ALGEBRA 1

Titolo nota 29/01/2009

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy+yz+zx)$$

con $\quad x+y+z=3 \qquad x,y,z \geq 0$

[1] AM-GM

$$\left(\frac{x^3}{y^3+8}, \frac{y+2}{27}, \frac{y^2-2y+4}{27}\right)$$

$$\frac{x^3}{y^3+8} + \frac{y+2}{27} + \frac{y^2-2y+4}{27} \geq 3\sqrt[3]{\frac{x^3}{\cancel{y^3+8}} \cdot \frac{\cancel{y+2}}{27} \cdot \frac{\cancel{y^2-2y}+4}{27}} =$$

$$3\sqrt[3]{\frac{x^3}{3^6}} = \frac{x}{3}$$

$$\sum_{cyc} \frac{x^3}{y^3+8} + \frac{y+2}{27} + \frac{y^2-2y+4}{27} \geq \sum_{cyc} \frac{x}{3}$$

$$\sum_{cyc}\left(\frac{x^3}{y^3+8} + \frac{y^2-y+6}{27}\right) =$$

$$= \sum_{cyc} \frac{x^3}{y^3+8} + \frac{\sum x^2 - \sum x + 18}{27} =$$

$$= \sum_{cyc} \frac{x^3}{y^3+8} + \frac{(\sum x)^2 - 2\sum xy + 15}{27} \geq \frac{\sum x}{3}$$

$$= \sum \frac{x^3}{y^3+8} + \frac{9 - 2\sum xy + 15}{27} \geq 1$$

$$\sum \frac{x^3}{y^3+8} \geq 1 - \frac{16}{27} + \frac{2}{27}\sum xy$$

$$\frac{1}{9} + \frac{2}{27}\sum xy$$

$\boxed{2}$ $lhs \geq \frac{1}{3} \geq rhs$

$$\sqrt{\frac{xy+yz+zx}{3}} \leq \frac{x+y+z}{3} = 1$$

$$xy+yz+zx \leq 3$$

$$RHS = \frac{1}{9} + \frac{2}{27}\sum xy \leq \frac{1}{9} + \frac{2}{27}\cdot 3 = \frac{1}{3}$$

$$\frac{x^3}{x^3+8} \qquad \sum \frac{x^3}{y^3+8} \geq \sum \frac{x^3}{x^3+8} \geq \frac{1}{3} \qquad x^3 \ldots \quad \frac{1}{x^3+8} \ldots$$

wlog $x \geq \sqrt[3]{4}$ $\qquad \frac{x^3}{x^3+8} \geq \frac{1}{3}$

$$x, y, z \leq \sqrt[3]{4}$$

$$\sum \frac{x^3}{x^3+8} = f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{3}{3}\right) =$$

$$= 3f(1) = 3\,\frac{1}{1+8} = \frac{1}{3}$$

Lagrange $\qquad \frac{x^3}{x^3+8} = 1 - \frac{8}{x^3+8}$

$$\sum \frac{x^3}{x^3+8} = 3 - \sum \frac{8}{x^3+8} \geq \frac{1}{3}$$

$$\sum \frac{1}{x^3+8} \leq \frac{1}{3}$$

$$(x, y, z) \in [0, 3]^3 \qquad f(x,y,z) = \sum_{cyc} \frac{1}{x^3+8}$$

$$g(x,y,z) = x+y+z-3$$

$$\Lambda(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$\begin{cases} \dfrac{\partial \Lambda}{\partial x} = -\dfrac{3x^2}{(x^3+8)^2} + \lambda = 0 \\ \text{-- -- -- -- --} \\ \overline{} \end{cases}$$

$$\frac{\cancel{3}x^2}{(x^3+8)^2} = \frac{\cancel{3}y^2}{(y^3+8)^2} \qquad \frac{x}{x^3+8} = \frac{y}{y^3+8}$$

$$(x-y)\left[xy(x+y)-8\right] = 0$$

$$xy(x+y) \le \left(\frac{x+y}{2}\right)^2 (x+y) = \frac{1}{4}(x+y)^3 \le \frac{1}{4}3^3 = \frac{27}{4} < 8$$

$x = y = z = 1$ $\qquad f(1) = \frac{1}{9}$

$$\sum \frac{1}{x^3+8} \le \frac{1}{3}$$

$z = 0$

$x = y = \frac{3}{2}$ $\qquad \sum = \frac{219}{728} < \frac{1}{3}$

$x = 3 \qquad y = z = 0$

$$\sum = \frac{39}{140} < \frac{1}{3}$$