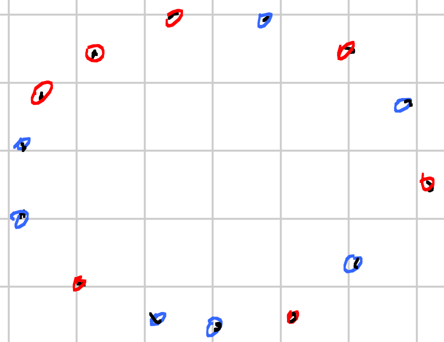
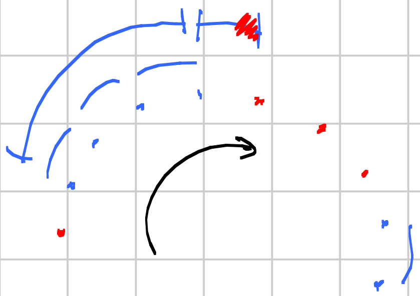
WC13 – Misc + avanzi di TdN

Titolo nota 01/02/2013



È equivalente contare le distanze "k"

Comincio con $k=1$ Considero R rossi e $B = 2n - R$ blu.



dist. blu da 1 = somma (lunghezze degli intervalli blu (-1))

dist. rosse = analogo

transizioni blu-rosso = # interv. blu

transizioni rosso-blu = " rossi

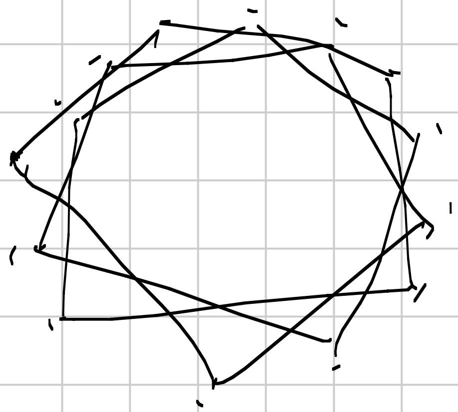
Σ lungh. interv. blu + Σ lungh. interv. rossi +
+ # transiz. blu-rosso + # transiz. rosso-blu = $2n$

Σ lungh. interv. blu + # transiz. blu-rosso =
= # blu = B

Σ " rossi + " rosso-blu =
= # rossi = R

dist. blu − # dist. rosse = $B - R$

Così le distanze 1 sono ok in generale.



Saltando di k in k i punti vengono ripartiti in classi. Ciascuna classe è "isomorfa" a un poligono con punti colorati a dist. 1 se prima era k.

In ciascuna classe le dist. blu − dist. rosse = $B_i - R_i$

\# dist. blu da k iniziali − \# dist. rosse " " = $\sum$ \# dist. blu da 1 nelle classi − $\sum$ \# dist. rosse " " =

$= \sum (B_i - R_i) = \sum B_i - \sum R_i = 0.$

$\boxed{Es. \; 5}$

PRIMO LEMMA | se $q(x)$ ha grado $< n$, allora

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot q(k) = \Delta_1^n q(0) = 0$$

Dim. (PER INDUZIONE) $\Delta_1 q(x) = q(x+1) - q(x)$

So che il grado di $\Delta_1 q$ è esattamente il grado di q meno 1

$$\deg(\Delta_1 q) = \deg(q) - 1$$

$$q(x) = a_d x^d + \text{(termini di grado inferiore)}$$

$$q(x+1) - q(x) = \underbrace{a_d (x+1)^d - a_d x^d}_{\substack{a_d \cdot \left((x+1)^d - x^d\right) \\ \shortparallel \\ a_d \cdot \left(\boxed{d \cdot x^{d-1}} + \dots\right)}} + \text{(termini di grado} \dots)$$

$\downarrow$ ci sono solo termini di grado $\leq d-2$

$$\deg(\Delta_1 q) = \deg(q) - 1$$

Coefficiente di testa di $\Delta_1 q$ $= d \cdot$ (coeff. di testa di q)

$$\Delta_1^n q = \underbrace{\Delta_1(\Delta_1(\Delta_1 \dots}_{n \text{ volte}} (\Delta_1 q)))) = 0$$

$\downarrow$ se e solo se $\deg(q) < n$

$$\Delta_1 q(x) = q(x+1) - q(x)$$

$$\Delta_1^2 q(x) = \big(q(x+2) - q(x+1)\big) - \big(q(x+1) - q(x)\big)$$

$$= q(x+2) - 2q(x+1) + q(x)$$

$$\Delta_1^3 q(x) = q(x+3) - 3q(x+2) + 3q(x+1) - q(x)$$

$$\Delta_1^n q(x) = \sum_{k=0}^{n} \binom{n}{k} q(x+k) \cdot (-1)^k$$

$$\Delta_\alpha q(x) = q(x+1) - \alpha q(x)$$

$$\Delta_\alpha \Delta_\beta q(x) = 1\, q(x+2) - (\alpha+\beta)\, q(x+1) + \alpha\beta\, q(x)$$

$$p(x) = \begin{cases} a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ a_n (x-\alpha_1)(x-\alpha_2) \dots (x-\alpha_n) \end{cases}$$

$$\sum_{k=0}^{n} a_k q(x+k) = a_n \Delta_{\alpha_1} \Delta_{\alpha_2} \underbrace{\Delta_{\alpha_3} \dots \Delta_{\alpha_n}}_{k+1} q(x)$$

Primo lemma (generalizzato) Se $(x-1)^{k+1} \mid p(x)$, allora, $p(x) = x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ $(a_n = 1)$

$$\sum_{k=0}^{n} a_k q(x+k) = 0$$

se (e solo se) $\deg q \leq k$.

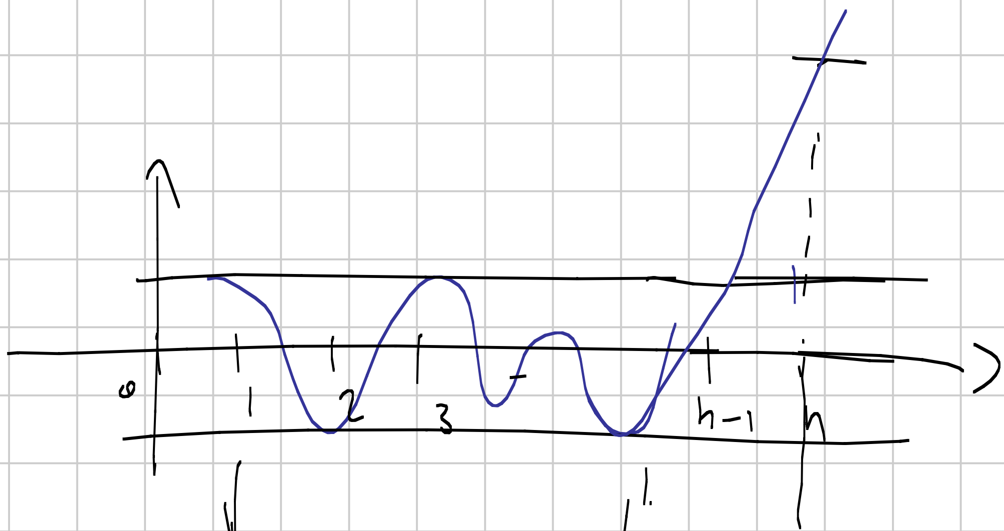
$$\sum |a_k| \geq ? \qquad \sum_{k=0}^{n} a_k q(k) = 0$$

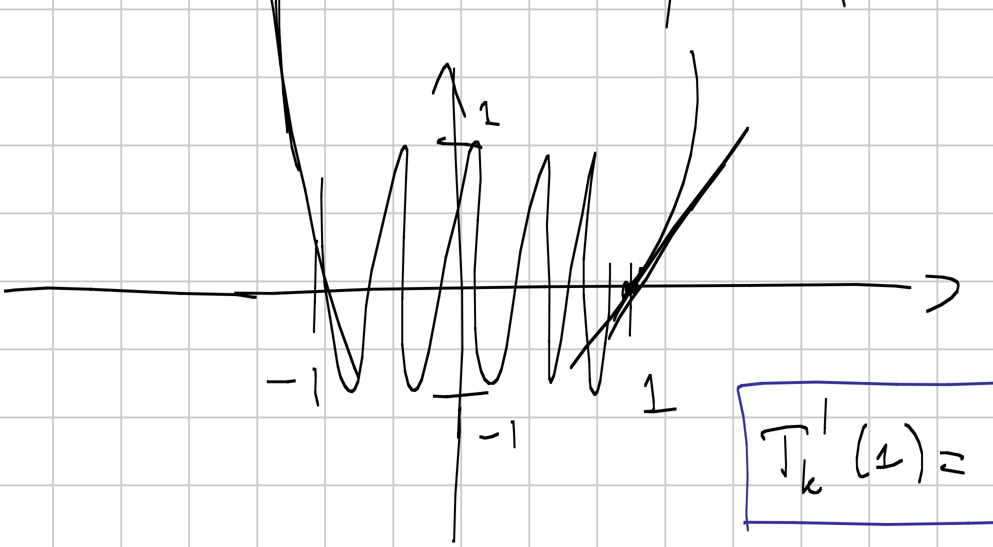
$$\sum_{k=0}^{n-1} a_k q(k) = -q(n) \qquad (*)$$

q di grado $\leq k$ quindi soddisfa $(*)$,

e tale che $|q(i)| \leq 1$ se $i=0,\dots,n-1$

$$1 + \sum_{k=0}^{n-1} |a_k| \geq -\sum a_k q(k) = q(n)+1$$





$$\boxed{T_k'(1) = k^2}$$

$$T_k(1+\varepsilon) \geq 1 + k^2 \varepsilon$$

Polinomi di Tchebyschev: $T_k(x)$ di grado k, tali che $T_k([-1,1]) = [-1,1]$

$$\underbrace{T_n\left(\frac{2}{n-1}x-1\right)}_{-1,\ \dots,\ 1}$$

$x \in [0, n-1] \longrightarrow \frac{2}{n-1}x - 1 \in [-1,1]$

$T_k(x) = 0$

$k\theta = \frac{\pi}{2} + 3\pi$

$\theta = \left(\frac{\frac{\pi}{2}+3\pi}{k}\right)$

$$\boxed{T_k(\cos\theta) = \cos(k\theta)}$$

$T_1(\cos\theta) = \cos\theta$ $\qquad T_1(x) = x$

$T_2(\cos\theta) = \cos 2\theta = 2\cos^2\theta - 1$ $\qquad T_2(x) = 2x^2 - 1$

$T_3(\cos\theta) = \cos 3\theta = 4\cos^3\theta - 3\cos\theta$ $\qquad T_3(x) = 4x^3 - 3x$

$$\cos(k+1)\theta + \cos(k-1)\theta = 2\cos(k\theta)\cos\theta$$

$$T_{k+1}(x) + T_{k-1}(x) = 2x\, T_k(x)$$

$$T_k\left(\frac{t+\frac{1}{t}}{2}\right) = \frac{t^k + \frac{1}{t^k}}{2}$$

$t = e^{i\theta}$

$$T_k\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \frac{e^{ik\theta}+e^{-ik\theta}}{2} = \cos k\theta$$

$$\left|T_k\left(\frac{2}{n-1}x-1\right)\right| \le 1 \quad \text{per} \quad x=0,1,2,\dots,n-1$$

$$T_k\left(\frac{2}{n-1}n-1\right) = T_k\left(\frac{n+1}{n-1}\right)$$

Somma che voglio stimare $\geq 1 + \boxed{T_k\left(\frac{n+1}{n-1}\right)} \geq 2 + \frac{k^2\cdot 2}{n-1}$

$(T_k \geq k^2\varepsilon)$

$$T_k\left(\frac{\rho+\frac{1}{\rho}}{2}\right) = \frac{\rho^k+\frac{1}{\rho^k}}{2}$$

$$\frac{\rho+\frac{1}{\rho}}{2} = \frac{n+1}{n-1} = 1+\varepsilon = 1+\frac{2}{n-1}$$

$$\rho^2+1-2(1+\varepsilon)\rho=0$$

$$\rho^k = \left(1+\varepsilon+\sqrt{\varepsilon^2+2\varepsilon}\right)^k$$

$$\frac{1}{2}\left(\left(1+\varepsilon+\sqrt{\varepsilon^2+2\varepsilon}\right)^k + \frac{1}{\left(1+\varepsilon+\sqrt{\varepsilon^2+2\varepsilon}\right)^k}\right) =$$

$$= \frac{1}{2}\left(\left(1+\varepsilon+\sqrt{\varepsilon^2+2\varepsilon}\right)^k + \left(1+\varepsilon-\sqrt{\varepsilon^2+2\varepsilon}\right)^k\right) \geq$$

$$(\alpha+\sqrt{\beta})^k + (\alpha-\sqrt{\beta})^k$$

$$\alpha^k \cancel{+ k\alpha^{k-1}\sqrt{\beta}} + \binom{k}{2}\alpha^{k-2}(\sqrt{\beta})^2 \cancel{+ \dots} \qquad \alpha^k \cancel{- k\alpha^{k-1}\sqrt{\beta}} + \binom{k}{2}\alpha^{k-2}(\sqrt{\beta})^2 \cancel{- \dots}$$

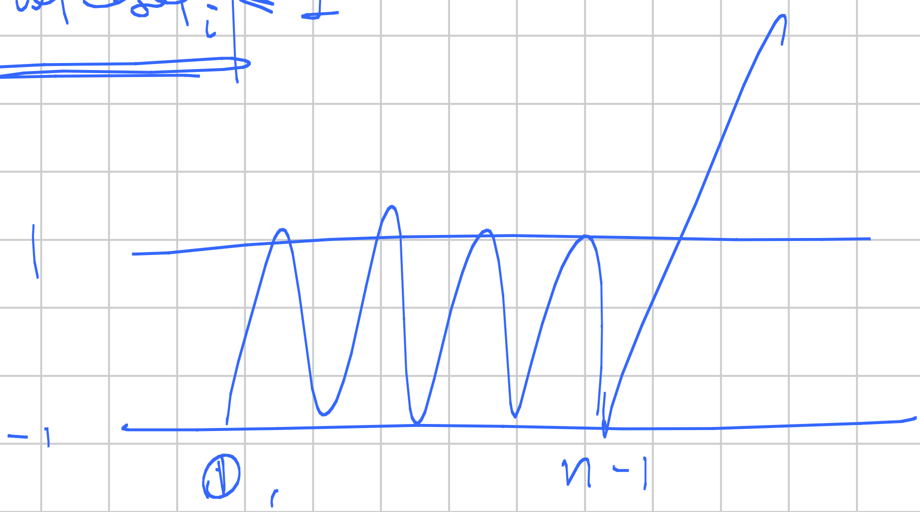
$$= \frac{1}{\cancel{2}}\left(\cancel{2}\cdot(1+\varepsilon)^k + \cancel{2}\binom{k}{2}\underbrace{\left(\sqrt{\varepsilon^2+2\varepsilon}\right)^2}_{\geq 2\varepsilon}\cdot\underbrace{(1+\varepsilon)^{k-2}}_{\geq 1}\right)$$

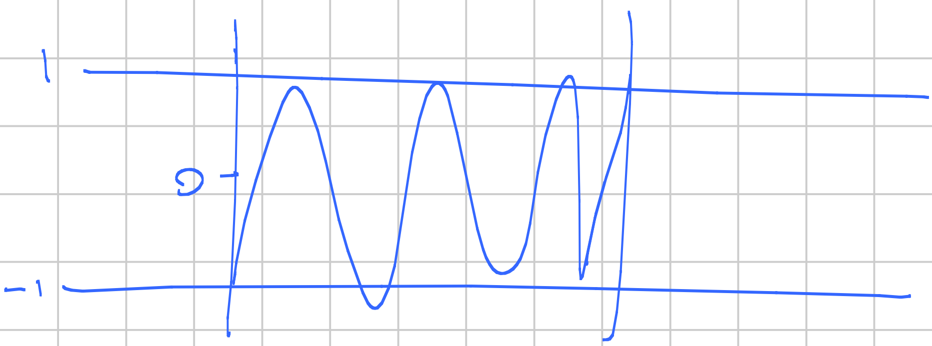
$$\geq 1 + k\varepsilon + \binom{k}{2}2\varepsilon = 1 + k^2\varepsilon$$

$$1 + |a_1| + |a_2| + \dots \geq$$

$$\geq 1 + a_1\cdot(\text{qualcosa})_1 + a_2\cdot(\text{qualcosa})_2 + \dots$$

$\underline{|\text{qualcosa}_i| \leq 1}$





__Problema__: tra tutti i poly. __monici__ di grado K, qual è quello che ha $\max_{x \in [-1,1]} |p(x)|$ più piccolo?

R: polinomi di Chebyshev

$\boxed{\text{Esercizio n. 6}}$

Scacchiera $n \times n$

caselle bianche e nere

$\begin{matrix} c_1 & & c_2 \\ B & & N \\ & & c_3 \\ & & B \end{matrix}$ $\quad T = \#\{(c_1, c_2, c_3)\}$

$(i, j) \to$ casella nera

a_i = n° bianche nella riga i

b_j = n° bianche nella colonna j

Data $c_2 \to$ $a_i b_j$ scelte per (c_1, c_3)

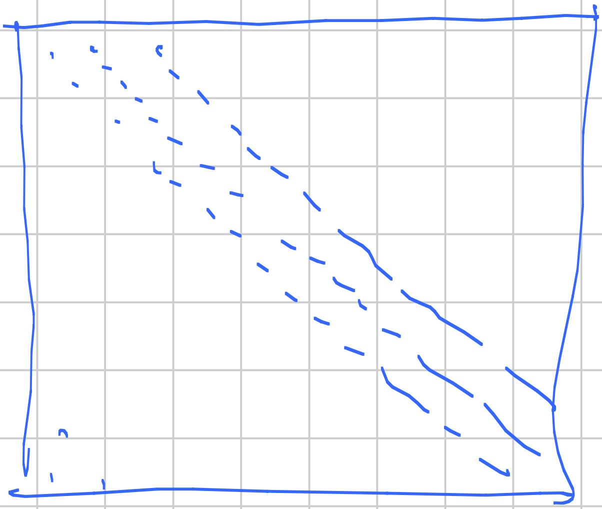
$$T = \sum_{(i,j) \in N} a_i b_j$$

$$T \leq \frac{1}{2} \sum_{(i,j) \in N} (a_i^2 + b_j^2)$$

$$= \frac{1}{2}\left\{\sum_{i=1}^{n} (n-a_i)a_i^2 + \sum_{j=1}^{n} (n-b_j)b_j^2\right\}$$

$$\frac{1}{2} \quad 2(n-a_i) \quad a_i \quad a_i$$

$$\leq \frac{n}{4}\left(\frac{2n}{3}\right)^3 + \frac{n}{4}\left(\frac{2n}{3}\right)^3 = \frac{4n^4}{27}.$$



2/3 diagonali

TdN 6

n intero positivo

Equazione $n x^2 + y^3 = z^4$

Ts. infinite sol. con $(x,y) = (y,z) = (z,x) = 1$

$$y^3 = z^4 - n x^2$$

$$u^2 - d v^2 = 1 \qquad (u + \sqrt{d}v)^k (u - \sqrt{d}v)^k = 1$$

Scegliamo y di una forma speciale

$$y = s^2 - nt^2 = (s - \sqrt{n}t)(s + \sqrt{n}t)$$

$$y^3 = (s + \sqrt{n}t)^3 (s - \sqrt{n}t)^3$$

$$z^2 = s^3 + 3nst^2 \qquad x = 3s^2t + nt^3$$

Di nuovo, caso particolare : $s = 1$

$$z^2 - \underbrace{3n}t^2 = 1 \qquad \text{se } 3n \neq \square .$$

Esempio $(x, z) = 1$?

$$(x, z) \mid (x, z^2) = (t(3 + nt^2), 1 + 3nt^2)$$

$\Downarrow$ divide 8

Infine, se $3n = m^2$

Una scelta possibile è

$$s = u^2 \qquad z = uv$$

$$u^4 = v^2 - mt^2 = \underset{\underset{u^4}{\shortparallel}}{(v + mt)} \underset{\underset{1}{\shortparallel}}{(v - mt)}$$