4 $\displaystyle\sum_{cyc} \frac{1}{x+y^{20}+z^{12}} \le 1$ $\qquad x,y,z>0 \qquad xyz=1$

$$\frac{1}{x+y^{20}+z^{12}} \le \frac{x^{2c-1}+y^{2c-20}+z^{2c-12}}{(x^c+y^c+z^c)^2}$$

$$\left(\sum ax\right)^2 \le \sum a^2 \sum x^2$$

$$\frac{1}{\sum x^2} \le \frac{\sum a^2}{\left(\sum ax\right)^2}$$

$$\sum_c \frac{1}{x+y^{20}+z^{12}} \le \sum_c \frac{x^{2c-1}+y^{2c-20}+z^{2c-12}}{(x^c+y^c+z^c)^2} = \frac{\sum_c \left(x^{2c-1}+x^{2c-20}+x^{2c-12}\right)}{\left(\sum_c x^c\right)^2} \overset{?}{\le} 1$$

$$(xyz)^{\frac13} = 1$$

$$\boxed{\frac12 \sum_s x^{2c-\frac23} y^{\frac13} z^{\frac13}} + \boxed{\frac12 \sum_s x^{2c-\frac23 20} y^{\frac13 20} z^{\frac13 20}} + \boxed{\frac12 \sum_s x^{2c-8} y^4 z^4} \overset{?}{\le}$$

$$(x^c+y^c+z^c)^2 = \boxed{\frac12 \sum_s x^{2c}} + \boxed{\frac12 \sum_s x^c y^c} + \boxed{\frac12 \sum_s x^c y^c}$$

$c \ge 2c - \frac{40}{3} \qquad c \ge \frac{20}{3} \qquad \frac{20}{3} \le c \le \frac{40}{3}$

$c \ge 2c - 8 \qquad c \ge 4 \qquad 4 \le c \le 8$

Va bene $\qquad c \in \left[\frac{20}{3} ; 8\right]$

$\boxed{5}$ IRAN 1996

$$a,b,c>0 \qquad (ab+bc+ca)\sum_{cyc}\frac{1}{(a+b)^2} \geq \frac{9}{4}$$

omogenea ciclica + simmetrica → bunching + Schur

$$\overset{LHS}{\overset{\parallel}{4}}\sum_c ab \sum_c (a+b)^2(b+c)^2 - 9\prod_c (a+b)^2 \overset{?}{\geq} 0$$

$$4\sum_c (a^2+b^2+2ab)(b^2+c^2+2bc)(ab+bc+ca)$$

$$2\cancel{4}\sum_{\cancel{c}\,s}\left(a^2c^2+b^4+b^2a^2+b^2c^2+2ab^3+2abc^2+4ab^2c+2a^2bc+2b^3c\right)(ab+bc+ca)$$

$$2\sum_s\left(a^3bc^2+a^3c^3+ab^5+b^4ca+a^4b^2+a^2b^2c^2\right)$$

a^3bc^2	a^3c^3	ab^5	b^4ca	a^4b^2	$a^2b^2c^2$
26	3	2	5	4	8

a b
b c
c a

$$3\prod_c(a+b) = 3\left(2abc+\sum_s a^2b\right) = \sum_s abc + 3\sum_s a^2b$$

$$9\prod_c(a+b)^2 = 36a^2b^2c^2 + 36\sum_s a^3b^2c + 9\sum_s a^2b\left(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\right)$$

$$= 6\sum_s a^2b^2c^2 + 36\sum_s a^3b^2c + 9\sum_s\left(a^4b^2+a^4bc+a^3b^3+2a^3b^2c+a^2b^2c^2\right)$$

$$= \sum_s\left(15a^2b^2c^2+54a^3b^2c+9a^4b^2+9a^4bc+9a^3b^3\right)$$

$$LHS = \sum_s\left(-2a^3b^2c-3a^3c^3+4a^5b+a^4bc-a^4b^2+a^2b^2c^2\right) \overset{?}{\geq} 0$$

$$\underline{4\sum_s a^5b} + \sum_s a^4bc + \sum_s a^2b^2c^2 \overset{?}{\geq} 2\sum_s a^3b^2c + \underline{3\sum_s a^3b^3} + \underline{\sum_s a^4b^2}$$

$$\sum_s a^3 + \sum_s abc \overset{?}{\geq} 2\sum_s a^2 b \qquad \text{SCHUR}$$

$$p = ab+bc+ca$$

$$(a+b)(b+c) = b^2+p$$

$$(a+b)(b+c)(c+a) = (a+b+c)p - abc$$

$$4p\left(\sum (b^2+p)^2\right) \geq 9\underbrace{\left((a+b+c)p - abc\right)^2}_{p\left((a+b+c)^2 p - 2abc(a+b+c)\right) + a^2b^2c^2}$$

$$p\left(\overbrace{4\sum (b^2+p)^2 - 9(a+b+c)^2 p + 18abc(a+b+c)}\right)$$

$$\geq 9a^2b^2c^2$$

Strada 2 : metodo SPQ (uvw, ...)

Fatto : ogni polinomio simmetrico in più variabili si scrive in modo unico come polinomio delle funzioni simmetriche elementari

a, b, c

$$(x-a)(x-b)(x-c) = x^3 - \overbrace{(a+b+c)}^{S}x^2 + \overbrace{(ab+bc+ca)}^{Q}x - \overbrace{abc}^{P} = t(x)$$

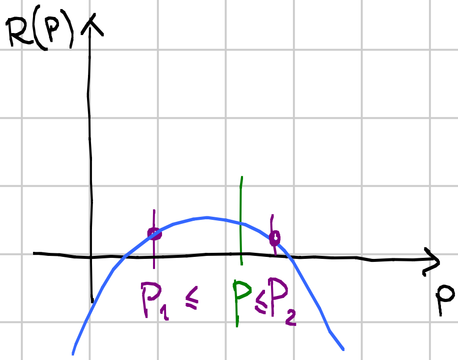
$$4\underbrace{\sum_c ab}_{Q} \sum_c (a+b)^2(b+c)^2 - 9\prod_c (a+b)^2 = R(S,P,Q) = \alpha P^2 + \beta P + \gamma$$

α, β, γ polinomi in S, Q

$$\prod (a+b)^2 = \left(2abc + \sum_s a^2 b\right)^2 = (SQ - P)^2$$

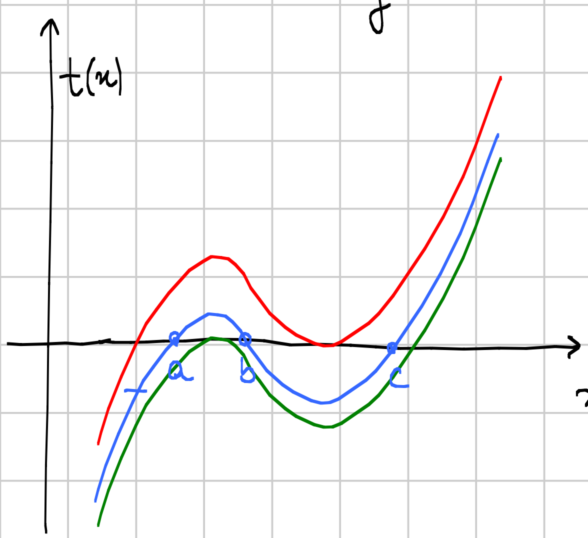
$$SQ = \sum_c a \sum_c ab = 3abc + \sum_s a^2 b$$

deduco che $\alpha = -9$ e $R(P) = -9P^2 + \beta P + \gamma \overset{?}{\gtrless} 0$



Sono dati $a, b, c \to P, Q, S$

Se trovo a_1, b_1, c_1 e a_2, b_2, c_2 tali che $S_1 = S_2 = S$
$Q_1 = Q_2 = Q$ $P_1 \leq P \leq P_2$ e inoltre riesco a dimostrare che la disug. è vera per a_1, b_1, c_1 e per a_2, b_2, c_2



P_1 corrisponde certamente ad un caso in cui $b_1 = c_1$

P_2 può essere o $b_2 = c_2$ oppure $a_2 = 0$

Devo solo verificare i casi

i. $a = 0$

ii. $b = c = 1$ (facili)

Strada 3

$$0 \le \sum_{cyc} a(a-b)(a-c) - \frac{1}{4}\,\frac{(a-b)^2(b-c)^2(c-a)^2}{(a+b)(b+c)(c+a)} = (ab+bc+ca)\sum_{cyc}\frac{(b+c)(c+a)}{(a+b)} - \frac{9}{4}(a+b)(b+c)(c+a)$$

$a \le b \le c$

$u = b-a \qquad v = c-b$

a, u, v esercizio per casa

$\boxed{6}$ $a, b, c, d > 0$

$$(a+b)(b+c)(c+d)(d+a)\Big(1+\overset{\substack{G\\ \shortparallel}}{\sqrt[4]{abcd}}\Big)^4 \ge 16abcd(1+a)(1+b)(1+c)(1+d)$$

$$\frac{\prod_{cyc}(1+a)}{(1+G)^4} \overset{?}{\le} \prod_{cyc}\frac{a+b}{2\sqrt{ab}} \qquad \alpha = \frac{a}{G} \qquad \beta = \frac{b}{G} \quad \dots$$

$$\alpha\beta\gamma\delta = \frac{abcd}{G^4} = 1$$

$$\frac{\prod_{cyc}(1+G\alpha)}{(1+G)^4} \overset{?}{\le} \prod_{cyc}\frac{\alpha+\beta}{2\sqrt{\alpha\beta}}$$

$$\overbrace{(\alpha+\beta)(\beta+\gamma)(\gamma+\delta)(\delta+\alpha)}^{16y}(1+G)^4 \ge 16(1+\alpha G)(1+\beta G)(1+\gamma G)(1+\delta G)$$

$$y(1+4G+6G^2+4G^3+G^4) - \Big(1+\sum_{\alpha}\alpha G + \sum_{\alpha<\beta}\alpha\beta G^2 + \sum_{\delta}\alpha\beta\gamma G^3 + G^4\Big) \overset{?}{\ge} 0$$

$$(y-1) + \Big(4y-\sum\alpha\Big)G + \Big(6y-\sum\alpha\beta\Big)G^2 + \Big(4y-\sum\alpha\beta\gamma\Big)G^3 + (y-1)G^4 \overset{?}{\ge} 0$$

$y \overset{?}{\ge} 1 \qquad 4y \overset{?}{\ge} \sum\alpha \qquad 4y \overset{?}{\ge} \sum\alpha\beta\gamma \qquad 6y \overset{?}{\ge} \sum\alpha\beta$

$$16y = 2 + \alpha^2(\gamma^2+\beta\delta+\beta\gamma+\gamma\delta) + \dots$$
$$= 2 + \frac{1}{2}\sum_{s}\alpha^2\beta\gamma + \alpha^2\gamma^2 + \beta^2\delta^2$$

$\alpha \quad \beta$
$\beta \quad \gamma$
$\gamma \quad \delta$
$\delta \quad \alpha$

$$(\sum \alpha\beta)^2 \qquad (\alpha\beta\gamma\delta)^{3/4} \sum \alpha$$

$$\sum\alpha \sum\alpha\beta\gamma = 4 + \frac{1}{2}\sum_{s} \alpha^2\beta\gamma$$

$$16\eta - \sum\alpha\sum\alpha\beta\gamma = \alpha^2\gamma^2 + \beta^2\delta^2 - 2\alpha\beta\gamma\delta = (\alpha\gamma - \beta\delta)^2$$

$$16\eta \geq \sum\alpha\sum\alpha\beta\gamma \qquad \sum\alpha^m \geq 4 \qquad m \in \mathbb{R}$$

$$\sqrt[4]{\alpha\beta\gamma\delta} \leq \sqrt[3]{\frac{\sum\alpha\beta\gamma}{4}} \leq \boxed{\sqrt{\frac{\sum\alpha\beta}{6}} \leq \frac{1}{4}\sum\alpha}$$

$$16\eta \geq \sum\alpha\sum\alpha\beta\gamma \geq 4\sum\alpha \geq 16 \qquad \sum\alpha\beta\gamma \geq 4$$

$$\geq 4\sum\alpha\beta\gamma$$

HOPE

$$\sqrt{\frac{\sum\alpha\beta}{6}} \leq \frac{1}{4}\sum\alpha\beta\gamma$$

$$\frac{1}{4}\sum\alpha\beta\gamma = \frac{1}{4}\sum\frac{1}{\alpha} \geq \sqrt{\frac{\sum\frac{1}{\alpha\beta}}{6}} = \sqrt{\frac{\sum\alpha\beta}{6}}$$