$k \in \mathbb{Z}$ $\quad k>0$

a^2+kb^2

$$(a^2+kb^2)(c^2+kd^2)$$

$$(a-\sqrt{k}\,b\,i)(a+\sqrt{k}\,b\,i)(c-\sqrt{k}\,d\,i)(c+\sqrt{k}\,d\,i)$$

$$((ac-kbd)-\sqrt{k}\,i\,(bc+ad))((ac-kbd)+\sqrt{k}\,i\,(bc+ad))$$

$$(ac-kbd)^2+k(bc+ad)^2=(a^2+kb^2)(c^2+kd^2)$$

$$a^2 + ab + b^2 \qquad\qquad \omega, \bar{\omega}$$

$$(a - \omega b)(a - \bar{\omega} b) \qquad\qquad x^2 + x + 1 = 0$$

$$(x - \omega)(x - \bar{\omega}) = 0$$

$$(a^2 + ab + b^2)(c^2 + cd + d^2)$$

$$(a - \omega b)(a - \bar{\omega} b)(c - \omega d)(c - \bar{\omega} d)$$

$$\left(ac + \omega^2 bd - \omega(bc + ad)\right)\left(ac + \bar{\omega}^2 bd - \bar{\omega}(bc + ad)\right)$$

$\omega^2 + \omega + 1 = 0$ PER DEFINIZIONE

$$\omega^2 = -1 - \omega$$

$$\left(ac - bd - \omega(bc + ad + bd)\right)\left(ac - bd - \bar{\omega}(bd + bc + ad)\right)$$

TEMA UTILE PER

$\forall P(x)$ t.c. se $x \geq 0 \Rightarrow P(x) \geq 0$

$P(x) = (A(x))^2 + x(B(x))^2$

$$\underbrace{a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a}$$

$$\sum_{sym} a^3 \geq \sum_{sym} a^2 b \qquad \text{NO BUNCHING}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\frac{\underbrace{a+a+a+\dots+a}_{n \text{ volte}} + b}{n+1} \geq \sqrt[n+1]{a^n b}$$

$$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$$

$$\sum a^3 \geq \sum a^2 b$$

$$\sum \frac{2a^3+b^3}{3} \geq \sum a^2 b$$

$$\sum \frac{x_1 a^3+x_2 b^3+x_3 c^3}{x_1+x_2+x_3} = a^3+b^3+c^3$$

$\downarrow$

PER AM-GM PESATO

$$\sum \frac{x_1 a^3+x_2 b^3+x_3 c^3}{x_1+x_2+x_3} \geq \sum a^{\frac{3x_1}{x_1+x_2+x_3}} b^{\frac{3x_2}{x_1+x_2+x_3}} c^{\frac{3x_3}{x_1+x_2+x_3}}$$

$$\begin{cases} 3x_1 = (x_1+x_2+x_3)\cdot 2 \\ 3x_2 = (x_1+x_2+x_3)\cdot 1 \\ 3x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = 2x_2 \\ 2x_2 = x_1 \end{cases} \qquad \begin{array}{l} x_2 = 1 \\ x_1 = 2 \end{array}$$

$$\{x_1, x_2, \ldots, x_s\}$$

$$\sum_{i=1}^{s} x^m \geq \sum_{i=1}^{s} x^n \quad \text{se} \quad m > n \quad \text{e} \quad \prod_{i=1}^{s} x_i = 1$$

— o — o —

$$p(x+p(x)) = p(x) + p(p(x))$$

$$y = p(x)$$

$$p(x+y) = p(x) + p(y) \qquad \forall x \ \forall y$$

$$p(p(x)+x) - p(p(x)) = p(x)$$

$$x \cdot p'(c_1)$$

LAGRANGE

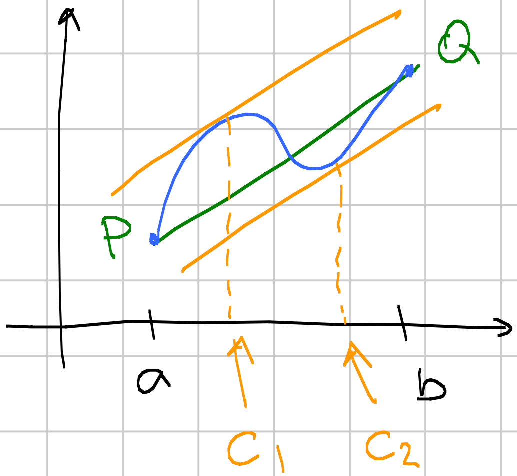
f derivabile

$$f(b) - f(a) = (b-a)\, f'(c) \qquad c \in (a,b)$$

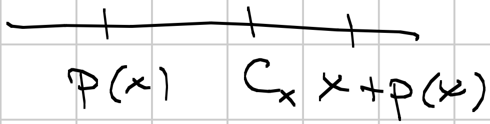
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

coeff. ang. retta

coeff. ang. della tg. al grafico in $x=c$



$$p(p(x)+x) - p(p(x)) = p(x)$$



$$p(a) - p(b) = (a-b) \cdot p'(c)$$

$$= x \cdot p'(c)$$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Supp. $a_n > 0$ per x grande $p(x) \sim a_n x^n$

$$(1-\varepsilon) a_n x^n < p(x) < a_n x^n (1+\varepsilon)$$

$$p(x) = x^n \left\{ a_n + a_{n-1} \frac{1}{x} + a_{n-2} \frac{1}{x^2} + \dots \right\}$$

$$\underbrace{p(x)}_{a_n x^n} = \underbrace{x}_{x} \cdot p'(c)$$

$p'(x) = n a_n x^{n-1} + \dots$

se $n > 1$
cresce troppo

c cresce come $p(x)$

$c \sim a_n x^n$

$p'(c) \sim n a_n (a_n x^n)^{n-1}$

$$\frac{p(x)}{x^m} = \frac{\cancel{x} \cdot p'(c(x))}{x^{m-1}[c(x)]^{m-1}} \qquad [c(x)]^{m-1} \to +\infty \qquad c(x) \geq p(x)$$

$$\frac{p(x)}{x^m} \to a_m \qquad x^{m-1}[c(x)]^{m-1} \to m a_m$$

$$\frac{p(x)}{x} \leq \frac{c(x)}{x} \leq \frac{p(x)+x}{x}$$

$$\frac{p(x)}{x} \to +\infty \qquad \frac{c(x)}{x} \to +\infty$$

Lemma (per Poloni)

$P(x)$ e $Q(x)$ a coeff. interi $\qquad$ $P(x)$ MONICO

$P(x) \mid Q(x)$ (come numeri) per infiniti valori interi di x,

Allora $\quad P(x) \mid Q(x) \quad$ come polinomi

$$Q(x) = P(x) \cdot A(x) + R(x)$$

Serve $P(x)$ MONICO per avere $R(x)$ a coeff. interi!

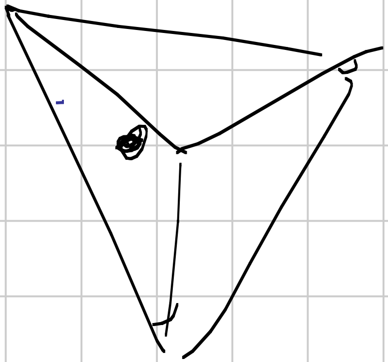
Divido per $P(x)$

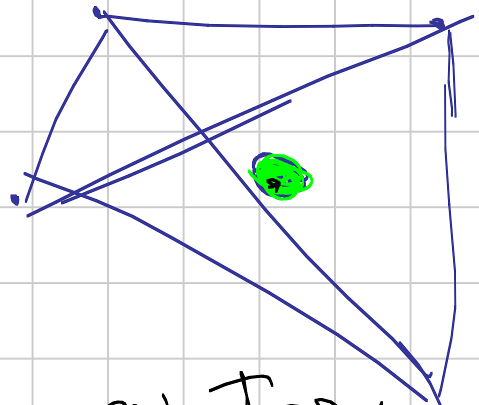
$$\frac{Q(x)}{P(x)} = A(x) + \frac{R(x)}{P(x)}$$

deg R < deg P dunque $\to 0$ per $x \to \pm\infty$

intero per ∞ valori di x

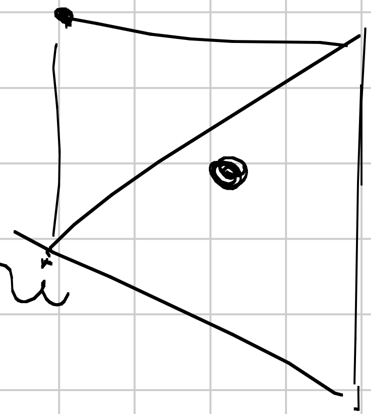
intero $\forall x$ intero



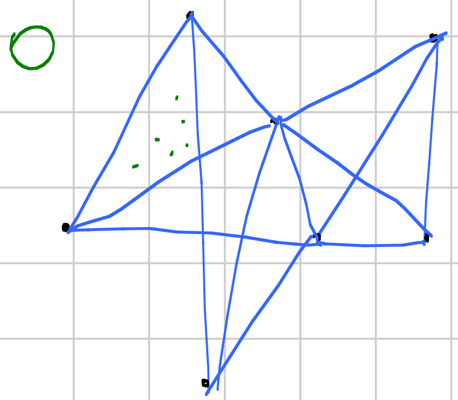


2 · numero quaterne che lo contengano

2003



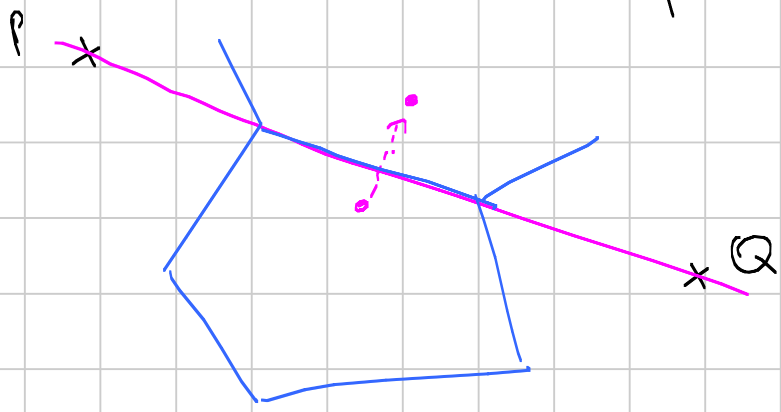
$$\frac{2 \cdot \text{numero di quaterne}}{2003} = \text{pari}$$



2006 punti

Tutti i punti di una stessa regione sono contenuti nello stesso numero di triangoli (negli stessi triangoli)

Tesi ⟺ 2006 punti dividono il piano in regioni con indice pari



Perdo i triangoli che hanno lato rosso (PQ) e 3 vertice da una parte e guadagno quelli con PQ e stesso vertice dall'altra

$$\frac{(2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1}}{4} = S_n$$

Ibero - American 88 : $(2+\sqrt{3})^{2n+1} = 1 + m^2 + k\sqrt{3}$

IAMC $\Rightarrow$ Tesi

$$S_n = \frac{1+m^2+\cancel{k\sqrt{3}} + 1+m^2-\cancel{k\sqrt{3}}}{4} = \frac{1+m^2}{2} = a^2+(a+1)^2$$

$1+m^2 = 2a^2+2a^2+4a+2$; $4a^2+4a+1 = m^2$

$m = 2a+1$

$\Rightarrow$ Trovo a purchè m sia pure dispari (e lo è chiaramente)

$$(2+\sqrt{3})^{2n+1} = 1+m^2+k\sqrt{3}$$

$$\frac{(2+\sqrt{3})^{2n+1}+(2-\sqrt{3})^{2n+1}}{2} \overset{?}{=} 1+m^2$$

$$2+\sqrt{3} = \frac{(\sqrt{3}+1)^2}{2} = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^2$$

$$1+m^2 \overset{?}{=} \frac{1}{2}\left\{\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{4n+2}+\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{4n+2}\right\}$$

Provo con $m = \frac{1}{\sqrt{2}}\left[\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{2n+1} \overset{-}{+} \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{2n+1}\right]$

DOPPIO PRODOTTO

$$m^2+1 = \frac{1}{2}\left\{\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{4n+2}+\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{4n+2} \overset{-}{+} \cancel{2\cdot 1}\right\} + \cancel{1}$$

Se n fosse intero ....

Idea m risolve una ricorrenza del 2° ordine

$$m = \frac{1}{\sqrt{2}} \left[\left(\frac{\sqrt{3}+1}{\sqrt{2}} \right)^{2n+1} \overset{-}{+} \left(\frac{\sqrt{3}-1}{\sqrt{2}} \right)^{2n+1} \right]$$

$$m = \frac{1}{\sqrt{2}} \left[\left(2+\sqrt{3}\right)^{n} \frac{\sqrt{3}+1}{\sqrt{2}} - \left(2-\sqrt{3}\right)^{n} \frac{\sqrt{3}-1}{\sqrt{2}} \right]$$

$$= \alpha R_1^n + \beta R_2^n$$

R_1, R_2 radici di pol. di II grado a coeff. interi MONICO

Basta provare $n=0$ e $n=1$.

$$f(x) = a_k X^k + a_{k-1} X^{k-1} + \dots + a_0$$

$$f(x) \equiv X^k + X^{k-1} + \dots + 1 \quad \text{modulo } 2$$

$$a(x) = b(x)\, c(x)$$

$$\Downarrow$$

$$a(x) \equiv b(x)\, c(x) \quad \text{mod } 2$$

Ipotesi più debole

$$X^k + X^{k-1} + \dots + 1 \;\Big|\; (X+1)^n - 1 \quad \text{modulo } 2$$

$$A(x) = a_m X^m + \dots + a_0 \qquad a_0 \neq 0$$

$$\tilde{A}(x) = a_0 X^m + \dots + a_m \qquad a_m \neq 0$$

$$A(x)\,B(x) = C(x)$$

$$\Downarrow$$

$$\tilde{A}(x)\,\tilde{B}(x) = \tilde{C}(x)$$

$$A(x) = a_m X^m + \dots + a_0$$

$$B(x) = b_n X^n + \dots + b_0$$

$$C(x) = c_k X^k + \dots + c_0$$

$$(a_m X^m + a_{m-1} X^{m-1} Y + \dots + a_0 Y^m)\,\cdot$$

$$(b_n X^n + b_{n-1} X^{n} Y + \dots + b_0 Y^n)$$

$$= c_k X^k + c_{k-1} X^{k-1} Y + \dots + c_0 Y^k$$

$$Y = 1$$

$$X = 1$$

$$a_n X^n + a_{n-1} X^{n-1} + \dots + a_0$$

$$\downarrow$$

$X \to \frac{1}{X}$

$$\left(a_n \frac{1}{X^n} + a_{n-1} \frac{1}{X^{n-1}} + \dots + a_0\right) X^n$$

$$\tilde{A}(X) = X^n A\left(\frac{1}{X}\right)$$

$$A(X) = a_n \prod_{i=1}^{n} (X - \alpha_i)$$

$$\tilde{A}(X) = X^n a_n \prod_{i=1}^{n} \left(\frac{1}{X} - \alpha_i\right) =$$

$$= a_n \prod_{i=1}^{n} (1 - \alpha_i X)$$

$$X^k + X^{k-1} + \dots + X + 1 \mid (X+1)^n - 1$$

$$X^k + X^{k-1} + \dots + X + 1 \mid (X+1)^n - X^n$$

$$X^n\left[\left(\frac{1}{X}+1\right)^n - 1\right] \qquad \hat{A}(X) = X^n A\left(\frac{1}{X}\right)$$

$$\shortparallel$$

$$(X+1)^n - X^n$$

$$\Rightarrow \quad X^k + X^{k-1} + \dots + X + 1 \mid X^n - 1 \quad \mathrm{mod}\ 2$$

$$X - 1 \mid X^n - 1$$

$$X^{k+1} - 1 \mid X^n - 1 \quad \mathrm{mod}\ 2$$

<u>TS</u>: $\quad k+1 \mid n$

$$n = q(k+1) + r \qquad 0 \leq r < k+1$$

Is. $r=0$

$$X^{k+1} - 1 \mid \begin{matrix} X^{2(k+1)} - 1 \\ X^{3(k+1)} - 1 \\ X^{q(k+1)} - 1 \end{matrix} \Rightarrow X^{k+1} - 1 \mid X^n - X^{q(k+1)}$$

$$X^n - X^{q(k+1)} = X^{q(k+1)}\left(X^r - 1\right)$$

$$X^{k+1} - 1 \mid X^r - 1$$

$$\begin{matrix} \parallel \\ 0 \end{matrix} \qquad X^r = 1 \qquad r = 0$$