WC 2014

ALGEBRA

Titolo nota 31/01/2014

$\boxed{1}$ $$\sum_{cyc} \frac{1}{a(a+1)+ab(ab+1)} \geq \frac{3}{4} \qquad abc=1 \qquad a,b,c>0$$

$$a=\frac{x}{y} \qquad b=\frac{y}{z} \qquad c=\frac{z}{x} \qquad x,y,z>0$$

$$\sum_{cyc} \frac{1}{\frac{x}{y}\left(\frac{x}{y}+1\right)+\frac{x}{z}\left(\frac{x}{z}+1\right)} = \sum_{cyc} \frac{y^2z^2}{x^2z^2+xyz^2+x^2y^2+xzy^2} \overset{?}{\geq} \frac{3}{4}$$

simmetrica in 3 variabili $\longrightarrow$ conti ... Bunching + Schur (384 termini di grado 12)

$\hookrightarrow$ PQS $\qquad P=xyz \quad Q=\sum_c xy \quad S=\sum_c x$

Provo $2xy \leq x^2+y^2$

$$LHS \geq \sum_{cyc} \frac{2\overbrace{y^2z^2}^{\alpha}}{2\underbrace{x^2z^2}_{\beta}+x^2z^2+y^2z^2+2\underbrace{x^2y^2}_{\gamma}+x^2y^2+z^2y^2} = \sum_{cyc} \frac{2\alpha}{2\alpha+3\beta+3\gamma} \overset{?}{\geq} \frac{3}{4}$$

$\rightarrow$ conti grado 3 162 termini (fattibile)

Lemma di Titu (Cauchy-Schwarz)

$$\sum_{cyc} \frac{a^2}{x} \geq \frac{(\sum a)^2}{\sum x}$$

fiducia $\leftarrow$ troppo

$$\sum_{cyc} \frac{2y^2z^2}{2y^2z^2+3x^2z^2+3x^2y^2} \geq \frac{2\left(\sum yz\right)^2}{\sum_{cyc} 8x^2y^2} = \frac{\sum y^2z^2+2\sum x^2yz}{4\sum x^2y^2} \overset{?}{\geq} \frac{3}{4} \quad \text{NO}$$

$$\sum \frac{2\alpha^2}{2\alpha^2+3\alpha\beta+3\alpha\gamma} \geq \frac{\sum\alpha^2+2\sum\alpha\beta}{\sum\alpha^2+3\sum\alpha\beta} \overset{?}{\geq} \frac{3}{4}$$

Titu più furbo

$$4\sum\alpha^2+8\sum\alpha\beta \overset{?}{\geq} 3\sum\alpha^2+9\sum\alpha\beta \qquad \sum\alpha^2 \overset{!}{\geq} \sum\alpha\beta$$

$\boxed{2}$ $f(xf(y)) + y + f(x) = f(x+f(y)) + yf(x)$ $x, y \in \mathbb{R}$

$x \leftarrow 0$ $2f(0) + y = f(f(y)) + yf(0)$

$2f(0) + y(1 - f(0)) = f(f(y))$

se $f(0) \neq 1$ f è biunivoca

caso $f(0) = 1$ $f(f(y)) \equiv 2$ $\forall y$

$y \leftarrow 0$ $2f(x) = f(x+1)$ $x \leftarrow 0$, $x \leftarrow 1$ $4 = 2f(1) = f(2) = f(f(1)) = 2$

$y \leftarrow \alpha$: $f(\alpha) = 0$ $\alpha = f^{-1}(0)$

$f(0) + \alpha + \cancel{f(x)} = \cancel{f(x)} + \alpha f(x)$ $\alpha f(x) \equiv f(0) + \alpha$ $\forall x$

$\Rightarrow \alpha = 0$

$\boxed{f(0) = 0}$ $\boxed{f(f(y)) = y}$ $\boxed{f = f^{-1}}$

$y \leftarrow f(y) = f^{-1}(y)$

$$\boxed{f(xy) + f(x) + f(y) = f(x+y) + f(x)f(y)}$$

$x \leftarrow 1$ $2f(y) + f(1) = f(y+1) + f(1)f(y)$

$y \leftarrow 1$ $3f(1) = f(2) + f(1)^2$ $f(1) = 1$

$xy = x + y \iff (x-1)(y-1) = 1$

$x = y = 2$ $2f(2) = f(2)^2$ $\Rightarrow f(2) = 2$

$\cancel{2}f(y) + 1 = f(y+1) + \cancel{f(y)}$ $\boxed{f(x+1) = f(x) + 1}$

$y \leftarrow y+1$ $f(xy+x) + \cancel{f(x)} + f(y) + \cancel{1} = f(x+y) \cancel{+1} + f(x)f(y) + \cancel{f(x)}$

$= f(xy) + f(x) + f(y)$

$$f(xy+x) = f(xy) + f(x)$$

$a,b \in \mathbb{R}$ $\quad a=b=0 \quad f(a+b) = f(a)+f(b)$

wlog $a \neq 0 \quad x \leftarrow a \quad y \leftarrow \frac{b}{a} \quad xy = b \qquad f(a+b) = f(a)+f(b)$

$f(xy) = f(x)f(y) \qquad f(x^2) = f(x)^2 \qquad$ IV quadrante del grafico è $\emptyset$

$f(x) = \lambda x \quad$ sostituisco .... trovo $\quad f(x) = x$

(3) $\quad a,b,c,d \geq 0$

$$(ab)^{\frac{1}{3}} + (cd)^{\frac{1}{3}} \overset{?}{\leq} (a+c+d)^{\frac{1}{3}}(a+c+b)^{\frac{1}{3}}$$

somme , prodotti (qualche radice per compensare)

- Voglio usare AM-GM !
  (prodotto $\leq$ somma)

A destra ho dei prodotti $\Rightarrow$ non va bene

IDEA 1 togli quei prodotti

Se $a+b+c=0$ o $a+d+c=0$ facile

Altrimenti divido

$$\left(\frac{ab}{(a+b+c)(a+d+c)}\right)^{\frac{1}{3}} + \left(\frac{cd}{(a+b+c)(a+d+c)}\right)^{\frac{1}{3}} \overset{?}{\leq} 1$$

- MOLTIPLICO E DIVIDO $\underbrace{}_{\text{per }(a+c)}$ PER AVERE 3 TERMINI PER AM-GM

$$\overset{\text{AM-GM}}{\leq} \frac{1}{3}\left(\frac{a}{a+c} + \frac{a+c}{a+c+d} + \frac{b}{a+c+b}\right)$$

$$\leq \frac{1}{3}\left(\frac{c}{a+c} + \frac{a+c}{a+c+b} + \frac{d}{a+c+d}\right)$$

(MAGIA!)

SOMMO : $\quad$ LHS $\leq \frac{1}{3}(1+1+1) = 1 \quad$ OK

- IDEA 2 AM-GM al contrario!

$$\sqrt{x\cdot y} \leq \frac{x+y}{2}$$

MA! $\sqrt{xy} = \frac{x+y}{2}$ quando $x=y$

$A = a+c+b$
$B = a+c+d$

se $Ax = By = \frac{1}{xy} \left(= \sqrt[3]{AB} \right)$

$$\sqrt[3]{AB} = \sqrt[3]{Ax\,By \cdot \frac{1}{xy}} = \frac{Ax + By + \frac{1}{xy}}{3} \overset{?}{\geq} \sqrt[3]{ab} + \sqrt[3]{cd}$$

$\rightarrow x = \frac{\sqrt[3]{AB}}{A} \quad y = \frac{\sqrt[3]{AB}}{B}$

$$\frac{1}{3}\left\{ (a+c+b)\,x + (a+c+d)\,y + \left(\frac{1}{x}+\frac{1}{y}\right)\frac{1}{x+y} \right\}$$

$$\frac{1}{3}\left(a\,(x+y) + \frac{1}{x}\frac{1}{x+y} + bx\right) + \frac{1}{3}\left(c(x+y) + \frac{1}{y}\frac{1}{x+y} + dy\right) \geq$$ AM-GM vera

$$\sqrt[3]{ab} + \sqrt[3]{cd}$$

C. S. : esercizio, dimostratelo con questo trucco!

$$\left(\sum a_i b_i\right) \leq^{x} \left(\sum a_i^2\right)^{\frac{1}{2}} \left(\sum b_i^2\right)^{\frac{1}{2}} \cdot \frac{1}{x}$$

Esercizio : dimostrate Hölder generalizzato con questo trucchetto!

$$\left(\sum a_i^{(1)} \cdot a_i^{(2)} \cdot \ldots \cdot a_i^{(n)}\right)^{\frac{1}{r}} \leq \left(\sum \left(a_i^{(1)}\right)^{p_1}\right)^{\frac{1}{p_1}} \cdot \ldots$$

$$\ldots \left(\sum \left(a_i^{(n)}\right)^{p_n}\right)^{\frac{1}{p_n}}$$ se $\boxed{\frac{1}{p_1} + \ldots + \frac{1}{p_n} = \frac{1}{r}}$

$\left(\text{C.S.} \qquad \frac{1}{2} + \frac{1}{2} = 1 \right)$

- sol. semplice con Hölder

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\sqrt[3]{ab} + \sqrt[3]{cd} = \qquad \text{Hölder}$$

$$= \sqrt[3]{\frac{a}{a+c}} \sqrt[3]{b} \sqrt[3]{a+c} + \sqrt[3]{\frac{c}{a+c}} \sqrt[3]{d} \sqrt[3]{a+c} \le$$

$$\underset{1}{} \quad \underset{2}{} \quad \underset{3}{} \qquad \underset{1}{} \quad \underset{3}{} \quad \underset{2}{}$$

$$\underbrace{\left(\frac{a}{a+c} + \frac{c}{a+c} \right)}_{\substack{\shortparallel \\ 1}}^{\frac{1}{3}} \left(b + a + c \right)^{\frac{1}{3}} \left(a + c + d \right)^{\frac{1}{3}} \qquad \checkmark$$

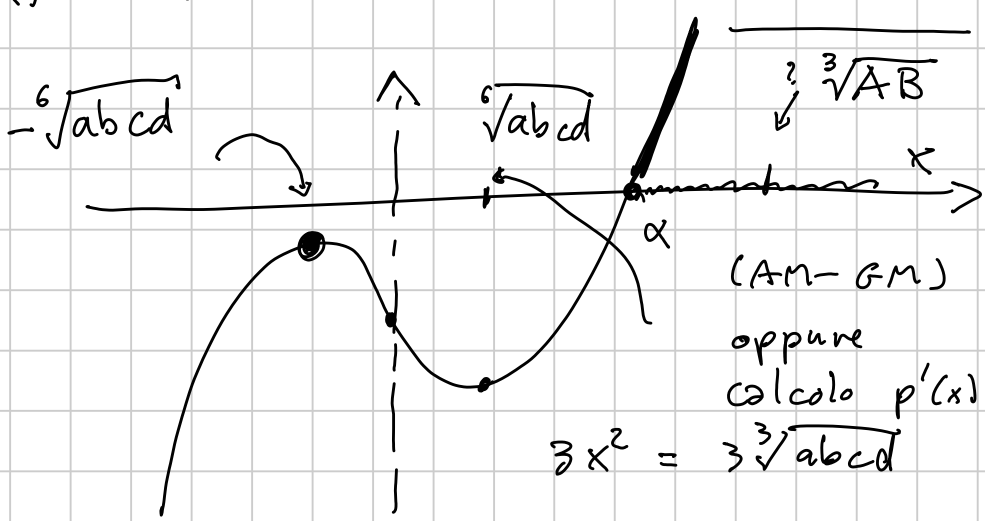
- $\boxed{\text{VERTECHI WAY}}$

$$\alpha = \text{LHS} = \sqrt[3]{ab} + \sqrt[3]{cd}$$

$$\alpha^3 = ab + cd + 3 \sqrt[3]{abcd}\, \alpha$$

$\longrightarrow$ α e' radice del polinomio

$$p(x) = x^3 - 3\sqrt[3]{abcd}\, x - (ab+cd)$$



(AM-GM)
oppure
calcolo $p'(x) = 0$

$$3x^2 = 3\sqrt[3]{abcd}$$

$$p\left(-\sqrt[6]{abcd}\right) = -\sqrt{abcd} + 3\sqrt{abcd} \mp ab - cd \overset{?}{\leq} 0$$

AM-GM

abbiamo:

$$\sqrt[3]{AB} \geq \alpha \qquad \text{(tesi)}$$

$$\Updownarrow$$

$$p\left(\sqrt[3]{AB}\right) \geq 0$$

fai il conto AM-GM

$\blacksquare$