WC 2015

ALGEBRA

Titolo nota 29/01/2015

$\boxed{A4}$ $f:\mathbb{Q}^+ \longrightarrow \mathbb{R}^+$ $\forall x,y$

$$f(xy) = f(x+y)\big(f(x)+f(y)\big) \qquad f(x)+f(y) = \frac{f(xy)}{f(x+y)}$$

$f(2) = \frac{1}{2}$ $\qquad f(x) = \frac{1}{x}$ è una soluz.

$g(x) := \frac{1}{f(x)}$ $\qquad h(x) := f\left(\frac{1}{x}\right)$

$$\boxed{g(x+y)g(x)g(y) = g(xy)\big(g(x)+g(y)\big)}$$

$y \leftarrow 1$ $\qquad g(x+1) = 1 + a\,g(x)$ $\qquad a = \frac{1}{g(1)}$

$g(2) = 2$ $\qquad g(3) = 1+2a$ $\qquad g(4) = 1+a+2a^2$

$$\boxed{g(n) = 1+a+\dots+a^{n-3}+2a^{n-2}}$$

$$1+a+a^2+a^3+2a^4 = g(6) = \frac{\overset{\deg=3}{g(5)}\,\overset{2}{g(2)}\,g(3)}{\underset{2}{g(2)}+\underset{\deg=1}{g(3)}} \quad \dots \; p(a)=0$$

$p(1) = 0$ $\qquad p(x) = (x-1)\,q(x)$

Caso $a>1$ $\qquad g(n) = a^{n-2} + \frac{a^{n-1}-1}{a-1} = \frac{2a^{n-1}-a^{n-2}-1}{a-1}$

$$\frac{1}{a-1}a^{n-1} \leq g(n) \leq \frac{2}{a-1}a^{n-1}$$

vera per n abbastanza grande

$x \leftarrow n$ $\qquad y \leftarrow n$

$$\frac{4}{(a-1)^2}a^{3n-2} \geq g(2n)\,g(n) = 2g(n^2) \geq 2\cdot\frac{1}{a-1}a^{n^2-1}$$

assurdo per $n >> 1$

$a<1$ $\qquad n$ grande $\qquad g(n) \approx \frac{1}{1-a}$ $\qquad \frac{1}{1-a} = 2 \qquad a = \frac{1}{2}$

$x+y=xy \qquad xy-x-y=0 \qquad (x-1)(y-1)=1 \qquad y=\frac{x}{x-1}$

$$g(x)\,g\left(\frac{x}{x-1}\right)=g(x)+g\left(\frac{x}{x-1}\right)$$

$\frac{p}{q}\in(1,2) \qquad \frac{p}{q}=\frac{\frac{p}{p-q}}{\frac{q}{p-q}}=\frac{\frac{p}{p-q}}{\frac{p}{p-q}-1} \qquad x=\frac{p}{p-q} \quad y=\frac{p}{q}$

$g(y)=\frac{g(x)}{g(x)-1} \qquad g\left(\frac{p}{q}\right)=\frac{g\left(\frac{p}{p-q}\right)}{g\left(\frac{p}{p-q}\right)-1}$

$x=3$

$$\underset{1+2a}{g(3)}\;\underset{1+2g\left(\frac12\right)}{g\left(\frac32\right)}=\underset{1+2a}{g(3)}+\underset{1+2g\left(\frac12\right)}{g\left(\frac32\right)} \qquad g\left(\frac12\right)=\frac{1}{2a^2}$$

$x=2 \quad y=\frac12 \quad \dots \quad 2a^2-3a+1=0 \qquad a=\frac12 \quad a=1$

$\boxed{A5}$ $\quad a,b,c>0 \qquad \sum ab=1$

$$\sum\sqrt[4]{\frac{\sqrt3}{a}+6\sqrt3\,b}\overset{?}{\le}\frac{1}{abc}$$

$$\sum\sqrt[4]{\frac{1+6ab}{a}}\overset{?}{\le}\frac{3^{-1/8}}{abc}$$

$$\sum\sqrt[4]{\frac{7ab+bc+ca}{a}}\overset{?}{\le}\frac{3^{-1/8}}{abc}$$

$$\sum\sqrt[4]{a^3b^4c^4(7ab+bc+ca)}\overset{?}{\le}3^{-1/8} \qquad \sqrt[3]{abc}\le\underbrace{\sqrt{\frac{ab+bc+ca}{3}}}_{=\frac{1}{\sqrt3}}\le\frac{a+b+c}{3}$$

$$\sum\sqrt[4]{bc(7ab+bc+ca)}\overset{?}{\le}3 \qquad (abc)^{3/4}\le 3^{-9/8}$$

$$\sum\sqrt{bc}\,\sum\sqrt{7ab+bc+ca}\overset{?}{\le}9$$

$$3\underbrace{\sum bc}_{1}\cdot 3\cdot\underbrace{\sum(7ab+bc+ca)}_{9}\overset{?}{\le}81$$

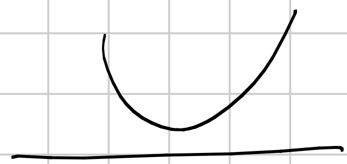
$\boxed{A6}$ $a,b,c>0$ $\quad 1 \overset{?}{\le} \sum(1-a)^2 + \frac{2\sqrt{2}\,abc}{\sqrt{\sum a^2}}$

$$\underbrace{\sum a^2}_{\mathrm{II}} - 2\underbrace{\sum a}_{\mathrm{I}} + \underbrace{2}_{0} + 2\sqrt{2}\,\underbrace{\frac{abc}{\sqrt{\sum a^2}}}_{\mathrm{II}} \overset{?}{\ge} 0$$

$$\left(\sum a^2 - 6\sqrt{\tfrac{1}{3}\sum a^2} + 2 + 2\sqrt{2}\,\frac{abc}{\sqrt{\sum a^2}} \overset{?}{\ge} 0\right)$$

$$q = \sqrt{\tfrac{1}{3}\sum a^2} \qquad \alpha = \frac{\sqrt[3]{abc}}{q} \qquad \beta = \frac{\frac{1}{3}\sum a}{q}$$

$$3q^2 - 6\beta q + 2 + \frac{2\sqrt{2}}{\sqrt{3}}\alpha^3 q^2 \overset{?}{\ge} 0$$



$$\Delta \overset{?}{\le} 0 \qquad \left(3 + \frac{2\sqrt{2}}{\sqrt{3}}\alpha^3\right)q^2 - 6\beta q + 2 \ge 0$$

$$9\beta^2 - 6 - \frac{4\sqrt{2}}{\sqrt{3}}\alpha^3 \overset{?}{\le} 0$$

questa verrà omogenea in a,b,c

$$9\,\frac{\frac{1}{9}(\sum a)^2}{q^2} - 6 - \frac{4\sqrt{2}}{\sqrt{3}}\,\frac{abc}{q^3} \overset{?}{\le} 0$$

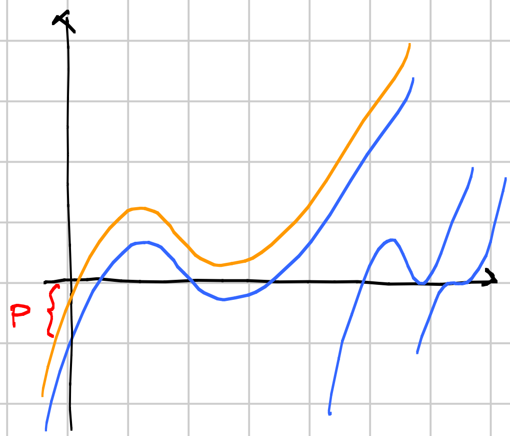
$$\underbrace{q}_{S,Q}\,(\underbrace{\sum a}_{S})^2 - 6\underbrace{q^3}_{S,Q} - \frac{4\sqrt{2}}{\sqrt{3}}\,\underbrace{abc}_{P} \overset{?}{\le} 0 \qquad q = \sqrt{\tfrac{1}{3}\sum a^2}$$

Metodo S,P,Q (∘ ABC ∘ PQR) $\qquad S = \sum a \quad P = abc \quad Q = \sum ab$

$$S^2 = \left(\sum a\right)^2 = \sum a^2 + 2\sum ab = \sum a^2 + 2Q \qquad \sum a^2 = S^2 - 2Q$$

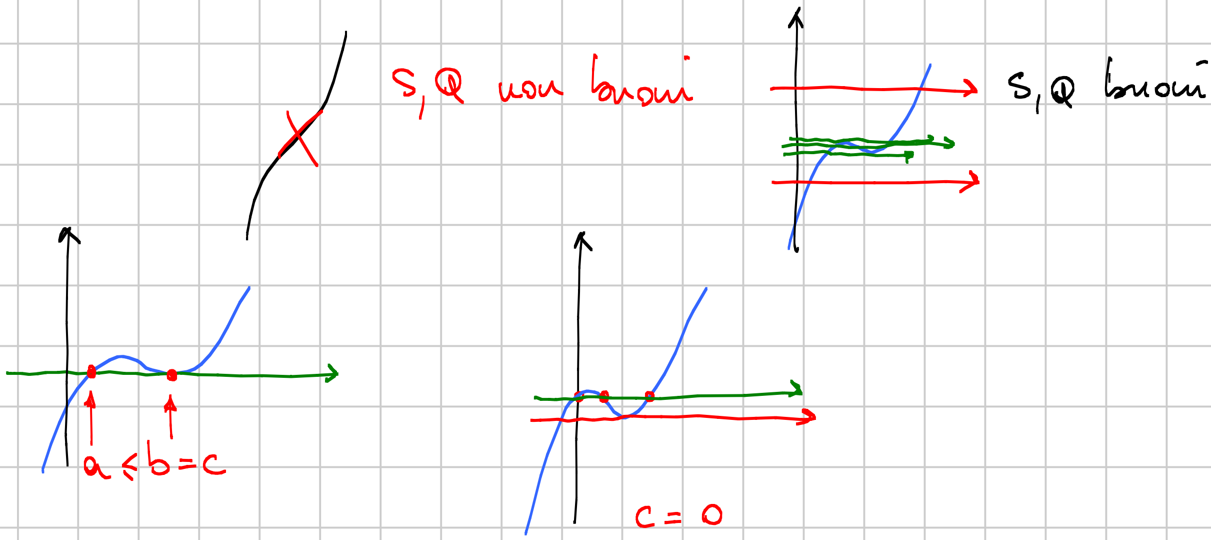
$$q = \frac{1}{\sqrt{3}}\sqrt{S^2 - 2Q}$$

$$p(x) := (x-a)(x-b)(x-c) = x^3 - Sx^2 + Qx - P$$



Per ogni coppia S, Q dimostro la disug. per tutti i valori di P compatibili

In questo caso basta P minimo



Caso $c=0$

$$(\Sigma a)^2 \overset{?}{\leq} 6q^2 \qquad q^2 = \frac{1}{3}(a^2+b^2)$$

$$(a+b)^2 \overset{?}{\leq} 2(a^2+b^2) \qquad \text{ok}$$

Caso $a \leq b = c$

$$q\left((\Sigma a)^2 - 6q^2\right) - \frac{4\sqrt{2}}{\sqrt{3}}ab^2 \overset{?}{\leq} 0 \qquad q^2 = \frac{1}{3}(a^2+2b^2)$$

$$\sqrt{a^2+2b^2}\left((a+2b)^2 - 2a^2 - 4b^2\right) - 4\sqrt{2}ab^2 \overset{?}{\leq} 0$$

$$a^2+4ab+\cancel{4b^2}-2a^2-\cancel{4b^2}$$

e divido per a

$$\sqrt{a^2+2b^2}\,(4b-a) - 4\sqrt{2}\,b^2 \overset{?}{\leq} 0 \qquad 0 < x := \frac{a}{b} \leq 1$$

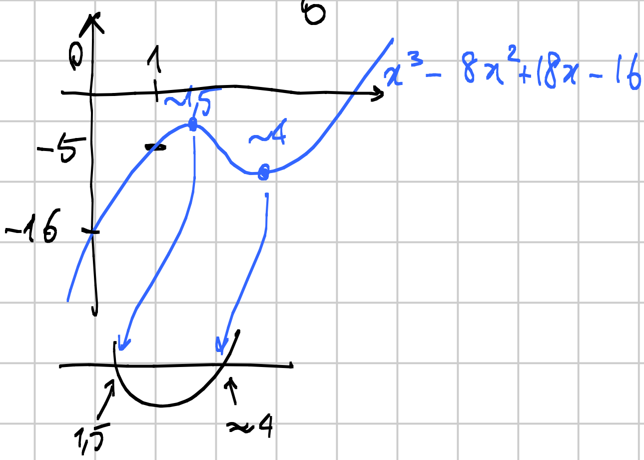
$$\sqrt{x^2+2}\,\underbrace{(4-x)}_{>0} \overset{?}{\leq} 4\sqrt{2}$$

$$(x^2+2)(16-8x+x^2) \overset{?}{\leq} 32$$

$$x^3 - 8x^2 + 18x - 16 \overset{?}{\leq} 0$$

derivo : $3x^2 - 16x + 18$

$$x_{1,2} = \frac{8 \pm \sqrt{10}}{3}$$



è crescente da $-\infty$ a $\simeq 1{,}5$ quindi anche tra 0 e 1

in 1 vale -5 in 0 vale -16