemoine.



Oss: Permutando i 6 punti ottengo rette diverse

Oss: L'inverso è falso.

## Teo (NEWTON)



$AC, BD, EG, FH$ concorrono.

dim: $X = EG \cap FH \qquad Y = EH \cap FG$

Pascal su $EGGFHH \longrightarrow \underset{X}{\underset{\shortparallel}{EG \cap FH}},\ \underset{C}{\underset{\shortparallel}{GG \cap HH}},\ \underset{Y}{\underset{\shortparallel}{GF \cap HE}}$ sono allineati

$EEHFFG \longrightarrow \underset{A}{\underset{\shortparallel}{EE \cap FF}},\ \underset{Y}{\underset{\shortparallel}{EH \cap FG}},\ \underset{X}{\underset{\shortparallel}{HF \cap GE}}$ sono allineati

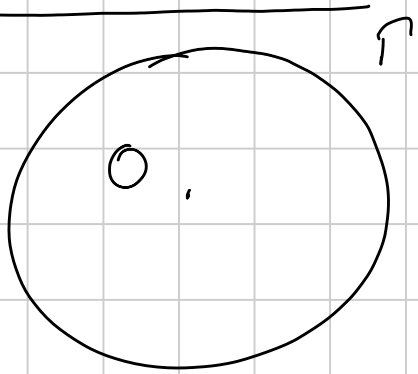
$\to$ $A, C, X, Y$ sono allineati

Similmente definisco $HG \cap EF = Z$ e con 2 pascal dimostro che $B, D, X, Z$ sono allineati. $\square$

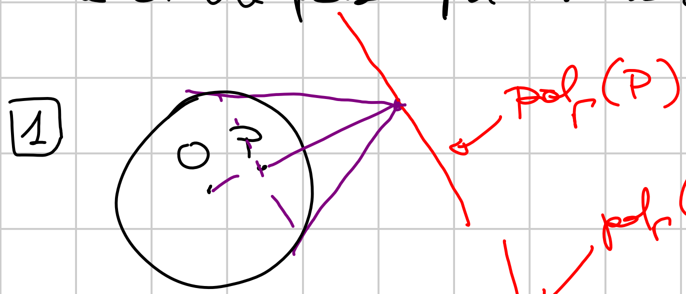
**Teo (BRIANCHON)**: $ABCDEF$ circoscritto a $\Gamma \Rightarrow AD, BE, CF$ concorrono

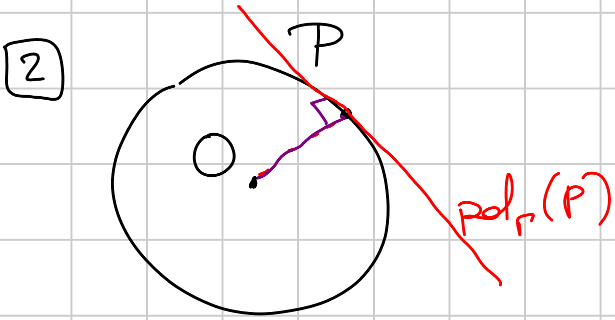
dim x esercizio.

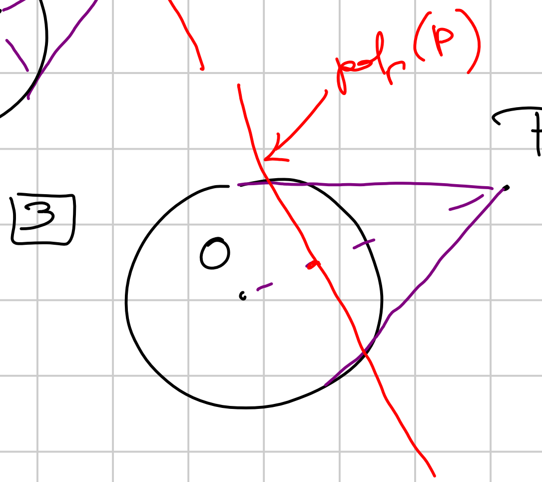
## 3 - Poli & Polari




La polare di $P$ risp. a $\Gamma$ è la perpendicolare a $OP$ che passa per l'inverso di $P$ in $\Gamma$.




$\Gamma$ cfr, $r$ retta, $pol_\Gamma(r) = P \iff r = pol_\Gamma(P)$

$P$ si dice allora **POLO** di $r$

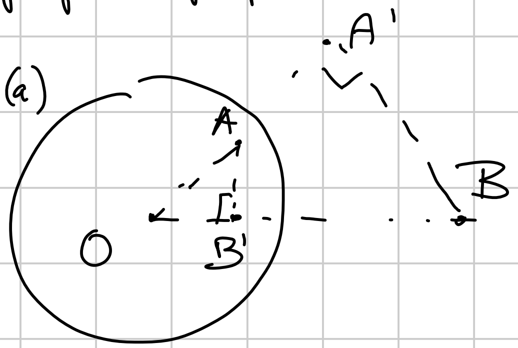
## Proprietà

(a) $A \in pol_\Gamma(B) \iff B \in pol_\Gamma(A)$

(b) $pol_\Gamma(r \cap s) =$ retta per $pol_\Gamma(r)$ e $pol_\Gamma(s)$

(c) $pol_\Gamma(A) \cap pol_\Gamma(B) = pol_\Gamma(AB)$

dim: (a)




$A \in pol_\Gamma(B) \qquad B \in pol_\Gamma(A)$

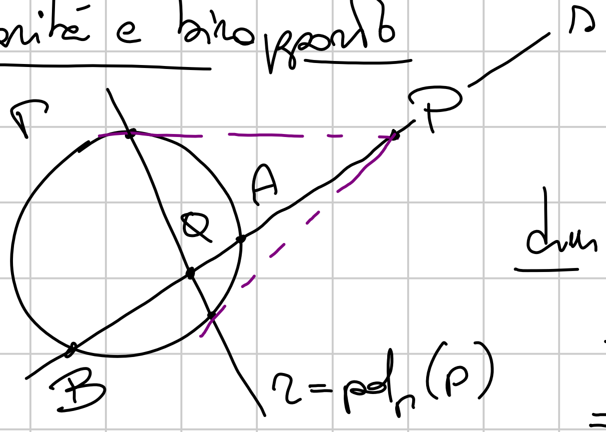
$\Updownarrow \qquad \Updownarrow$

$\widehat{AB'O} = \frac{\pi}{2} \iff \widehat{BA'O} = \frac{\pi}{2}$

(c), (b) x esercizio

Oss: $pol_\Gamma(P) = \{ pol_\Gamma(r) \mid P \in r \}$

## Polarità e birapporto




$(A, B; Q, P) = -1$

dim: Siano $\{C, D\} = pol_\Gamma(P) \cap \Gamma$

$\Rightarrow CC \cap DD = P$

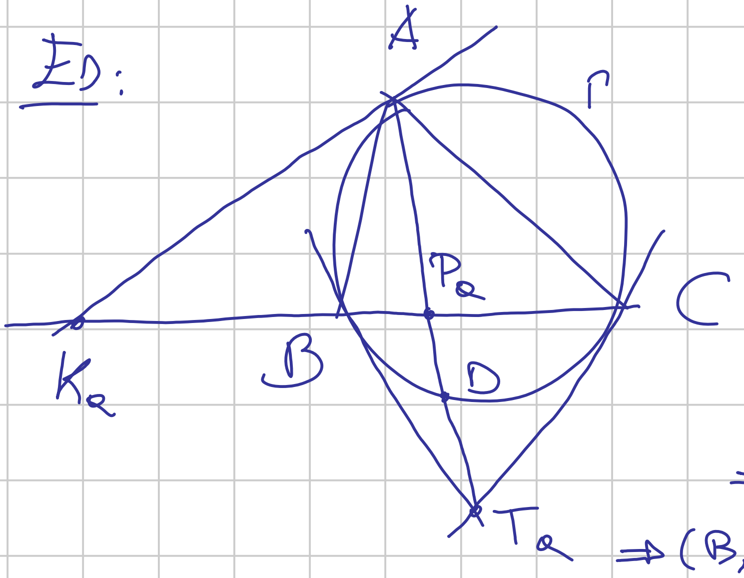
$\Rightarrow CDBA$ quad armonico

$\Rightarrow (A, B; D, C) = -1$

$\Rightarrow$ proiettando da $C$ su $s$

$(A, B; Q, P) = -1$.

Es:




$BC = pol_\Gamma(T_a)$

$\Rightarrow K_a \in pol_\Gamma(T_a)$

$\Rightarrow T_a \in pol_\Gamma(K_a)$

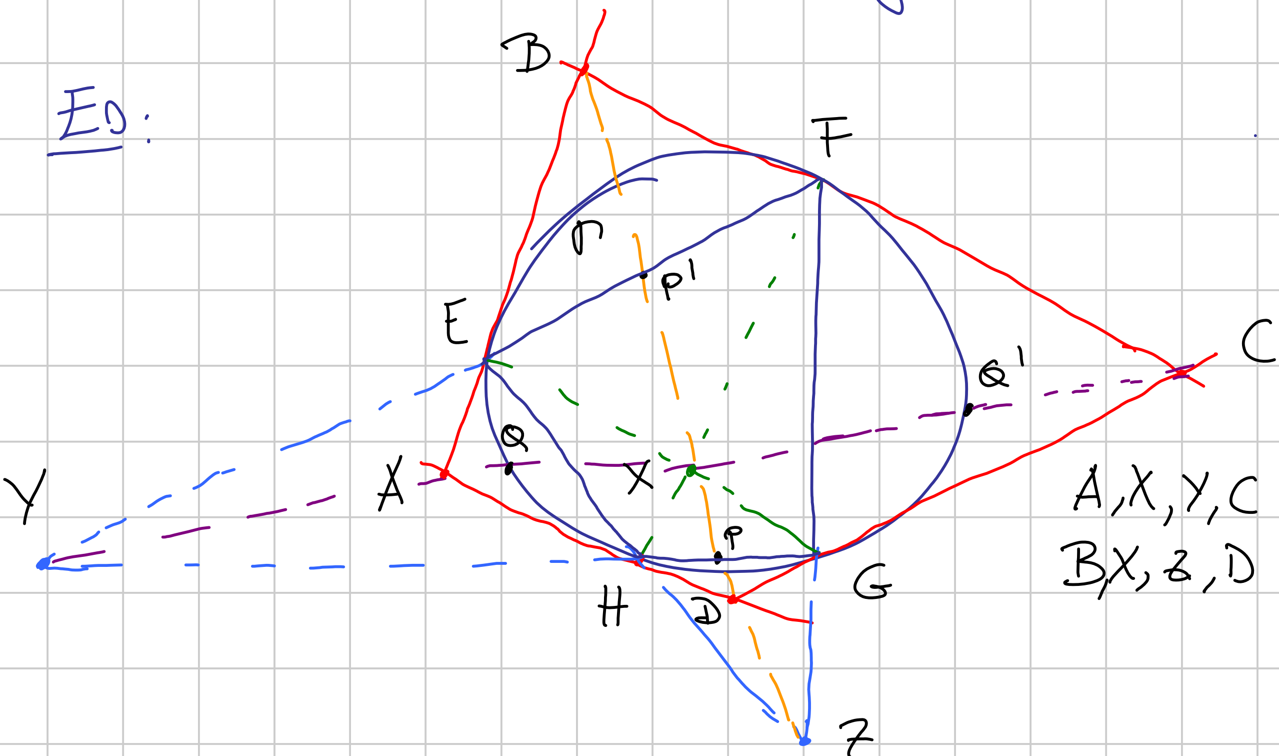
e $A \in pol_\Gamma(K_a)$

$\Rightarrow AT_a = pol_\Gamma(K_a)$

$\Rightarrow (B, C; P_a; K_a) = -1$

e KoD tg a $\Gamma$ in D

Es:




A, X, Y, C allineati
B, X, Z, D

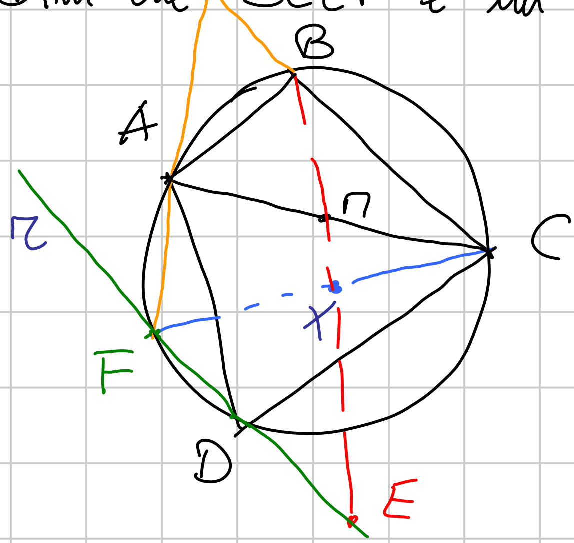
$B = pol_\Gamma(EF)$ $\quad Y = EF \cap HG = pol_\Gamma(BD) \Rightarrow (H,G;Y,P) = -1$

$D = pol_\Gamma(HG)$ $\quad (E,F;Y,P') = -1$

$(Q,Q';Y,X) = -1$

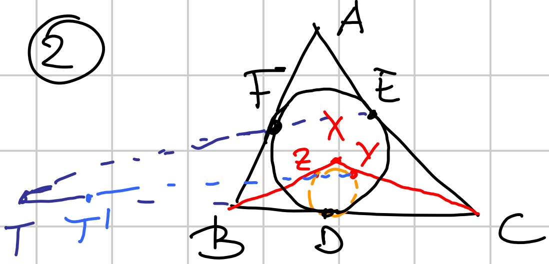
## 4 - Esercizi

① ABCD ciclico con le bisettrici di $\hat{ABC}$ e $\hat{ADC}$ che si intersecano in AC. Sia $M$ il pt medio di AC.
La retta parallela a BC per D incontra BM in E e la circonf. circoscritta ad ABCD in $F \neq D$.
Dim che BCEF è un parallelogramma.




dim: BCEF //gm $\Leftrightarrow$ X pt med di FC
$\Leftrightarrow AF \parallel BE \Leftrightarrow$ B pt med di PC

$(B,D;A,C) = -1$
$\parallel_F$
$(B, r_\infty; P, C) \Rightarrow$ B pt med di PC.
$\square$

② 

$X$ t.c. la cf. inscritta in $X\hat{B}C$ è tangente a $BC$ in $D$

$\Rightarrow EFZY$ ciclico.

Sol: $T = FE \cap BC$  $TF \cdot TE = TD^2$

$T' = ZY \cap BC$  $T'Z \cdot T'Y = T'D^2$

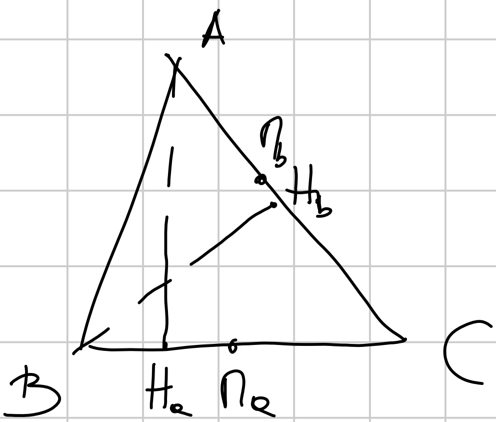
Hp: $EF, ZY, BC$ concorrono cioè $T = T'$

$AD, BE, CF$ concorrono $\Rightarrow (B, C; D, T) = -1$

$XD, BY, CZ$ concorrono $\Rightarrow (B, C; D, T') = -1$

$\Rightarrow T = T'$!

## ③ Retta di Eulero



$A_\infty$ = pt all'$\infty$ di $H_aA$

$B_\infty$ = pt all'$\infty$ di $H_bB$

$AM_aA_\infty$  $BM_bB_\infty$

$AB$

$M_aM_b$ concorrono

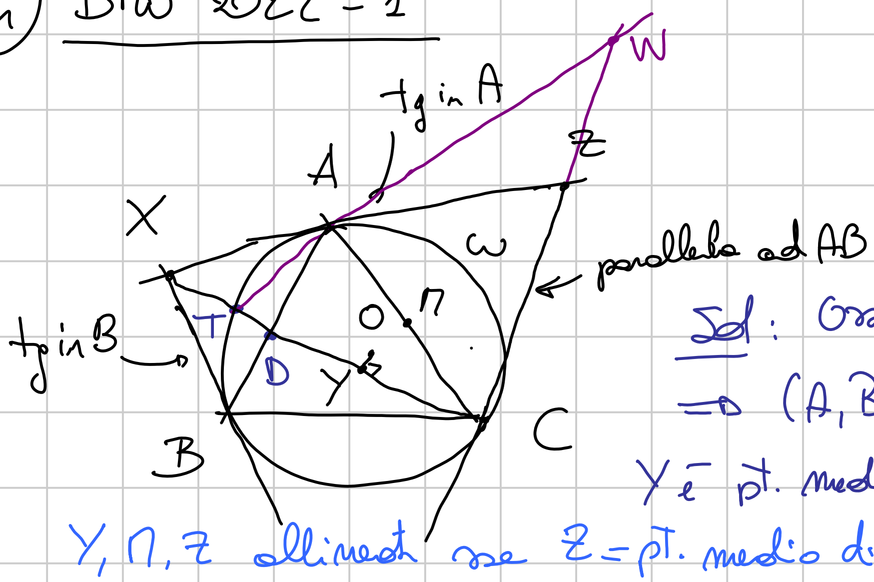
$A_\infty B_\infty$

Desargues

$\Rightarrow AA_\infty \cap BB_\infty$  $AM_a \cap BM_b$  $M_aA_\infty \cap M_bB_\infty$ sono allineati

$\| \quad H$  $\| \quad G$  $\| \quad O$

## ④ BMO 2022 - 1



th: $YZ$ biseca $AC$

Sol: Osservo che $CX$ è armonica

$\Rightarrow (A, B; C, T) = -1$

$Y$ è pt. medio di $CT$.

$Y, M, Z$ allineati sse $Z$ = pt. medio di $CW$ con $W = TA \cap CZ$.

Sia $r_\infty$ il pt all'$\infty$ di $CZ$, voglio $(W, C; Z, r_\infty) = -1$

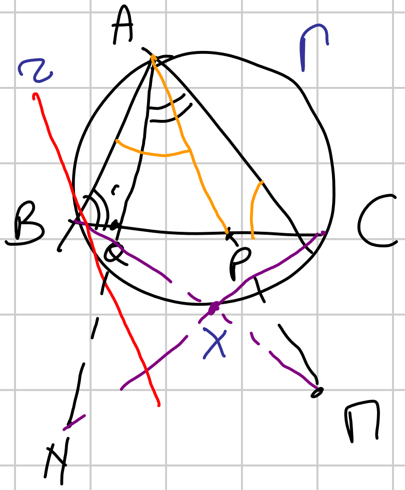
$$(W, C; Z, r_\infty) \underset{A}{=} (T, C; A, B) = -1. \ \square$$

⑤ <u>IMO 2014-4</u>

$P, Q$ su $BC$ in $ABC$ acutangolo t.c. $\widehat{PAB} = \widehat{BCA}$, $\widehat{CAQ} = \widehat{ABC}$

$M, N$ punti su $AP$ e $AQ$ t.c. $P$ pt. medio di $AM$, $Q$ pt. medio di $AN$.

Dim che $BM, CN$ si intersecano sulla circonferenza di $ABC$




<u>Sol</u>: $\widehat{AQC} = \widehat{BAC}$

$\widehat{APB} = \widehat{BAC}$ = l'angolo tra la tg a $\Gamma$ in $B$ e il lato $BC$

$\Rightarrow$ tg in $B \parallel AP$

$X = BM \cap \Gamma$

$(A, M; P, r_\infty) = -1$

$\Downarrow$ $(BA, BM; BP, BB) = -1$

$\Downarrow$ $(A, X; C, B) = -1$

Idem con $Y = CN \cap \Gamma$ e la tg in $C$ $\Rightarrow (A, Y; C, B) = -1$

$\Rightarrow X = Y$.

<u>Es</u>:

IMO SL 2017 - G4, APMO 2012-4, ELMO SL 2018-G1 (TST Romania 2018)